INTEGRATION ON HYPERSPHERES IN $\boldsymbol{R}^n$

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Abstract. The subject of this paper are the hyperspheres in n-dimensional Euclidean space, which are the intersection of sphere and few planes. The paper concerns only the spheres with the center in the origin of coordinate system and the planes crossing through this point. Hypersphere parametrization and some integration formulas will be shown.

1. Explanation of notations used

In the beginning we would like to explain some of the symbols and theorems used in the first part of this paper.

Section of the vector $v = [V_1, V_2, ..., V_n] \in \boldsymbol{R}^n$ from the first to k-th coefficients is the vector $\tilde{v}^k$ given by:

$$\tilde{v}^k = [V_1, V_2, ..., V_k] \in \boldsymbol{R}^k \tag{1}$$

Gramian matrix $G(a_1, a_2, ..., a_k)$ of elements $a_1, a_2, ..., a_k \in R^n$ is a symmetrical square matrix, which elements are the scalar products of corresponding vectors:

$$G(a_1, a_2, ..., a_k) = \begin{bmatrix} \langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \dots & \langle a_1, a_k \rangle \\ \langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \dots & \langle a_2, a_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_k, a_1 \rangle & \langle a_k, a_2 \rangle & \dots & \langle a_k, a_k \rangle \end{bmatrix} \tag{2}$$

Gramian determinant of elements $a_1, a_2, ..., a_k \in R^n$ is a determinant of gramian matrix $G(a_1, a_2, ..., a_k)$:

$$\left|G(a_1,a_2,...,a_k)\right| = \begin{vmatrix} \|a_1\|^2 & \langle a_1,a_2\rangle & \cdots & \langle a_1,a_k\rangle \\ \langle a_1,a_2\rangle & \|a_2\|^2 & \cdots & \langle a_2,a_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_1,a_k\rangle & \langle a_2,a_k\rangle & \cdots & \|a_k\|^2 \end{vmatrix} \tag{3}$$

If vector b is orthogonal to every vector $a_1,a_2,..,a_k$ then

$$\left|G(b,a_1,a_2,...,a_k)\right| = \|b\|^2 \left|G(a_1,a_2,...,a_k)\right| \tag{4}$$

Theorem 1.1 (of the norm of vector product)

If c is a vector product of $a_1,a_2,...,a_k \in \boldsymbol{R}^n$ ($c = a_1 \times a_2 \times ... \times a_{n-1}$), *then its norm is equal to square root of gramian determinant of* $a_1,a_2,...,a_k \in \boldsymbol{R}^n$:

$$\|c\| = \sqrt{\left|G(a_1,a_2,...,a_{n-1})\right|} \tag{5}$$

2. Parametrization of hyperspheres

Consider a set of k+1 equations with n variables:

$$\begin{cases} x_1^{\ 2} + x_2^{\ 2} + ... + x_n^{\ 2} = r^2 \\ A_{11}x_1 + A_{12}x_2 + ... + A_{1n}x_n = 0 \\ A_{21}x_1 + A_{22}x_2 + ... + A_{2n}x_n = 0 \\ \ldots\ldots \\ A_{k1}x_1 + A_{k2}x_2 + ... + A_{kn}x_n = 0 \end{cases} \tag{6}$$

We assume that $2 \le k \le n-2$ (when $k=1$ refer to paper [3]) and

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{vmatrix} \ne 0 \tag{7}$$

Let $a_1 = [A_{11}, A_{12},..., A_{1n}]$, $a_2 = [A_{21}, A_{22},..., A_{2n}]$, $\ldots$, $a_k = [A_{k1}, A_{k2},..., A_{kn}]$. Now we can begin solving the equation (6). We start by finding the set of vectors orthogonal to each other and orthogonal to every vector a_i. This set will be a basis for the solutions of (6).

