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COMPARISON OF ANALYTIC HIERARCHY PROCESS AND SOME NEW OPTIMIZATION PROCEDURES FOR RATIO SCALING

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Abstract. Deriving true priority vectors from intuitive pairwise comparison matrices constitutes a key part of the Analytic Hierarchy Process. The Eigenvalue Method, commonly applied in the Analytic Hierarchy Process, is the most popular concept in the process of ratio scaling. It is known that the Eigenvalue Method captures transitivity in matrices that are not consistent in a unique way. However, there are other methods such as statistical estimation techniques and methods based on constrained optimisation models that are equally interesting. This article compares two novel methods for priority vectors deriving, which combine the eigenvalue concept with a constrained optimisation based approach. Evidence is provided that contrary to the logarithmic least squares method, they coincide with the Eigenvalue Method in capturing the ratio scale rank order inherent in inconsistent pairwise comparison judgments.

Introduction

Plenty of methods designed for the purpose of priorities establishment on the basis of intuitive judgments can be found in literature. Some of them are based on different statistical concepts [1-3], while others focus on constrained optimization models [4-8]. Obviously, every method proposed in the literature has its own pros and cons debate and thus one can find supporters and adversaries for each of them. Comparative studies of different prioritization methods [9-15], as well as suggestions to blend various prioritization techniques for better true priority vector estimates [16], can be found as well. It seems that most of the known prioritization methods can be numbered among constrained optimization ones [17]. However, there are also a few others, including the most popular Eigenvalue Method and two recently introduced ones, which combine the eigenvalue approach with a certain constrained optimization procedure.

1. Constrained optimization methods

These methods can be described in the following manner. Let us presume that we have only judgments (estimates) of the relative weights of a set of activities.

Then we can express them in a pairwise comparison matrix (PCM) denoted as ***A*** with elements $a_{ij} = a_i/a_j$ that can be presented as follows:

$$\boldsymbol{A} = \begin{bmatrix} a1/a1 & a1/a2 & a1/a3 & \ldots & a1/an \\ a2/a1 & a2/a2 & a2/a3 & \ldots & a2/an \\ a3/a1 & a3/a2 & a3/a3 & \ldots & a3/an \\ \vdots & \vdots & \vdots & & \vdots \\ an/a1 & an/a2 & an/a3 & \ldots & an/an \end{bmatrix} \tag{1}$$

Let us also denote ***A*(*w*)** as the symbol of a matrix with elements $w_{ij} = w_i/w_j$ that can be presented as follows:

$$A(w) = \begin{bmatrix} w1/w1 & w1/w2 & w1/w3 & \ldots & w1/wn \\ w2/w1 & w2/w2 & w2/w3 & \ldots & w2/wn \\ w3/w1 & w3/w2 & w3/w3 & \ldots & w3/wn \\ \vdots & \vdots & \vdots & & \vdots \\ wn/w1 & wn/w2 & wn/w3 & \ldots & wn/wn \end{bmatrix} \tag{2}$$

Now, if we would like to recover the vector of weights $\boldsymbol{w} = [w_1, w_2, w_3, \ldots, w_n]^T$ whose true relative weights of a set of activities can be created from, as in the case of the above matrix ***A*(*w*)**, we can apply an optimization method which seeks a vector ***w*** as a solution to the following minimization problem:

$$min\, D(\boldsymbol{A}, \boldsymbol{A}(\boldsymbol{w})) \tag{3}$$

subject to some assigned constraints such as positive coefficients and the normalization condition.

As the distance function ***D*** measures an interval between matrices ***A*** and ***A*(*w*)**, various ways of its definition lead to different prioritization concepts. It seems that the most popular one is called the *logarithmic least squares method* (LLSM), known also as the geometric mean method [2, 5, 6, 15]. In this method the objective function measuring the distance between ***A*** and ***A*(*w*)** is given by:

$$\min D(\boldsymbol{A}, \boldsymbol{A(w)}) = \sum_{i,j=1}^{n} (\ln a_{ij} - \ln w_i + \ln w_j)^2 \tag{4}$$

In order to receive the estimate of the priority vector, objective function (4) needs to be minimized with subjection to the following constraints:

$$\prod_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1, \ldots, n$$

The LLSM solution also has the following closed form and is given by the normalized products of the elements in each row:

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n} \Bigg/ \sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n} \tag{5}$$

2. The eigenvalue method

There is a method that cannot be recognized as one of those characterized as constrained optimization ones. This method is a fundamental part of the mathematical theory for deriving ratio scale priority vectors (PV) from positive reciprocal matrices with entries set on the basis of pairwise comparisons. The theory is called the Analytic Hierarchy Process (AHP) and it uses the principal Eigenvalue Method (EM) to derive priority vectors [13, 14, 18-20].

