Scientific Research of the Institute of Mathematics and Computer Science, 1(10) 2011, 139-144

IDENTIFICATION OF BOUNDARY HEAT FLUX USING SEQUENTIAL AND GLOBAL FUNCTION SPECIFICATION METHODS AND FDM ALGORITHM

Sylwia Lara-Dziembek, Edyta Pawlak

Institute of Mathematics, Czestochowa University of Technology, Poland
sylwia.lara@im.pcz.pl, edyta.pawlak@im.pcz.pl

Abstract. In the paper the global function specification method is used for identification of the time dependent boundary heat flux on the external boundary of the domain considered. The additional information necessary to solve an inverse problem results from the knowledge of heating (cooling) curves at points selected from the interior of the domain. The mathematical model of thermal processes proceeding in the system is based on the Fourier equation. As an example, a 1D problem is considered, but generalization of the algorithm on 2D or 3D problems is not complicated. At the stage of the numerical solution of a direct problem and an additional one, the finite difference method has been applied.

Introduction

In the group of inverse problems connected with the heat transfer proceeding in the domains of solid bodies, estimation of the parameters appearing in the boundary conditions plays an essential role. As is well known, on the external boundary limiting the system considered, different forms of boundary conditions can be given. The Knowledge of boundary temperature corresponds to the first type of boundary condition. When the boundary heat flux (of normal derivative of temperature) is given, then the second type of boundary condition appears. The continuity of heat flux on the external surface of the system corresponds to the third kind of boundary condition (the Robin one). In this paper the problems of boundary heat flux estimation are discussed. To solve the problem, the global function specification method is used, while at the stage of computations, the finite difference method is applied.

Let us consider the following boundary-initial problem. The transient temperature field is described by the Fourier equation

$$x \in \Omega: \quad c(T)\frac{\partial T(x,t)}{\partial t} = \mathrm{div}\left[\lambda(T)\ \mathrm{grad}\, T(x,t)\right], \tag{1}$$

where c is the volumetric specific heat, λ is the thermal conductivity, T, x, t denote the temperature, geometrical co-ordinates and time.

To simplify further considerations, the 1D linear task (infinite plate) will be analyzed and then

$$x \in \Omega: \quad \frac{\partial T(x,t)}{\partial t} = a \frac{\partial^2 T(x,t)}{\partial x^2} \tag{2}$$

where a is the heat diffusion coefficient.
The following boundary conditions are given for $x = 0$ and $x = L$

$$x = 0: \quad \lambda \frac{\partial T(x,t)}{\partial x} = 0 \tag{3}$$

and

$$x = L: \quad -\lambda \frac{\partial T(x,t)}{\partial x} = q_b(t) \tag{4}$$

Additionally for $t = 0$ (initial condition)

$$t = 0: \quad T(x,t) = T_0(x) \tag{5}$$

The solution of the inverse problem requires the knowledge of temperature history at the set of points (sensors) selected from the domain considered, in particular

$$T_{di}^{f} = T\left(x_i, t^f\right), \quad i = 1, 2, ..., n, \quad f = 0, 1, 2, ..., F \tag{6}$$

The above additional input data result from the measurement given on a real system, but at the stage of numerical algorithm construction, the exact solution of a direct problem (or disturbed one) constitutes a base for unknown parameters estimation. The domain considered is shown in Figure 1.

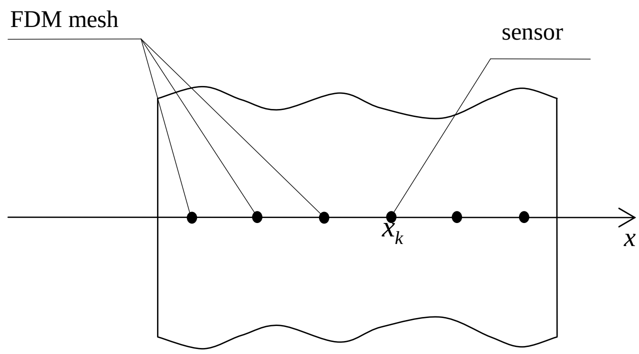

Fig. 1. Domain considered

Sequential and global function specification methods

The aim of the considerations presented is the estimation of the time-dependent heat flux $q(t)$ corresponding to the boundary condition given on the external surface of domain $(x=L)$. Unknown function $q(t)$ is assumed in the form of a piece-wise constant one (Fig. 2), this means

$$t \in \left[t^{f-1}, t^f \right]: \qquad q(t) = q\left(t^f\right) = q^f, \qquad f = 0,1,2,...,F \tag{7}$$