Step 1
In the first step, we calculate the $(k+1)$-dimensional vector w_1, according to formula:

$$w_1 = \tilde{a}_1^{k+1} \times \tilde{a}_2^{k+1} \times ... \times \tilde{a}_k^{k+1} \tag{8}$$

Every coefficient of w_1 can be calculated by the formula:

$$w_{1i} = (-1)^{i-1} D_i \tag{9}$$

where D_i is a determinant created by omitting the *i*-th coefficient in every vector

$$D_i = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1\,i-1} & A_{1\,i+1} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2\,i-1} & A_{2\,i+1} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{k\,i-1} & A_{k\,i+1} & \cdots & A_{kn} \end{vmatrix} \tag{10}$$

According to theorem 1.1 the norm of vector w_1 is equal to:

$$\|w_1\| = \sqrt{\left|G\left(\tilde{a}_1^{k+1}, \tilde{a}_2^{k+1}, ..., \tilde{a}_k^{k+1}\right)\right|} \tag{11}$$

Step 2
In the second step, we calculate the $(k+2)$-dimensional vector w_2, according to formula:

$$w_2 = w_1 \times \tilde{a}_1^{k+2} \times \tilde{a}_2^{k+2} \times ... \times \tilde{a}_k^{k+2} \tag{12}$$

The norm of vector w_2 is equal to:

$$\|w_2\| = \sqrt{\left|G\left(w_1, \tilde{a}_1^{k+2}, \tilde{a}_2^{k+2}, ..., \tilde{a}_k^{k+2}\right)\right|} = \|w_1\| \sqrt{\left|G\left(\tilde{a}_1^{k+2}, \tilde{a}_2^{k+2}, ..., \tilde{a}_k^{k+2}\right)\right|} \tag{13}$$

Step *n–k*–1
In this step we calculate $(n-1)$-dimensional vector

$$w_{n-k-1} = w_{n-k-2} \times ... \times w_2 \times w_1 \times \tilde{a}_1^{n-1} \times \tilde{a}_2^{n-1} \times ... \times \tilde{a}_k^{n-1} \tag{14}$$

with the norm

$$\|w_{n-k-1}\| = \|w_{n-k-2}\| ... \|w_2\| \; \|w_1\| \sqrt{\left|G\left(\tilde{a}_1^{k+2}, \tilde{a}_2^{k+2}, ..., \tilde{a}_k^{k+2}\right)\right|} \tag{15}$$

Step *n–k*
Finally we calculate *n*-dimensional vector

$$w_{n-k} = w_{n-k-1} \times ... \times w_2 \times w_1 \times a_1 \times a_2 \times ... \times a_k \tag{16}$$

with the norm

$$\|w_{n-k}\| = \|w_{n-k-1}\| \ldots \|w_2\| \, \|w_1\| \sqrt{|G(a_1, a_2, \ldots, a_k)|} \tag{17}$$

Next we replace the orthogonal set of vectors $w_1, w_2, \ldots, w_{n-k}$ by the orthonormal set, according to the formula:

$$v_i = \frac{\tilde{w}_i}{\|w_i\|} \tag{18}$$

where $\tilde{w}_i = [w_i, 0, \ldots, 0] \in \boldsymbol{R}^n$ (0 at $(n-k-i)$ -places).
Set of vectors v_i is a basis for solutions of (6). Let us define $l = n - k$ as a codimension of our solution, and a matrix V as follows:

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{l1} & v_{l2} & \cdots & v_{ln} \end{bmatrix} \tag{19}$$

Now, let us recall an *l*-dimensional sphere parametrization:

$$\begin{cases} x_1 = r \cos t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\ x_2 = r \sin t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\ x_3 = r \quad \sin t_2 \cos t_3 \ldots \cos t_{l-1} \\ \vdots \\ x_l = r \quad \sin t_{l-1} \end{cases} \tag{20}$$

where $t_1 \in [0, 2\pi]$, $t_2, \ldots, t_{l-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
By implementing a vector function $F(t)$:

$$F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_l(t) \\ f_{l+1}(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} \cos t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\ \sin t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\ \vdots \\ \sin t_{l-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{21}$$

where $t=[t_1,t_2,...t_{l-1}]$, we can simply write sphere parametrization as $rF(t)$.
We calculate the parametrization of hypersphere by solving the following equation:

$$V\cdot\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=rF(t) \tag{22}$$

Because matrix V is orthogonal, the solution of (6) is given by:

$$x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=V^T rF(t)=\begin{bmatrix}r\left(v_{11}f_1(t)+v_{21}f_2(t)+...+v_{l1}f_l(t)\right)\\r\left(v_{12}f_1(t)+v_{22}f_2(t)+...+v_{l2}f_l(t)\right)\\\vdots\\r\left(v_{1n}f_1(t)+v_{2n}f_2(t)+...+v_{ln}f_l(t)\right)\end{bmatrix} \tag{23}$$

