Scientific Research of the Institute of Mathematics and Computer Science, 1(10) 2011, 151-161

SHAPE SENSITIVITY ANALYSIS. IMPLICIT APPROACH USING BOUNDARY ELEMENT METHOD

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Abstract. The Laplace equation (2D problem) supplemented by boundary conditions is analyzed. To estimate the changes of temperature in the 2D domain due to the change of local geometry of the boundary, the implicit method of sensitivity analysis is used. In the final part of the paper, the example of numerical computations is shown.

Introduction

To estimate the changes of temperature in a 2D domain due to the change of local geometry of the boundary, the methods of sensitivity analysis can be applied [1-5]. There are two basic approaches to sensitivity analysis using boundary element formulation: the continuous approach and the discretized one [6]. In the continuous approach (explicit differentiation method), the analytical expressions for sensitivities are derived and then they are calculated numerically using the BEM. They have the form of boundary integrals with integrands that depend only on the variables of the primary as well as additional problems. The implicit differentiation method, which belongs to the discretized approach, is based on the differentiation of algebraic boundary element matrix equations. The derivatives of boundary element system matrices can be calculated either analytically or semi-analytically. In the paper the implicit differentiation method of sensitivity analysis for a steady state problem is presented

$$(x, y) \in \Omega: \quad \lambda \frac{\partial^2 T(x, y)}{\partial x^2} + \lambda \frac{\partial^2 T(x, y)}{\partial y^2} = 0 \tag{1}$$

where λ [W/(mK)] is the thermal conductivity, T is the temperature and x, y are the geometrical co-ordinates. Equation (1) is supplemented by boundary conditions

$$
\begin{aligned}
(x, y) \in \Gamma_1 : &\quad T(x, y) = T_b \\
(x, y) \in \Gamma_2 : &\quad q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y) = q_b \\
(x, y) \in \Gamma_3 : &\quad q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y) = \alpha[T(x, y) - T_\infty]
\end{aligned}
\tag{2}
$$

where T_b is the known boundary temperature, q_b is the known boundary heat flux, α [W/(m^2 K)] is the heat transfer coefficient and T_∞ is the ambient temperature.

1. Boundary element method for Laplace equation

The boundary integral equation for the problem described by equations (1), (2) is the following [7, 8]

$$
(\xi, \eta) \in \Gamma : \quad B(\xi, \eta) T(\xi, \eta) + \int_\Gamma T^*(\xi, \eta, x, y)\, q(x, y) \mathrm{d}\Gamma = \int_\Gamma q^*(\xi, \eta, x, y)\, T(x, y) \mathrm{d}\Gamma
\tag{3}
$$

where $B(\xi, \eta) \in (0,1)$ is the coefficient connected with the local shape of the boundary, (ξ,η) is the observation point, $q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y)$, $T^*(\xi, \eta, x, y)$ is the fundamental solution

$$
T^*(\xi, \eta, x, y) = \frac{1}{2\pi\lambda} \ln \frac{1}{r}
\tag{4}
$$

where r is the distance between points (ξ,η) and (x,y)

$$
r = \sqrt{(x - \xi)^2 + (y - \eta)^2}
\tag{5}
$$

Function $q^*(\xi, \eta, x, y)$ is defined as follows:

$$
q^*(\xi, \eta, x, y) = -\lambda\, \mathbf{n} \cdot \nabla T^*(\xi, \eta, x, y)
\tag{6}
$$

and it can be calculated analytically

$$
q^*(\xi, \eta, x, y) = \frac{d}{2\pi r^2}
\tag{7}
$$

where

$$
d = (x - \xi) n_x + (y - \eta) n_y
\tag{8}
$$

while n_x, n_y are the directional cosines of normal outward vector $\mathbf{n}$.

2. Numerical realization of boundary element method

In numerical realization of the BEM, the boundary is divided into N boundary elements and integrals appearing in equation (3) are substituted by the sums of integrals over these elements

$$B(\xi_i,\eta_i)T(\xi_i,\eta_i)+\sum_{j=1}^{N}\int_{\Gamma_j} q(x,y)T^*(\xi_i,\eta_i,x,y)\mathrm{d}\Gamma_j = \sum_{j=1}^{N}\int_{\Gamma_j} T(x,y)q^*(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j \tag{9}$$

For linear boundary element Γ_j, it is assumed that

$$(x,y)\in\Gamma_j: \quad \begin{cases} T(\theta)=N_pT_p^j+N_kT_k^j \\ q(\theta)=N_pq_p^j+N_kq_k^j \end{cases} \tag{10}$$

where

$$N_p=\frac{1-\theta}{2}, \quad N_k=\frac{1+\theta}{2} \tag{11}$$

are the shape functions, $\theta\in[-1,1]$ and (x_j^p, y_j^p), (x_j^k, y_j^k) are the co-ordinates of the beginning and end of element Γ_j.