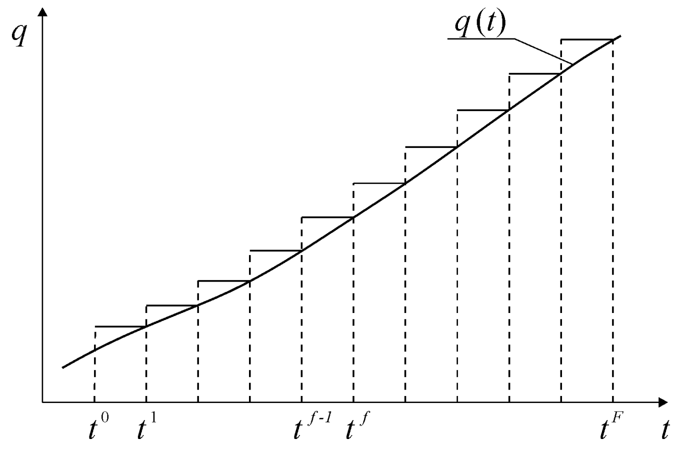

Fig. 2. Approximation of boundary heat flux

In this place, several approaches to solving the problem can be taken into account. The first one is called the sequential function specification method (SFSM) [1, 3]. Using this algorithm, we assume the knowledge of heat fluxes $q^1,..., q^{f-1}$ and only the value of q^f should be determined. The first stage of SFSM resolves itself in the construction of a sensitivity model with respect to heat flux *q* and using the direct approach of sensitivity analysis, one should differentiate the energy equation and boundary-initial conditions with respect to *q*. The sensitivity model is of the form

$$x \in \Omega: \qquad \frac{\partial U(x,t)}{\partial t} = a \frac{\partial^2 U(x,t)}{\partial x^2}, \;\; U(x,t) = \frac{\partial T(x,t)}{\partial q}. \tag{8}$$

The following boundary conditions are given for $x=0$ and $x=L$

$$x = 0: \quad \lambda \frac{\partial U(x,t)}{\partial x} = 0 \tag{9}$$

and

$$x = L: \quad -\lambda \frac{\partial U(x,t)}{\partial x} = 1 \tag{10}$$

For $t = 0$

$$t = 0: \quad U(x,t) = 0 \tag{11}$$

The sensitivity model is not coupled with the basic one and it can be solved before the main part of computations. Therefore, the distribution of function U for successive time-levels is known. Now, the Taylor formula should be used:

$$t \in \left[t^{f-1}, t^f\right]: \qquad T_i^f = T_{ai}^f + U_i^f \left(q^f - q_a^f\right). \tag{12}$$

Where q_a^f is an arbitrarily assumed value of the boundary heat flux, T_{ai}^f is the temperature at the point, and x_i is calculated under the assumption that $q = q_a^f$. In the simplest case corresponding to the application of only one sensor, $x = x_k$, this means

$$T_k^f = T_d\left(x_k, t^f\right) = T_d^f, \qquad f = 0,1,2,...,F \tag{13}$$

and one has

$$t \in \left[t^{f-1}, t^f\right]: \qquad T_d^f = T_{ak}^f + U_k^f \left(q^f - q_a^f\right) \tag{14}$$

or

$$t \in \left[t^{f-1}, t^f\right]: \qquad q^f = q_a^f + \frac{T_d^f - T_{ak}^f}{U_k^f} \tag{15}$$

The application of formula (15) leads to a simple numerical procedure which allows one to determine the successive ‘stairs’ of function q (e.g. [1, 2]), the computer program (or analytical solution) affords possibilities for calculations of the temperature distribution for different values of boundary heat fluxes, which must be repeatedly used, of course.

The other variant of the method discussed has been proposed by Beck [4], but this approach will not be presented here.

The sequential function specification method works ‘step by step’.

In the case of the global function specification method, unknown values $q^1, q^2, ..., q^{f-1}, q^f, ..., q^F$ are identified simultaneously [5-7].

Hence, the time interval $\left[0, t^F\right]$ is divided into intervals $\left[t^{f-1}, t^f\right]$ with constant step $\Delta t = t^f - t^{f-1}$ and for $t \in \left[t^{f-1}, t^f\right]$: $q(t) = q\left(t^f\right) = q^f$ as shown in Figure 2.

