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OPERATIONS ON INTUITIONISTIC FUZZY VALUES IN MULTIPLE CRITERIA DECISION MAKING

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Abstract. This paper presents an analysis of the basic definitions of the theory of intuitionistic fuzzy sets. Some undesirable properties of commonly used operations on intuitionistic fuzzy values are revealed and the ways to approve the properties of intuitionistic fuzzy arithmetic are proposed. The aim of the analysis presented in the paper is to propose a set of operations on intuitionistic fuzzy values, which provides non-controversial results of the solution of multiple criteria decision making problems in the intuitionistic fuzzy setting. The theoretical analysis is illustrated with numerical examples.

Introduction

The intuitionistic fuzzy set introduced by Atanassov [1] may be treated as a generalization of fuzzy sets theory which currently is used mainly for solving multiple criteria decision making problems (*MCDM*) [2-8] and group decision making problems [9-11] when the values of local criteria (attributes) of alternatives and/or their weights are intuitionistic fuzzy values (IFV).

As the so-called "intuitionistic fuzzy set theory" was independently introduced by Takeuti and Titani [12], there are some terminological difficulties in fuzzy set theory. Dubois et al. [13] noted that "Takeuti-Titani's intuitionitic fuzzy logic is simply an extension of intuitionistic logic [14], i.e., all formulas provable in the intuitionistic logic are provable in their logic. Intuitionistic fuzzy set theory by Takeuti and Titani is an absolutely legitimate approach, in the scope of intuitionistic logic, but it has nothing to do with Atanassov's intuitionistic fuzzy sets." Therefore, to avoid a misunderstanding, in this paper, Atanassov's intuitionistic fuzzy sets is abbreviated as *A-IFS*. Generally, Atanassov's model (*A-IFS*) may be treated as a classification model subject to a valuation space with three classes and defining certain structure [15].

The concept of *A-IFS* is based on the simultaneous consideration of membership μ and non-membership ν of an element of a set in the set itself [1].

It is postulated that $0 \le \mu + \nu \le 1$. A similar approach, the so-called vague sets, proposed by Gau and Buehrer in [16] is proved to be equivalent to *A-IFS* (see [17]). Since vague sets were proposed later than *A-IFS*, in this paper, we shall always speak of *A-IFS*. To make the basic definitions of *A-IFS* clearer and more transparent, consider an illustrative example from [18].

Let E be the set of all countries with elective governments. Assume that we now for every country $x \in E$ the percentage of the electorate that has voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = M(x)/100$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number corresponds to the part of the electorate who has not voted for the government. Using fuzzy set theory alone, we cannot consider this value in more detail. However, if we define ν (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who has not voted at all or has spoiled their ballots, and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, hesitation degree, etc.). Thus we can construct the set $\left\{\left\langle x, \mu(x), \nu(x)\right\rangle \middle| x \in E\right\}$ and obviously, $0 \le \mu + \nu \le 1$. It is clear that for every ordinary fuzzy set $\pi(x) = 0$ for each $x \in E$ and these sets have the form $\left\{\left\langle x, \mu(x), 1 - \mu(x)\right\rangle \middle| x \in E\right\}$.

As the most important applications of *A-IFS* are decision making problems when the values of local criteria (attributes) of alternatives and/or their weights are *IFVs*, it seems quite natural that the resulting alternative evaluation should be an *IFV* as well. Therefore, appropriate operations on *IFVs* used for aggregating local criteria should be properly defined. Obviously, if the final scores of alternatives are *IFVs,* then appropriate methods for their comparison are needed to select the best alternative.

Since there are many different operations on *IFVs* and methods for their comparison and aggregation have been proposed in the literature, the aim of this paper is to analyze their merits and drawbacks and extract those which provide the results of operations on *IFVs* and aggregation with acceptable properties. It is safe to say that currently *A-IFS* is an open theory, since there are many competing definitions of operations on *IFVs* which often lead to non-intuitive or non-acceptable results of the aggregation of *IFVs.*

Therefore, the rest of the paper is set out as follows. Section 1 presents the basic definition of *A-IFS* , the commonly used arithmetical operations on *IFVs* and the method for their comparison. In Section 2, we provide a critical analysis of the operations presented in Section 2 to elicit their disadvantages and propose a compromising set of operations with at least satisfactory algebraic properties and reasonable results of *IFVs* aggregation. The last section concludes with some remarks.