The integrals appearing in equation (9) can be written in the form of [7, 8]

$$\int_{\Gamma_j} T^*(\xi_i,\eta_i,x,y)\; q(x,y)\mathrm{d}\Gamma_j = G_{ij}^p q_p^j + G_{ij}^k q_k^j \tag{12}$$

and

$$\int_{\Gamma_j} q^*(\xi_i,\eta_i,x,y)\; T(x,y)\mathrm{d}\Gamma_j = \hat{H}_{ij}^p T_p^j + \hat{H}_{ij}^k T_k^j \tag{13}$$

where

$$G_{ij}^p=\frac{l_j}{2}\int_{-1}^{1} N_pT^*(\xi^i,\eta^i,N_px^p+N_kx^k,N_py^p+N_ky^k)\mathrm{d}\theta \tag{14}$$

$$G_{ij}^k=\frac{l_j}{2}\int_{-1}^{1} N_kT^*(\xi^i,\eta^i,N_px^p+N_kx^k,N_py^p+N_ky^k)\mathrm{d}\theta \tag{15}$$

and

$$\hat{H}_{ij}^{p} = \frac{l_j}{2}\int_{-1}^{1} N_p q^*(\xi^i, \eta^i, N_p x^p + N_k x^k, N_p y^p + N_k y^k)\mathrm{d}\theta \tag{16}$$

$$\hat{H}_{ij}^{k} = \frac{l_j}{2}\int_{-1}^{1} N_k q^*(\xi^i, \eta^i, N_p x^p + N_k x^k, N_p y^p + N_k y^k)\mathrm{d}\theta \tag{17}$$

where

$$l_j = \sqrt{(x_j^k - x_j^p)^2 + (y_j^k - y_j^p)^2} = \sqrt{(l_x^j)^2 + (l_y^j)^2} \tag{18}$$

is the length of element Γ_j.

Taking into account dependencies (4), (7), one has

$$G_{ij}^{p} = \frac{l_j}{4\pi\lambda}\int_{-1}^{1} N_p \ln\frac{1}{r_{ij}}\,\mathrm{d}\theta \tag{19}$$

$$G_{ij}^{k} = \frac{l_j}{4\pi\lambda}\int_{-1}^{1} N_k \ln\frac{1}{r_{ij}}\,\mathrm{d}\theta \tag{20}$$

and

$$\hat{H}_{ij}^{p} = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\mathrm{d}\theta \tag{21}$$

$$\hat{H}_{ij}^{k} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\mathrm{d}\theta\,, \tag{22}$$

where

$$r_{ij} = \sqrt{(N_p x_j^p + N_k x_j^k - \xi_i)^2 + (N_p y_j^p + N_k y_j^k - \eta_i)^2} = \sqrt{(r_x^j)^2 + (r_y^j)^2}\,. \tag{23}$$

It should be pointed out that if (ξ_i,η_i) is the beginning of boundary element Γ_j, this means $(\xi_i,\eta_i) = (x_j^p, y_j^p)$ then

$$G_{ij}^{p} = \frac{l_j(3 - 2\ln l_j)}{8\pi\lambda}, \quad G_{ij}^{k} = \frac{l_j(1 - 2\ln l_j)}{8\pi\lambda}, \quad \hat{H}_{ij}^{p} = \hat{H}_{ij}^{k} = 0\,, \tag{24}$$

while if (ξ_i,η_i) is the end of boundary element Γ_j: $(\xi_i,\eta_i) = (x_j^k, y_j^k)$ then

$$G_{ij}^{p} = \frac{l_j (1 - 2\ln l_j)}{8\pi\lambda}, \quad G_{ij}^{k} = \frac{l_j (3 - 2\ln l_j)}{8\pi\lambda}, \quad \hat{H}_{ij}^{p} = \hat{H}_{ij}^{k} = 0\,. \tag{25}$$