Let us assume that temperatures T_{di}^f at points x_i are given (c.f. equation (6)). Applying the least squares criterion one obtains

$$S\left(q^1, q^2, \dots, q^F\right) = \sum_{f=1}^{F}\sum_{i=1}^{M}\left(T_i^f - T_{di}^f\right)^2 \rightarrow \text{MIN} \tag{16}$$

where M is the number of sensors, T_i^f are the calculated temperatures obtained from the solution of the direct problem by using the current available estimate for the unknown values q^f, $f = 0,1,2,\dots,F$.

At first, the direct problem should be solved under the assumption that $q^f = q_a^f$, $f = 0,1,2,\dots,F$, at the same time, q_a^f are the arbitrarily determined values of heat fluxes. The solution obtained, means the temperature distribution at points x_i for times t^f, $f = 0,1,2,\dots,F$ will be denoted as T_{ai}^f.

Function T_i^f is expanded into a Taylor series in the neighborhood of this solution

$$T_i^f = T_{ai}^f + \sum_{k=1}^{f}\frac{\partial T_i^f}{\partial q^k}\left(q^k - q_a^k\right). \tag{17}$$

Putting (17) into (16) one has

$$S = \sum_{f=1}^{F}\sum_{i=1}^{M}\left[T_{ai}^f + \sum_{k=1}^{f}\frac{\partial T_i^f}{\partial q^k}\left(q^k - q_a^k\right) - T_{di}^f\right]^2 \tag{18}$$

Using the necessary condition of the several variables function minimum, after mathematical manipulations one obtains the following system of equations

$$\sum_{f=1}^{F}\sum_{i=1}^{M}\sum_{k=1}^{f} Z_i^{f,k}\, Z_i^{f,p}\left(q^k - q_a^k\right) = \sum_{f=1}^{F}\sum_{i=1}^{M} Z_i^{f,p}\left(T_{ai}^f - T_{di}^f\right),\ p = 0,1,2,\dots,F \tag{19}$$

where

$$Z_i^{f,k} = \frac{\partial T_i^f}{\partial q^k}, \quad Z_i^{f,p} = \frac{\partial T_i^f}{\partial q^p} \tag{20}$$

are the sensitivity coefficients.

This system of equations allows one to find simultaneously values $q^1, q^2, \dots, q^F$.

Final remarks

To find the numerical solution to a basic problem and sensitivity, ones the finite difference method in the version presented in [8] has been used. A plate with

$L=0.02$ m, $\lambda=1$ W/mK, $c=10^6$ kJ/m^{3}K and initial temperature of 100°C has been considered. The boundary heat flux for $x=L$ has been assumed in the form (c.f. [9])

$$x=L: \quad q(t)=2271-11.421t+0.081t^2 \quad \left[\mathrm{W/m^2}\right] \tag{21}$$

Both methods have been tested. From the mathematical and numerical points of view, the sequential function specification method is essentially simpler (it is first of all, visible in the case of 2D or 3D problems). On the other hand, however, the method is very sensitive with respect to measurement errors (at the stage of testing computations, the basic solution can be randomly disturbed and in this case the numerical results are closer to the real ones). The global function specification method leads to a system of linear equations and the number of unknown parameters can be very big (e.g. 2D task). This method is also sensitive with respect to measurement errors and it should be supplemented by regularization procedures.

References

[1] Kurpisz K., Nowak A.J., Inverse Thermal Problems, Computational Mechanics Publications, Southampton, Boston 1995.

[2] Beck J.V., Blackwel B., Inverse Problems, Handbook of Numerical Heat Transfer, Wiley Inters., New York 1988.

[3] Majchrzak E., Freus K., Janisz D., Solution of heat conduction problem using the sequential function specification method, Scientific Research of the Institute of Mathematics and Computer Science 2004, 1(3), 115-120.

[4] Beck J.V., Blackwell B., Clair Ch.R.S., Inverse Heat Conduction, Ill-Posed Problems, J. Wiley and Sons, New York 1985.

[5] Majchrzak E., Mochnacki B., Suchy J.S., Estimation of boundary heat flux between casting and mould using the global function specification method, Archives of Foundry Engineering 2008, 8, 1, 193-198.

[6] Majchrzak E., Janisz D., Kałuża G., Freus K., Identification of the boundary condition on the skin surface subjected to external heat source, Int. J. of Engineering Modeling 2004, 17, 1-2, 33-39.

[7] Majchrzak E., Freus K., Mendakiewicz J., Identification of boundary heat flux using the global function specification method, Scientific Research of the Institute of Mathematics and Computer Science 2005, 1(4), 147-156.

[8] Mochnacki B, Suchy J.S., Numerical Methods in Computations of Foundry Processes, PFTA, Cracow 1995.

[9] Janisz D., Doctoral Theses, Silesian University of Technology, Gliwice 2004.