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APPLICATION OF EIGENFUNCTIONS METHOD FOR SOLVING TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the solution of a class of time fractional differential equations by using Green's function method is presented. Green's functions by using the Laplace transformation with respect to the time variable and the method of eigenfunction expansion with respect to space variables are derived. An analytical form of the solution to the problem in rectangular, polar and elliptical coordinates has been given.

Introduction

The theory of differential equations of non-integer order is widely used in modeling various physical processes [1, 2]. If possible, the analytical methods to solve initial-boundary problems with differential equations of non-integer order are applied and in particular Green's function method is used.

Green's functions for fractional differential operators are of great interest to many authors (for instance the book by Podlubny [3] and papers [4-6]). Fractional Green's functions for linear many-term fractional-order differential equations with constant coefficients are presented in book [3]. The explicit representation of a Green's function for a space-time fractional diffusion equation is given in paper [4]. Fractional Green's function associated with the fractional reaction-diffusion equation is considered in [5]. The fundamental solution for a fractional diffusion-wave equation is derived in paper [6]. Papers [4-6] concern one dimensional problems.

Here we propose the application of Green's function method to problems with partial differential equations including a Caputo derivative with respect to the time variable and standard Laplace operator with respect to space variables. Green's functions in the form of a series of eigenfunctions of a Laplace operator in rectangular, polar and elliptical coordinates are determined. Special cases of the presented fractional Green's function are Green's functions for a parameter denoting the order, which tends to an integer number. The obtained Green's functions in these cases agree with standard Green's functions [7, 8].

Problem formulation

The derivative of fractional-order α of function $g(t)$ in the sense of Caputo is defined by

$$ {}_0D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} g^{(n)}(\tau)\, d\tau \tag{1} $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α. In this paper we consider partial differential equations with a fractional-order derivative in the following form

$$ \left[{}_0D_t^{\alpha} - c^2 \nabla^2 \right] \Phi = f \tag{2} $$

where ∇^2 is the Laplace operator and c is a real coefficient. This equation completed with initial and boundary conditions will be considered in rectangular, polar and elliptical coordinates. Note that equation (2) for $\alpha = 1$ is the standard diffusion equation and for $\alpha = 2$ it is the standard wave equation.

In order to determine the solution of equation (2), in the first step we use the Laplace transform with respect to the t variable. For function $f(t)$ and its Laplace transform $\bar{f}(s)$ we have

$$ \mathrm{L}\left[f(t)\right] = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt\,, \quad \mathrm{L}^{-1}\left[\bar{f}(s)\right] = f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{f}(s)\, ds \tag{3} $$

where s is a complex parameter. Moreover, we have [3]

$$ \mathrm{L}\left[{}_0D_t^{\alpha} f(t) \right] = s^{\alpha} \bar{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}\left(0^+\right) \tag{4} $$

Taking the Laplace transform in equation (2) and assuming zero initial conditions, we obtain

$$ \left[s^{\alpha} - c^2 \nabla^2 \right] \overline{\Phi} = \bar{f} \tag{5} $$

The solution of equation (5) can be presented in the following form

$$ \overline{\Phi}(s,\mathbf{x}) = \int_D \bar{f}(s,\xi) \overline{G}(s,\mathbf{x};\tau,\xi)\, d\xi \tag{6} $$

where $\overline{G}$ is the Laplace transform of Green's function, which satisfies the following differential equation

$$ \left[s^{\alpha} - c^2 \nabla^2 \right] \overline{G}(s,x,\tau,\xi) = e^{-\tau s} \delta(x-\xi) \tag{7} $$

where $\delta(\cdot)$ is the Dirac delta function. Function $\overline{G}$ satisfies the same boundary conditions as function Φ. The solution of equation (2) on the basis of (6) can be obtained in the form of

$$\Phi(t,\mathbf{x}) = \int_0^t \int_D f(t-u,\xi)\, G(t,\mathbf{x};u,\xi)\, d\xi\, du \tag{8}$$

Our aim is firstly to derive the Laplace transform of Green's function $\overline{G}$ and next Green's function G.

Rectangular coordinates

Consider equation (7) in a rectangle $D = \{(x,y): 0 \le x \le a,\ 0 \le y \le b\}$. In this case, the rectangular coordinates are applied and the Laplace transform of Green's function is a solution of equation (7) with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We assume that at the boundary of rectangle D, the Dirichlet conditions are satisfied

$$\overline{G}\Big|_{x=0} = \overline{G}\Big|_{x=a} = 0\,, \quad \overline{G}\Big|_{y=0} = \overline{G}\Big|_{y=b} = 0 \tag{9}$$