As is well known, in the final system of algebraic equations, the values of temperatures or heat fluxes are connected with the boundary nodes. If the following numeration of boundary nodes $r = 1, 2, ..., R$ is accepted, then for $i = 1, 2, ..., R$ one obtains the system of equations (c.f. equation (9))

$$B_i T_i + \sum_{r=1}^{R} G_{ir} q_r = \sum_{r=1}^{R} \hat{H}_{ir} T_r \,, \tag{26}$$

where for single node r being the end of boundary element Γ_j and being the beginning of boundary element Γ_{j+1} (Fig. 1) we have

$$G_{ir} = G_{ij}^{k} + G_{i\,j+1}^{p}, \quad \hat{H}_{ir} = \hat{H}_{ij}^{k} + \hat{H}_{i\,j+1}^{p}, \tag{27}$$

while for double node r, $r+1$

$$\begin{aligned} G_{ir} &= G_{ij}^{k}, \quad G_{ir+1} = G_{i\,j+1}^{p} \\ \hat{H}_{ir} &= \hat{H}_{ij}^{k}, \quad \hat{H}_{ir+1} = \hat{H}_{i\,j+1}^{p} \end{aligned} \,. \tag{28}$$

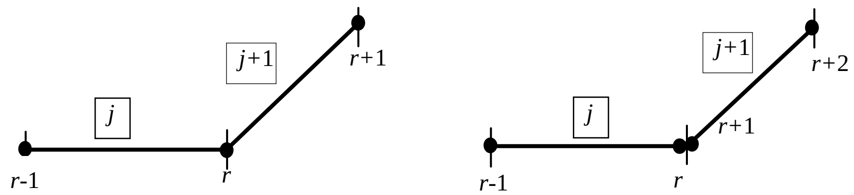

Fig. 1. Single and double nodes

The system of equations (26) can be written in the form of

$$\sum_{r=1}^{R} G_{ir} q_r = \sum_{r=1}^{R} H_{ir} T_r, \quad i = 1, 2, ..., R \tag{29}$$

or

$$\mathbf{Gq} = \mathbf{HT} \tag{30}$$

where

$$H_{ir} = \begin{cases} \hat{H}_{ir} & i \neq r \\ \hat{H}_{ir} - B_i & i = r \end{cases} \tag{31}$$

It should be pointed out that it is convenient to calculate values $H_{i\,i}$ using the formula

$$H_{ii} = -\sum_{\substack{r=1 \\ r\neq i}}^{R} H_{i\,r}, \quad i = 1, 2, ..., R . \tag{32}$$

Taking into account boundary conditions (2), system of equations (30) should be rebuilt to the form **AY** = **F**. The solution of this system allows one to determine the "missing" boundary temperatures and heat fluxes. Next, the temperatures in an optional set of internal nodes can be calculated using formula

$$T_i = \sum_{r=1}^{R} H_{i\,r} T_r - \sum_{r=1}^{R} G_{i\,r} q_r \; . \tag{33}$$

3. Implicit differentiation method of shape sensitivity analysis

We assume that b is the shape parameter, this means b corresponds to the x or y coordinate of one of the boundary nodes. The implicit differentiation method [6, 3] starts with the algebraic system of equations (30). The differentiation of (30) with respect to b leads to the following system of equations

$$\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} + \mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b} = \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} + \mathbf{H}\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b} \tag{34}$$

or

$$\mathbf{GW} = \mathbf{HU} + \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} - \frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} \tag{35}$$

where

$$\mathbf{U} = \frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b}, \quad \mathbf{W} = \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b} \tag{36}$$

The differentiation of boundary conditions (2) gives

$$\begin{aligned} (x,y)\in\Gamma_1 : &\quad U = \frac{\mathrm{D}T_b}{\mathrm{D}b} = 0 \\ (x,y)\in\Gamma_2 : &\quad W = \frac{\mathrm{D}q_b}{\mathrm{D}b} = 0 \\ (x,y)\in\Gamma_3 : &\quad \frac{\mathrm{D}q}{\mathrm{D}b} = \alpha\frac{\mathrm{DT}}{\mathrm{D}b} \;\rightarrow\; W = \alpha U \end{aligned} \tag{37}$$

Therefore, this approach of shape sensitivity analysis is connected with the differentiation of elements of matrices **G** and **H**.