1. Basic definitions of intuitionistic fuzzy set theory

In [1], Atanassov defined *A-IFS* as follows.

Definition 1. Let $A = \{x_1, x_2, ..., x_n\}$ be a finite universal set. An intuitionistic fuzzy set A in X is an object having the following form: A = $=\left\{\left\langle x_j, \mu_A(x_j), \nu_A(x_j)\right\rangle \middle| x_j \in X\right\}$, where functions $\mu_A : X \to [0,1]$, $x_j \in X \to \mu(x_j) \in [0.1]$

and $\nu_A : X \to [0,1]$, $x_j \in X \to \nu(x_j) \in [0.1]$ define the degree of membership and degree of nonmembership of element $x_j \in X$ to set $A \subseteq X$, respectively, and for every $x_j \in X$ we have $0 \le \mu_A(x_j) + \nu_A(x_j) \le 1$.

Following [1], we call $\pi_A(x_j) = 1 - \mu_A(x_j) + \nu_A(x_j)$ the intuitionistic index (or the hesitation degree) of element x_j in set A. It is obvious that for every $x_j \in X$ we have $0 \le \pi_A \le 1$.

As we noted above, *A-IFS* is a generalization of the standard fuzzy set. Therefore, all the results which are typical for ordinary fuzzy sets theory can be transformed in the framework of *A-IFS* as well as. Moreover, any research based on fuzzy sets can be described in terms of *A-IFS*. On the other hand, in the framework of *A-IFS*, there are not only operations similar to ordinary fuzzy set ones, but also operators that cannot be defined in the case of ordinary fuzzy sets.

The operations of addition $\oplus$ and multiplication $\otimes$ on *IFVs* were defined by Atanassov [19] as follows. Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be *IFVs*. Then

$$A \oplus B = \langle \mu_A + \mu_B - \mu_A \mu_B, \nu_A \nu_B \rangle, \tag{1}$$

$$A \otimes B = \langle \mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B \rangle. \tag{2}$$

These operations were constructed in such a way that they produce *IFVs* since it is easy to prove that $0 \le \mu_A + \mu_B - \mu_A\mu_B + \nu_A\nu_B \le 1$ and $0 \le \mu_A\mu_B + \nu_A + \nu_B - \nu_A\nu_B \le 1$.

Using expressions (1) and (2), in [20] the following equations were obtained for any integer n = 1,2,..:

$$nA = A \oplus ... \oplus A = \langle 1 - (1 - \mu_A)^n, \nu_A^n \rangle, \; A^n = A \otimes ... \otimes A = \langle \mu_A^n, 1 - (1 - \nu_A)^n \rangle$$

It was proved later that these operations produce *IFVs* not only for integer n, but also for all real values $\lambda > 0$, i.e.

$$\lambda A = \langle 1 - (1 - \mu_A)^\lambda, \nu_A^\lambda \rangle, \tag{3}$$

$$A^\lambda = \langle \mu_A^\lambda, 1 - (1 - \nu_A)^\lambda \rangle. \tag{4}$$

Operations (1)-(4) have the following algebraic properties [21]:

Theorem 1. Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be *IFVs* and $\lambda > 0$ be a real value. Then

$$A \oplus B = B \oplus A, \tag{5}$$

$$A \otimes B = B \otimes A, \tag{6}$$

$$\lambda(A \oplus B) = \lambda A \oplus \lambda B, \tag{7}$$

$$(A \otimes B)^{\lambda} = A^{\lambda} \otimes B^{\lambda}, \tag{8}$$

$$\lambda_1 A \oplus \lambda_2 A = (\lambda_1 + \lambda_2) A,\ \lambda_1, \lambda_2 > 0, \tag{9}$$

$$A^{\lambda_1} \otimes A^{\lambda_2} = A^{\lambda_1 + \lambda_2},\ \lambda_1, \lambda_2 > 0. \tag{10}$$

Operations (1)-(4) are used to aggregate local criteria for the solution of *MCDM* problems in the intuitionistic fuzzy setting.

Let $A_1,...,A_n$ be *IFV*s representing the values of local criteria and $w_1,\dots,w_n$, $\sum_{i=1}^{n} w_i = 1$, be their weights.