3. Notations and theorems used for integration on hyperspheres

Set of points given by (23) will be called an *l*-dimensional hypersphere and noted as S_l:

$$S_l=\begin{cases}r\left(v_{11}f_1(t)+v_{21}f_2(t)+...+v_{l1}f_l(t)\right)\\r\left(v_{12}f_1(t)+v_{22}f_2(t)+...+v_{l2}f_l(t)\right)\\\vdots\\r\left(v_{1n}f_1(t)+v_{2n}f_2(t)+...+v_{ln}f_l(t)\right)\end{cases} \tag{24}$$

Partial derivative of a vector x given by (23) with respect to variable t_j is a following vector:

$$\frac{\partial x}{\partial t_j}=\left[\frac{\partial x_1}{\partial t_j},\frac{\partial x_2}{\partial t_j},...,\frac{\partial x_n}{\partial t_j}\right] \tag{25}$$

where:

$$\frac{\partial x_i}{\partial t_j}=r\left(v_{1i}\frac{\partial f_1}{\partial t_j}+v_{2i}\frac{\partial f_2}{\partial t_j}+\ldots+v_{li}\frac{\partial f_l}{\partial t_j}\right) \tag{26}$$

We should note as a fact, that

$$\left\|\frac{\partial x}{\partial t_j}\right\| = \begin{cases} r\cos t_{j+1}\cos t_{j+2}...\cos t_{l-1} & when\ j < l-1 \\ r & when\ j = l-1 \end{cases} \tag{27}$$

For every given matrix

$$M = M_{k\times n} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kn} \end{bmatrix} \tag{28}$$

we can define a number $|M|$, called this matrix module, by

$$|M| = \sqrt{\sum_{i_1<i_2<...i_k} \begin{vmatrix} m_{i_1 1} & \dots & m_{i_1 k} \\ \vdots & \ddots & \vdots \\ m_{i_k 1} & \dots & m_{i_k k} \end{vmatrix}^2} \tag{29}$$

Theorem 3.1 (of matrix module, [1])

Let M *be a matrix given by (28), and let* $m_i = [m_{i1}, m_{i2}, ..., m_{in}]$ *be the i-th row of that matrix. Then the matrix module is equal to square root of gramian matrix of its rows*

$$|M| = \sqrt{M\,M^T} = \sqrt{|G(m_1, m_2, ..., m_n)|} \tag{30}$$

Theorem 3.2 (calculating the matrix determinant according to several rows, [1])

Let $k < n$ *and* $l = n-k$ *. Then we have*

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{l1} & b_{l2} & \dots & b_{ln} \end{vmatrix} = \sum_{\substack{1\le i_1<...<i_k\le n \\ 1\le j_1<...<j_l\le n}} \operatorname{sgn}(i_1,...,i_k, j_1,..., j_l) \begin{vmatrix} a_{1i_1} & \dots & a_{1i_k} \\ \vdots & \ddots & \vdots \\ a_{ki_1} & \dots & a_{ki_k} \end{vmatrix} \begin{vmatrix} b_{1j_1} & \dots & b_{1j_l} \\ \vdots & \ddots & \vdots \\ b_{lj_1} & \dots & b_{lj_l} \end{vmatrix} \tag{31}$$

Theorem 3.3 ([2], §49, Tw. 9)

Let $E \subset R^m$, $\gamma(E)$ *be a hypersurface of parametrization* $\gamma : E \to R^m$ *and* $t \in R^m$. *If function* $f : \gamma(E) \to R$ *is inregrable, then the following formula follows:*

$$\int_{\gamma(E)} f \, d\gamma(t) = \int_E (f \circ \gamma) \left| J\gamma(t) \right| dt \tag{32}$$

4. Integration on hyperspheres

In the last part of this paper we develop the subject explained in [3], concerning integrals of the differential forms. We show the proof to the hypothesis shown there and expand it to the case, where the number of planes is greater than 1.