It can be described in the following manner. Let us presume that we know the relative weights of a set of activities. Then we can express them in a PCM like $\boldsymbol{A}(\boldsymbol{w})$ which was described above. Now, if we would like to recover the vector of weights $\boldsymbol{w}$ which the ratios in $\boldsymbol{A}(\boldsymbol{w})$ can be created from, we could take the matrix product of matrix $\boldsymbol{A}(\boldsymbol{w}) = [w_{ij}]_{\mathrm{nxn}}$ with vector $\boldsymbol{w}$ in order to receive:

$$\begin{bmatrix} w1/w1 & w1/w2 & w1/w3 & \cdots & w1/wn \\ w2/w1 & w2/w2 & w2/w3 & \cdots & w2/wn \\ w3/w1 & w3/w2 & w3/w3 & \cdots & w3/wn \\ \vdots & \vdots & \vdots & & \vdots \\ wn/w1 & wn/w2 & wn/w3 & \cdots & wn/wn \end{bmatrix} \times \begin{bmatrix} w1 \\ w2 \\ w3 \\ \vdots \\ wn \end{bmatrix} = \begin{bmatrix} nw1 \\ nw2 \\ nw3 \\ \vdots \\ nwn \end{bmatrix} \tag{6}$$

If we know $\boldsymbol{A}(\boldsymbol{w})$, but not $\boldsymbol{w}$, we can solve this problem for $\boldsymbol{w}$. Solving for a nonzero solution for this set of equations is a very common procedure and is known as an eigenvalue problem:

$$A(w) \times w = \lambda \times w\,. \tag{7}$$

In order to find the solution to this set of equations, in general, one needs to solve an nth order equation for λ that, in general, leads to n unique values for λ, with an associated vector $\boldsymbol{w}$ for each of the n values. However, in the case of PCM based on priority weighting, matrix $\boldsymbol{A}(\boldsymbol{w})$ has a special form, since each row is a constant multiple of the first row. In this case, matrix $\boldsymbol{A}(\boldsymbol{w})$ has only one nonzero eigenvalue and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only nonzero eigenvalue in such case equals the size of the matrix and can be denoted as $\lambda_{max} = n$. If the elements of a matrix $\boldsymbol{A}(\boldsymbol{w})$ satisfy condition $w_{ij} = 1/w_{ji}$ for all $i, j = 1,\ldots,n$, then matrix $\boldsymbol{A}(\boldsymbol{w})$ is said to be *reciprocal*. If its elements satisfy condition $w_{ik}w_{kj} = w_{ij}$ for all $i, j, k = 1,\ldots,n$ and the matrix is reciprocal, then it is called *consistent*. Finally, matrix $\boldsymbol{A(w)}$ is said to be *transitive* if the following condition holds: if element w_{ij} is not less than element w_{ik} then $w_{ij} \geq w_{ik}$ for $i = 1,\ldots,n$.

It is obvious that in real life during priority weighing we do not have $\boldsymbol{A(w)}$ but only its estimate $\boldsymbol{A}$ containing our intuitive judgments, more or less close to $\boldsymbol{A(w)}$ in accordance to our skills, experience, etc. In such a case, the consistency property obviously does not hold and the relation between elements of $\boldsymbol{A}$ and $\boldsymbol{A(w)}$ can be expressed in the following form:

$$a_{ij} = e_{ij} w_{ij} \tag{8}$$

where e_{ij} is a perturbation factor which should be close to 1. It has been shown that for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues, that is why in order to estimate true priority vector $\boldsymbol{w}$, one needs to solve the following matrix equation:

$$\boldsymbol{A} \times \boldsymbol{w} = \lambda_{\max} \times \boldsymbol{w} \tag{9}$$

where λ_{max} is the principal eigenvalue, it is not smaller than n, and other characteristic values are close to zero. The estimates of true priority vector $\boldsymbol{w}$ can be found then by normalizing the eigenvector corresponding to the largest eigenvalue in equation (9) which is simple and its existence is guaranteed by Perron's Theorem.