Then the Intuitionistic Weighted Arithmetic Mean (*IWAM*) can be obtained using operations (1) and (3) as follows:

$$IWAM = w_1 A_1 \oplus w_2 A_2 \oplus ... \oplus w_n A_n = \left\langle 1 - \prod_{i=1}^{n} (1 - \mu_{A_i})^{w_i}, \prod_{i=1}^{n} \nu_{A_i}^{w_i} \right\rangle \tag{11}$$

This aggregating operator provides *IFV*s and currently is the most popular in the solution of *MCDM* problems in the intuitionistic fuzzy setting.

An important problem is the comparison of *IFV*s. This problem arises, e.g., when we have to choose the best alternative in the framework of *MCDM* and the final scores of alternatives are presented by *IFV*s, e.g., by *IWAM*. Bustince and Burillo [22] analyzed the general properties of intuitionistic fuzzy relations and showed that the definition of these properties does not always coincide with the definition of the properties of fuzzy relations. Therefore, the specific methods which are rather of the heuristic nature were developed to compare *IFVs*. For this purpose, Chen and Tan [23] proposed to use the so-called score function (or net membership) $S(x) = \mu(x) - \nu(x)$, where x is an *IFV*. Let a and b be *IFVs*. It is intuitively appealing that if $S(a) > S(b)$ then a should be greater (better) than b, but if $S(a) = S(b)$ this does not always mean that a is equal to b. Therefore, Hong and Choi [24] in addition to the above score function introduced the so-called accuracy function $H(x) = \mu(x) + \nu(x)$ and showed that the relation between functions S and H is similar to the relation between mean and variance in statistics. Xu [25] used functions S and H to construct order relations between any pair of intuitionistic fuzzy values as follows:

$$\begin{aligned}
&\textit{If } (S(a) > S(b)), \textit{ then } b \textit{ is smaller than } a;\\
&\textit{If } (S(a) = S(b)), \textit{ then}\\
&(1)\ \textit{If } (H(a) = H(b)), \textit{ then } a = b;\\
&(2)\ \textit{If } (H(a) < H(b)) \textit{ then } a \textit{ is smaller than } b.
\end{aligned} \tag{12}$$

Basing on these relations, Xu [25] introduced the concepts of intuitionistic preference relation, consistent intuitionistic preference relation, incomplete intuitionistic preference relation and acceptable intuitionistic preference relation. The method for *IFVs* comparison based on functions S and H seems to be intuitively obvious and this is its undeniable merit.

Since the approach described above is rather of a heuristic nature, there are some different definitions of the score function proposed in literature. They were presented in [26] as follows: $S_1 = \mu - \nu$, $S_2 = \mu - \nu\pi$, $S_3 = \mu - 0.5(\mu + \pi)$, $S_4 = 0.5(\mu + \nu) - \pi$, $S_5 = \gamma\mu + (1-\gamma)(1-\nu)$, $\gamma \in [0,1]$.

In [26], these score functions were analyzed and compared, and finally the author concludes: "The observed differences among the score functions will motivate further research on the question of the justification of the five score functions in real-world decision-making problems. For routine or limited decision-making problems, S_1 is suggested to be an appropriate score function. The reasons provided for the superiority of this score function are as follows: it is easily understandable, it takes little time to calculate the score value, and it is ideal for dealing with *MCDA* problems because of its high consistency".

Therefore, in the following, we shall use the score function $S = \mu - \nu$.

2. Limitations of operations on intuitionistic fuzzy values in context of multiple criteria decision making problem

The problems with the above defined operations are revealed when they are used with order relation (12). Hence addition (1) is not an addition invariant operation. To show this, consider the following example:

Example 1. Let $A = \langle 0.5,0.3 \rangle$, $B = \langle 0.4,0.1 \rangle$ and $C = \langle 0.1,0.1 \rangle$. Since $S(A) = 0.2$ and $S(B) = 0.3$ then according to (12) we have $A < B$. On the other hand, $A \oplus C = \langle 0.55,0.03 \rangle$, $B \oplus C = \langle 0.46,0.01 \rangle$, $S(A \oplus C)$ = 0.52, $S(B \oplus C)$ = = 0.45 and from $S(A \oplus C) > S(B \oplus C)$ we get $A \oplus C > B \oplus C$.

It is worth noting that these undesirable properties currently cannot be eliminated as there are no other definitions of *IFV*s sum and ordering proposed in literature.

Another undesirable property of ordering (12) is that it is not preserved under the multiplication by a scalar: $A < B$ does not necessarily imply $\lambda A < \lambda B$, $\lambda > 0$. To illustrate this, consider the following example.