A function ω given by

$$\omega_{l-1} = \sum_{1 \le i_1 < i_2 < \ldots < i_{l-1} \le n} f_{i_1, i_2, \ldots, i_l} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_{l-1}} \tag{33}$$

is called a differential form of $l-1$ order in R^n.

We define the integral from that form as follows:

$$\int_{S_l} \omega_{l-1} = \sum_{1 \le i_1 < i_2 < \ldots < i_{l-1} \le n} \int_\Omega f_{i_1, i_2, \ldots, i_l} \circ \left(x_1(t), x_2(t), \ldots, x_n(t) \right) \left| JS_l \left(t_1, t_2, \ldots, t_{l-1} \right) \right| dt \tag{34}$$

In last paragraph of [3] there was a following hypothesis stated:

If ω_{l-1} is a differential form given by

$$\omega_{n-1} = \frac{(-1)^{n-1}}{\left(x_1^2 + \ldots + x_n^2 \right)^{\frac{n-1}{2}}} \sum_{1 \le i_1 < i_2 < \ldots < i_{n-2} \le n} \sum_{j_1, j_2 \ne i_1, i_2, \ldots, i_{n-2}} \operatorname{sgn}(i_1, i_2, \ldots, i_{n-2}, j_1, j_2) \begin{vmatrix} A_{j_1} & A_{j_2} \\ x_{j_1} & x_{j_2} \end{vmatrix} dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_{n-2}}$$

then the integral over hypersphere S_{n-1} is equal to

$$\int_{S_{n-1}} \omega_{n-2} = \sigma_{l-1} \sqrt{A_1^2 + A_2^2 + \ldots + A_n^2}$$

where σ_{l-1} denotes the measure of the surface of $(l-1)$-dimensional sphere of radius $r = 1$.

For every l the formula follows:

$$\sigma_{l-1} = \frac{l \pi^{\frac{l}{2}}}{\Gamma\left(\frac{l}{2} + 1 \right)} \tag{35}$$

In paper [4], the hypothesis was extended and later proven, so the following theorem was introduced:

Theorem (of integrating the differential forms over hypersphere)

Let S_l be a l-dimensional hypersphere being an intersection of sphere in $\boldsymbol{R}^n$ and k planes (that is S_l be a solution of (6)), and ω_{l-1} be a differential form given by

$$\omega_{l-1} = r^{-1} \sum_{\substack{1 \le i_1 < i_2 < .. < i_{l-1} \le n \\ j_1 \ne i_1,...,i_{l-1} \\ ... \\ j_{k+1} \ne i_1,...,i_{l-1}}} \operatorname{sgn}\ (i_1,...,i_{l-1}, j_1,...,j_{k+1}) \begin{vmatrix} A_{1j_1} & A_{1j_2} & \cdots & A_{1j_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{kj_1} & A_{kj_2} & \cdots & A_{kj_{k+1}} \\ x_{j_1} & x_{j_2} & \cdots & x_{j_{k+1}} \end{vmatrix} dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_{l-1}} \tag{36}$$

then the integral of ω_{l-1} over S_l is equal to

$$\int_{S_l} \omega_{l-1} = \sigma_{l-1} \sqrt{\left| G(a_1, a_2, ..., a_k) \right|} \tag{37}$$

Proof:

Let $t = (t_1, t_2, ..., t_{l-1})$, $\Omega = \left\{ (t_1, t_2, ..., t_{l-1}) : 0 \le t_1 \le 2\pi, -\frac{\pi}{2} \le t_2 \le \frac{\pi}{2}, ..., -\frac{\pi}{2} \le t_{l-1} \le \frac{\pi}{2} \right\}$.