In order to solve equation (7) with boundary conditions (9), we consider at first the eigenproblem

$$\nabla^2 \Phi(x,y) = -\omega^2\, \Phi(x,y) \tag{10}$$

$$\Phi(0,y) = \Phi(a,y) = 0\,, \quad \Phi(x,0) = \Phi(x,b) = 0 \tag{11}$$

Eigenvalues ω_{mn} and eigenfunctions $\Phi_{mn}(x,y)$ are

$$\omega_{nm} = \sqrt{\left(\frac{n\pi x}{a}\right)^2 + \left(\frac{m\pi y}{b}\right)^2}\,,\ \Phi_{nm}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,,\ n = 1, 2, \ldots \tag{12}$$

We seek the solution to boundary problem (7), (9) in the form of a double series of eigenfunctions:

$$\overline{G}(s,x,y;\tau,\xi,\eta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\Phi_{n,m}(x,y) \tag{13}$$

We determine coefficients A_{mn} by substituting function $\overline{G}$ given by (13) into equation (7), multiplying both sides of the equation by $\Phi_{kl}(x,y)$ and integrating with respect to x and y in intervals $[0,a]$ and $[0,b]$, respectively. Using the orthogonality condition we obtain:

$$A_{mn} = \frac{4\Phi_{mn}(x,y)\Phi_{mn}(\xi,\eta)}{ab\left(s^{\alpha}+c^2\,\omega_{mn}^2\right)}e^{-s\tau} \tag{14}$$

As a result, we have the Laplace transform of Green's function in the following form:

$$\overline{G}(s,x,y;\tau,\xi,\eta) = \frac{4}{ab}\sum_{n=1}^{\infty}\frac{e^{-s\tau}}{s^{\alpha}+c^2\omega_n^2}\sin\frac{m\pi x}{a}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y}{b} \tag{15}$$

Finally, Green's function, as an inverse transform of (15), has the form of

$$G(t,x,y;\tau,\xi,\eta) = \frac{4(t-\tau)^{\alpha-1}}{ab}\sum_{n=1}^{\infty}E_{\alpha,\alpha}\left(-c^2\omega_{mn}^2\,(t-\tau)^{\alpha}\right)\sin\frac{m\pi x}{a}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi y}{b}\sin\frac{n\pi y}{b} \tag{16}$$

where $t>\tau$ and $E_{\alpha\beta}(z)$ is the Mittag-Lefler function defined by [3]:

$$E_{\alpha\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)} \tag{17}$$

Polar coordinates

Differential equation (5) in a circular/annular region should be written in polar coordinates. This equation for Green's function has the following form:

$$\left[s^{\alpha} - c^2\,\nabla^2\right]\overline{G}(s,r,\phi;\tau,\rho,\theta) = \frac{\delta(r-\rho)}{r}\delta(\phi-\theta)e^{-\tau s} \tag{18}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\phi^2}$. Equation (18) obliges in the circle region: $r\le b$. We assume the Dirichlet condition on the boundary, i.e. $\overline{G}\big|_{r=b}=0$. We find the solution of this equation in the form

$$\overline{G}(s,r,\phi;\tau,\rho,\theta) = \sum_{m=-\infty}^{\infty}g_m(s,r;\tau,\rho)\cos m(\phi-\theta) \tag{19}$$

where functions $g_m(s,r;\tau,\rho)$ satisfy the equation

$$\left[s^{\alpha} - c^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\right)\right]g_m(s,r;\tau,\rho) = \frac{\delta(r-\rho)}{2\pi r}e^{-\tau s} \tag{20}$$

We seek a solution of equation (20) in the series form of eigenfuctions of the following problem:

$$\left(\frac{d^2}{d\,r^2}+\frac{1}{r}\frac{d}{d\,r}-\frac{m^2}{r^2}\right)\Psi_m(r)=-\omega^2\,\Psi_m(r),\quad \Psi_m(r)<+\infty \quad \text{for} \quad 0\le r\le b,\quad \Psi_m(b)=0 \tag{21}$$

The solutions to this eigenproblem create a sequence of functions

$$\Psi_{mn}(r)=J_m(\omega_{mn}r)\ , \tag{22}$$

where J_m are Bessel functions of the first kind of order, *m* and ω_{mn} (*m* - integer, *n* - natural) are the roots of equation

$$J_m(\omega_{mn}b)=0\,. \tag{23}$$

Next we assume that

$$g_m(s,r;\tau,\rho)=\sum_{n=1}^{\infty}A_{mn}J_m(\lambda_{mn}r)\ . \tag{24}$$

Substituting series (24) into equation (20) and using the orthogonality condition of functions (22) in interval [0,*b*], we obtain coefficients A_{mn} in the form of

$$A_{mn}=\frac{J_m(\lambda_{mn}\rho)}{\left(s^\alpha+c^2\omega_{mn}^2\right)d_{mn}}e^{-s\tau} \tag{25}$$

where

$$d_{mn}=2\pi\int_0^b r\,J_0^2(\omega_{mn}r)dr=\pi\,b^2\,J_1^2(\omega_{mn}b). \tag{26}$$