3. Least absolute- and least squared deviation approximation method

It has been devised [21] that instead of solving eigenvalue equation (9), one may seek a vector $\boldsymbol{w}$ which best estimates equation (7). In order to satisfy equation (7) as accurately as possible, two new methods were recently proposed in communication [22], they were called: *least absolute deviation approximation* (denoted LADA) and *least squared deviation approximation* (denoted LSDA).

In order to estimate PV from the LADA, the following goal programming model was formulated:

$$\min \sum_{i=1}^{n} \left(d_i^+ + d_i^- \right) \tag{10}$$

subject to: $$d_i^- - d_i^+ + \sum_{j=1}^{n} a_{ij} w_j = n w_i$$

$$\sum_{j=1}^{n} w_j = 1, \quad w_i \geq 0, \quad d_i^+ \geq 0, \quad d_i^- \geq 0, \quad i = 1,\ldots,n$$

where $[d_1, d_2, d_3, \ldots, d_n]^T = Aw - nw$.

In order to estimate PV from the LSDA, the following constrained optimization model was formulated:

$$\min w^T B w \tag{11}$$

subject to: $\sum_{j=1}^{n} w_j = 1, \quad w_i \geq 0, \quad i = 1,\ldots,n$,

where $B = \left[(A - nI)^T (A - nI)\right]$ with $\boldsymbol{I}$ being an identity matrix of order n.

4. An example scenario based analysis

In this section of the article, we provide the LADA and LSDA efficacy analysis based on already published case studies. Some examples provided in the literature [13] showed a sequentially small and drastic discrepancy between the results obtained with the application of the EM and LLSM. We adopt here the AHP model presented there in order to analyze if there is the same discrepancy between the LADA, LSDA and EM. The first two scenarios are simple AHP models. For the overall goal, there are four criteria: c1, c2, c3, and c4. For each criterion, there are four alternatives: a1, a2, a3, and a4, which are the same for all the four criteria. The judgment matrices and corresponding estimation of PVs obtained with the application of the EM, LSDA, and LADA, respectively, are provided below. We start from scenario no. 1:

with respect to the GOAL:

$$\begin{array}{c} \\ c1 \\ c2 \\ c3 \\ c4 \end{array} \begin{array}{c} \begin{array}{cccc} c1 & c2 & c3 & c4 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 3 & 3 \\ 1/2 & 1/3 & 1 & 4 \\ 1/4 & 1/3 & 1/4 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{bmatrix} 0.412 \\ 0.316 \\ 0.194 \\ 0.079 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{LSDA} \\ \begin{bmatrix} 0.4248 \\ 0.3206 \\ 0.1876 \\ 0.0670 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{LADA} \\ \begin{bmatrix} 0.4249 \\ 0.3229 \\ 0.1898 \\ 0.0623 \end{bmatrix} \end{array}$$

with respect to criterion c1 and c4:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array} \begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 3 & 3 \\ 1/2 & 1/3 & 1 & 4 \\ 1/4 & 1/3 & 1/4 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{bmatrix} 0.412 \\ 0.316 \\ 0.194 \\ 0.079 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{LSDA} \\ \begin{bmatrix} 0.4248 \\ 0.3206 \\ 0.1876 \\ 0.0670 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{LADA} \\ \begin{bmatrix} 0.4249 \\ 0.3229 \\ 0.1898 \\ 0.0623 \end{bmatrix} \end{array}$$

with respect to criterion c2 and c3:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array} \begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 1/4 & 1/3 & 1/4 \\ 4 & 1 & 4 & 1/2 \\ 3 & 1/4 & 1 & 1/2 \\ 4 & 2 & 2 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} EM \\ \begin{bmatrix} 0.075 \\ 0.354 \\ 0.160 \\ 0.410 \end{bmatrix} \end{array} \quad \begin{array}{c} LSDA \\ \begin{bmatrix} 0.0637 \\ 0.3579 \\ 0.1549 \\ 0.4235 \end{bmatrix} \end{array} \quad \begin{array}{c} LADA \\ \begin{bmatrix} 0.0582 \\ 0.3598 \\ 0.1587 \\ 0.4233 \end{bmatrix} \end{array}$$