According to theorem 3.3 we can bring integral over S_l to integral over Ω:

$$\int_{S_l} \omega_{l-1} = \int_{\Omega} r^{-1} \sum_{\substack{1 \le i_1 < ... < i_{n-2} \le n \\ j_1 \ne i_1,...,i_{l-1} \\ ... \\ j_{k+1} \ne i_1,...,i_{l-1}}} \operatorname{sgn}\ (i_1,..., i_{l-1}, j_1,..., j_{k+1}) \Theta dt \tag{38}$$

where

$$\Theta = \begin{vmatrix} A_{1j_1} & A_{1j_2} & \cdots & A_{1j_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{kj_1} & A_{kj_2} & \cdots & A_{kj_{k+1}} \\ x_{j_1} & x_{j_2} & \cdots & x_{j_{k+1}} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_{i_1}}{\partial t_{i_1}} & \frac{\partial x_{i_2}}{\partial t_{i_1}} & \cdots & \frac{\partial x_{i_{l-1}}}{\partial t_{i_1}} \\ \frac{\partial x_{i_1}}{\partial t_{i_2}} & \frac{\partial x_{i_2}}{\partial t_{i_2}} & \cdots & \frac{\partial x_{i_{l-1}}}{\partial t_{i_2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{i_1}}{\partial t_{i_{l-1}}} & \frac{\partial x_{i_2}}{\partial t_{i_{l-1}}} & \cdots & \frac{\partial x_{i_{l-1}}}{\partial t_{i_{l-1}}} \end{vmatrix} \tag{39}$$

According to theorem 3.2, equation (38) can be simply written as:

$$\int_{S_l} \omega_{l-1} = \int_{\Omega} r^{-1} \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{l-1}} & \frac{\partial x_2}{\partial t_{l-1}} & \cdots & \frac{\partial x_n}{\partial t_{l-1}} \\ A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kn} \\ x_1 & x_2 & \cdots & x_n \end{vmatrix} dt \tag{40}$$

From the theorem 3.1 it follows, that the integral above can be written as

$$\int_{S_l} \omega_{l-1} = r^{-1} \int_{\Omega} \sqrt{\left| G\left(\frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, \ldots, \frac{\partial x}{\partial t_{l-1}}, a_1, a_2, \ldots, a_k, x \right) \right|} dt \tag{41}$$

Every vector $x, \frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, \ldots, \frac{\partial x}{\partial t_{l-1}}$ is a linear combination of $v_1, v_2, \ldots, v_l$, so following the property of gramian determinant (see (4)), we can write:

$$\int_{S_l} \omega_{l-1} = r^{-1} \int_{\Omega} \left\| \frac{\partial x}{\partial t_1} \right\| \cdot \left\| \frac{\partial x}{\partial t_2} \right\| \cdot \ldots \cdot \left\| \frac{\partial x}{\partial t_{l-1}} \right\| \cdot \|x\| \sqrt{\left| G(a_1, a_2, \ldots, a_k) \right|} dt \tag{42}$$

By inserting the values of vector norms, we obtain (see (27)):

$$\left\| \frac{\partial x}{\partial t_1} \right\| \cdot \left\| \frac{\partial x}{\partial t_2} \right\| \cdot \ldots \cdot \left\| \frac{\partial x}{\partial t_{l-1}} \right\| \cdot \|x\| = r^l \cos t_2 \cos^2 t_3 \ldots \cos^{l-2} t_{l-1} \tag{43}$$

Inserting this value to our integral yields:

$$\int_{S_l} \omega_{l-1} = \sqrt{\left| G(a_1, a_2, \ldots, a_k) \right|} \int_{\Omega} \cos t_2 \cos^2 t_3 \ldots \cos^{l-2} t_{l-1} dt \tag{44}$$

Calculating the following integral brings the proof to an end:

$$\int_{\Omega} \cos t_2 \cos^2 t_3 \ldots \cos^{l-2} t_{l-1} dt = \frac{l \pi^{\frac{1}{2}}}{\Gamma\left(\frac{l}{2} + 1\right)} = \sigma_{l-1} \tag{45}$$

$$\int_{S_l} \omega_{l-1} = \sigma_{l-1} \sqrt{\left| G(a_1, a_2, \ldots, a_k) \right|} \tag{46}$$

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