Finally, taking into account equations (19) and (24), (25), the Laplace transform of Green's function is

$$\overline{G}(s,r,\phi;\tau,\rho,\theta)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{d_{mn}}J_m(\omega_{mn}r)J_m(\omega_{mn}\rho)\frac{e^{-s\tau}}{s^\alpha+c^2\omega_{mn}^2}\cos m(\phi-\theta) \tag{27}$$

Hence, Green's function has the form

$$G(t,r,\phi;\tau,\rho,\theta)$$

$$=\frac{1}{\pi\,b^2}(t-\tau)^{\alpha-1}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{J_m(\omega_{mn}r)J_m(\omega_{mn}\rho)}{J_1^2(\lambda_{mn}b)}E_{\alpha,\alpha}\left(-c^2\omega_{mn}^2(t-\tau)^\alpha\right)\cos m(\phi-\theta) \tag{28}$$

Elliptical coordinates

Differential equation (5) in an elliptical space domain we write in the elliptical coordinates. This equation has the following form [9]

$$\left[s^{\alpha} - c^2 \nabla^2\right] \overline{G}(s,\xi,\eta;\tau,\zeta,\theta) = \frac{\delta(\xi-\zeta)\delta(\eta-\theta)}{h^2\left(\cosh^2\xi - \cos^2\eta\right)} e^{-\tau s} \tag{29}$$

where $\nabla^2 = \frac{1}{h^2\left(\cosh^2\xi - \cos^2\eta\right)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)$. The equation obliges in the elliptical region: $\xi \le \xi_0$, where $\xi_0 = \operatorname{arctgh}\frac{b}{a}$, a and b are half axes of the ellipse which is the edge of the region. We assume the Dirichlet condition on the boundary, i.e. $\left.\overline{G}\right|_{\xi=\xi_0} = 0$.

We seek the Laplace transform of Green's function in the form of a series:

$$\overline{G}(s,\xi,\eta;\tau,\zeta,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\,\Phi_{mn}(\xi,\eta) \tag{30}$$

where

$$\Phi_{mn}(\xi,\eta) = Me_m(\xi,q_{mn})\; me_m(\eta,q_{mn}) \tag{31}$$

are the eigenfunctions of the problem presented in paper [8], q_{mn} are roots of the equation which follows from boundary condition

$$Me_m(\xi_0,q_{mn}) = 0\,. \tag{32}$$

The angular and radial Mathieu functions $me_m(\eta,q)$ and $Me_m(\eta,q)$ were introduced in [9]. Substituting function (31) into equation (30) and using the orthogonality condition we have

$$A_{mn} = \frac{e^{-\tau s}}{\pi h^2\left(\cosh^2\zeta - \cos^2\theta\right)\left(s^{\alpha} + c^{2\lambda}\omega_{mn}^2\right)} Me_m(\zeta,q_{mn})\; me_m(\theta,q_{mn}) \tag{33}$$

Green's function has the form of

$$G(t,\xi,\eta;\tau,\zeta,\theta) = \frac{(t-\tau)^{\alpha-1}}{\pi h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \Phi_{mn}(\xi,\eta)\Phi(\zeta,\theta) E_{\alpha,\alpha}\left(-c^2\omega_{mn}^2(t-\tau)^{\alpha}\right) \tag{34}$$

Note, that for $\alpha = 1$ and for $\alpha = 2$, the Mittag-Lefler function has the form of [3]

$$E_{1,1}(z) = e^{z}\,, \quad E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}} \tag{35}$$

That way, Green's functions for the standard diffusion equation $(\alpha = 1)$ and standard wave equation $(\alpha = 2)$ as particular cases of the function given by (34) are:

- for $\alpha = 1$:

$$G(t,\xi,\eta;\tau,\zeta,\theta) = \frac{1}{\pi h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \Phi_{mn}(\xi,\eta)\Phi_{mn}(\zeta,\theta)\exp\left(c^2\omega_{mn}^2(t-\tau)\right) \tag{36}$$

- for $\alpha = 2$:

$$G(t,\xi,\eta;\tau,\zeta,\theta) = \frac{1}{\pi c h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\omega_{nm}} \Phi_{mn}(\xi,\eta)\Phi_{mn}(\zeta,\theta)\sin\left(c\,\omega_{mn}(t-\tau)\right) \tag{37}$$

Green's function (37) for the standard wave equation in an elliptical region is derived also in paper [8].

Conclusions

The analytical form of solutions to initial-boundary problems with a differential equation including a Caputo derivative with respect to time and the Laplace operator with respect to space variables is presented. Particular cases of the considered differential equation are classical diffusion and wave equations. The presented solutions, which concern 2D problems in rectangular, polar and elliptical coordinates are expressed by Green's functions corresponding to associated problems with homogeneous boundary conditions. The obtained Green's functions can be used to derive solutions to the considered differential problems with time-fractional derivatives.

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