After synthesis, we obtain the following overall ranking:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array} \begin{array}{c} EM \\ \begin{bmatrix} 0.240 \\ 0.335 \\ 0.177 \\ 0.248 \end{bmatrix} \end{array} \quad \begin{array}{c} LSDA \\ \begin{bmatrix} 0.2413 \\ 0.3396 \\ 0.1710 \\ 0.2482 \end{bmatrix} \end{array} \quad \begin{array}{c} LADA \\ \begin{bmatrix} 0.2369 \\ 0.3418 \\ 0.1739 \\ 0.2474 \end{bmatrix} \end{array}$$

We note that all three methods coincide with the alternatives ranks, resulting in a2 > a4 > a1 > a3. Now, we analyze scenario no. 2:

with respect to the GOAL:

$$\begin{array}{c} \\ c1 \\ c2 \\ c3 \\ c4 \end{array} \begin{array}{c} \begin{array}{cccc} c1 & c2 & c3 & c4 \end{array} \\ \begin{bmatrix} 1 & 4 & 2 & 2 \\ 1/4 & 1 & 1/3 & 1/4 \\ 1/2 & 3 & 1 & 3 \\ 1/2 & 4 & 1/3 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} EM \\ \begin{bmatrix} 0.412 \\ 0.079 \\ 0.316 \\ 0.194 \end{bmatrix} \end{array} \quad \begin{array}{c} LSDA \\ \begin{bmatrix} 0.4248 \\ 0.0670 \\ 0.3206 \\ 0.1876 \end{bmatrix} \end{array} \quad \begin{array}{c} LADA \\ \begin{bmatrix} 0.4249 \\ 0.0623 \\ 0.3229 \\ 0.1898 \end{bmatrix} \end{array}$$

with respect to criterion c1 and c2:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array} \begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 4 & 2 & 2 \\ 1/4 & 1 & 1/3 & 1/4 \\ 1/2 & 3 & 1 & 3 \\ 1/2 & 4 & 1/3 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} EM \\ \begin{bmatrix} 0.412 \\ 0.079 \\ 0.316 \\ 0.194 \end{bmatrix} \end{array} \quad \begin{array}{c} LSDA \\ \begin{bmatrix} 0.4248 \\ 0.0670 \\ 0.3206 \\ 0.1876 \end{bmatrix} \end{array} \quad \begin{array}{c} LADA \\ \begin{bmatrix} 0.4249 \\ 0.0623 \\ 0.3229 \\ 0.1898 \end{bmatrix} \end{array}$$

with respect to criterion c3 and c4:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array} \begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 1/4 & 1/4 & 1/3 \\ 4 & 1 & 2 & 2 \\ 4 & 1/2 & 1 & 1/3 \\ 3 & 1/2 & 3 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} EM \\ \begin{bmatrix} 0.079 \\ 0.412 \\ 0.194 \\ 0.316 \end{bmatrix} \end{array} \quad \begin{array}{c} LSDA \\ \begin{bmatrix} 0.0670 \\ 0.4248 \\ 0.1876 \\ 0.3206 \end{bmatrix} \end{array} \quad \begin{array}{c} LADA \\ \begin{bmatrix} 0.0623 \\ 0.4249 \\ 0.1898 \\ 0.3229 \end{bmatrix} \end{array}$$

After synthesis, we obtain the following overall ranking:

$$
\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array}
\begin{array}{c} \text{EM} \\ \begin{bmatrix} 0.242 \\ 0.248 \\ 0.253 \\ 0.256 \end{bmatrix} \end{array}
\quad
\begin{array}{c} \text{LSDA} \\ \begin{bmatrix} 0.2430 \\ 0.2488 \\ 0.2530 \\ 0.2552 \end{bmatrix} \end{array}
\quad
\begin{array}{c} \text{LADA} \\ \begin{bmatrix} 0.2390 \\ 0.2482 \\ 0.2547 \\ 0.2581 \end{bmatrix} \end{array}
$$

We note that all three methods again coincide with the alternatives ranks, resulting in: a4 > a3 > a2 > a1. We resume now with the following conclusions.

Conclusions

To summarize, there are other valid methods for deriving the priority vector from a pairwise comparison matrix, especially when the matrix is inconsistent, that are equally satisfying as the eigenvalue method. As was presented in this article, on the basis of an example scenario analysis, there are at least two such methods: the least absolute deviation approximation and least squared deviation approximation. What is more, the two latter methods, as optimization based, allow the decision maker to introduce additional constraints reflecting some additional requirements connected with the preference modelling.

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