Example 2. Let $A = \langle 0.5,0.4 \rangle$, $B = \langle 0.4,0.3 \rangle$ and $\lambda = 0.5$. Then $S(A) = S(B) =$ = 0.1,

$H(A)$ = 0.9, $H(B)$ = 0.7 and from (12) we get $A > B$. Using (3) we obtain

$\lambda A = \langle 0.2928, 0.632 \rangle$, $\lambda B = \langle 0.225, 0.5477 \rangle$, $S(\lambda A) = -0.3396$, $S(\lambda B) = -0.3227$.

Since $S(\lambda A) < S(\lambda B)$ we get $\lambda A < \lambda B$.

In [27], with the use of the Lukasiewicz *t*-conorm and *t*-norm, the following expression was inferred:

$$\lambda A = \langle \lambda \mu_A, 1 - \lambda(1 - \nu_A) \rangle, \lambda \in [0,1]. \tag{13}$$

It is easy to prove that the use of (13) guarantees that for *IFV*s A and B the inequality $A<B$ always implies $\lambda A < \lambda B$ ($\lambda \in [0,1]$), but unfortunately, properties (7) and (9) with operation (13) do not hold.

An important problem with aggregation operation (11) is that it is not consistent with the aggregation operation on ordinary fuzzy sets (when $\mu = 1 - \nu$). This can be easily seen from the following example.

Example 3. Let $A = \langle 0.95, 0.01 \rangle$ and $B = \langle 0.1, 0.6 \rangle$, $w_1 = w_2 = 0.5$. Then in the framework of ordinary fuzzy sets we get the Ordinary Weighted Arithmetic Mean $OWAM = w_1 \mu_A + w_2 \mu_B = 0.5 \cdot 0.95 + 0.5 \cdot 0.01 = 0.48$ and in the framework of *A-IFS*, from (11) we obtain $IWAM = \langle 0.78, 0.077 \rangle$. We can see that the resulting value of μ obtained using $IWAM$ is considerably greater than that obtained from $OWAM$.

In [27], using the corresponding *t*-norms and *t*-conorms, the following simple expression was inferred for $IWAM$:

$$IWAM = \left\langle \sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} w_i \nu_i \right\rangle. \tag{14}$$

It is easy to show that this operator is consistent with the aggregation operation on ordinary fuzzy sets. For Example 3 from (14) we obtain $IWAM = \langle 0.48, 0.305 \rangle$.

Another problem with aggregation operation (11) is that it is not monotonic with respect to the ordering in (12). Consider an illustrative example:

Example 4. Let $A = \langle 0,1 \rangle$, $B = \langle 0.5, 0.4 \rangle$ and $C = \langle 0.3, 0.2 \rangle$. Since $S(A) = -1$, $S(B) = 0.1$, $S(C) = 0.1$ and $H(A) = 1$, $H(B) = 0.9$, $H(C) = 0.5$, from (12) we obtain $B > C > A$. Suppose $w_1 = w_2 = 0.5$. Then from (11) we obtain $IWAM(A,C) = \langle 0.1634, 0.4472 \rangle$, $IWAM(A,B) = \langle 0.2928, 0.6423 \rangle$. The score functions of these results are as follows: $S(IWAM(A,C)) = -0.2838$, $S(IWAM(A,B)) = -0.3396$. We can see that $S(IWAM(A,C)) > S(IWAM(A,B))$.

It is important that this problem does not arise when we use aggregation (14).

Summarizing, we can say that for the solution of *MCDM* problems in the intuitionistic fuzzy setting, the set of operations (1), (2), (4), (13) and (14) should be recommended as they do not provide controversial results. Nevertheless, this set of operations cannot be considered as inherently consistent as aggregation (14) cannot be directly obtained from (1) and (13).

Conclusion

The aim of this paper is to present the set of operations on intuitionistic fuzzy values which provides non-controversial results of the solution of multiple criteria decision making problems in the intuitionistic fuzzy setting. For this purpose, the properties of commonly used operations on intuitionistic fuzzy values have been analyzed and some of their undesirable properties were revealed. The ways to approve the properties of intuitionistic fuzzy arithmetic are proposed. Finally, the set of operations providing non-controversial results of solving multiple criteria decision making problems in the intuitionistic fuzzy setting is proposed. The analysis is illustrated with numerical examples throughout the paper.

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