Taking into account dependencies (19), (20), one has

$$\frac{\partial G_{ij}^{p}}{\partial b}=\frac{1}{4\pi\lambda}\left[\frac{\partial l_j}{\partial b}\int_{-1}^{1}N_p\ln\frac{1}{r_{ij}}\,\mathrm{d}\theta+l_j\int_{-1}^{1}N_p\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)\mathrm{d}\theta\right] \tag{38}$$

and

$$\frac{\partial G_{ij}^{k}}{\partial b}=\frac{1}{4\pi\lambda}\left[\frac{\partial l_j}{\partial b}\int_{-1}^{1}N_k\ln\frac{1}{r_{ij}}\,\mathrm{d}\theta+l_j\int_{-1}^{1}N_k\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)\mathrm{d}\theta\right] \tag{39}$$

where

$$\begin{gathered}\frac{\partial l_j}{\partial b}=\frac{1}{l_j}\left(l_x^j\frac{\partial l_x^j}{\partial b}+l_y^j\frac{\partial l_y^j}{\partial b}\right)\\ \frac{\partial l_x}{\partial b}=\frac{\partial}{\partial b}\left(x_j^k-x_j^p\right),\ \frac{\partial l_y}{\partial b}=\frac{\partial}{\partial b}\left(y_j^k-y_j^p\right)\end{gathered} \tag{40}$$

and

$$\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)=-\frac{1}{r_{ij}}\frac{\partial r_{ij}}{\partial b}, \tag{41}$$

where

$$\begin{gathered}\frac{\partial r_{ij}}{\partial b}=\frac{1}{r_{ij}}\left(r_x^j\frac{\partial r_x^j}{\partial b}+r_y^j\frac{\partial r_y^j}{\partial b}\right)\\ \frac{\partial r_x^j}{\partial b}=N_p\frac{\partial x_j^p}{\partial b}+N_k\frac{\partial x_j^k}{\partial b}-\frac{\partial \xi_i}{\partial b}\\ \frac{\partial r_y^j}{\partial b}=N_p\frac{\partial y_j^p}{\partial b}+N_k\frac{\partial y_j^k}{\partial b}-\frac{\partial \eta_i}{\partial b}\end{gathered} \tag{42}$$

Next, using formulas (21), (22), one obtains

$$\begin{aligned}\frac{\partial \hat{H}_{ij}^{p}}{\partial b}=\frac{1}{4\pi}\int_{-1}^{1}N_p\Bigg[&\frac{1}{r_{ij}^2}\left(\frac{\partial r_x^j}{\partial b}l_y^j+r_x^j\frac{\partial l_y^j}{\partial b}-\frac{\partial r_y^j}{\partial b}l_x^j-r_y^j\frac{\partial l_x^j}{\partial b}\right)-\\ &\frac{2}{r_{ij}^4}\left(r_x^j\frac{\partial r_x^j}{\partial b}+r_y^j\frac{\partial r_y^j}{\partial b}\right)\left(r_x^j l_y^j-r_y^j l_x^j\right)\Bigg]\mathrm{d}\theta\end{aligned} \tag{43}$$

and

$$\frac{\partial \hat{H}_{ij}^{k}}{\partial b} = \frac{1}{4\pi}\int_{-1}^{1} N_k \left[\frac{1}{r_{ij}^2}\left(\frac{\partial r_x^j}{\partial b} l_y^j + r_x^j \frac{\partial l_y^j}{\partial b} - \frac{\partial r_y^j}{\partial b} l_x^j - r_y^j \frac{\partial l_x^j}{\partial b} \right) - \frac{2}{r_{ij}^4}\left(r_x^j \frac{\partial r_x^j}{\partial b} + r_y^j \frac{\partial r_y^j}{\partial b} \right)\left(r_x^j l_y^j - r_y^j l_x^j \right) \right] \mathrm{d}\theta \tag{44}$$

In the case when shape parameter b corresponds to node $(\xi_i, \eta_i) = (x_j^p, y_j^p)$ or to node $(\xi_i, \eta_i) = (x_j^k, y_j^k)$, then formulas (24), (25) should be differentiated with respect to b.

The way of creating matrices $\partial G_{ir}/\partial b$ and $\partial H_{ir}/\partial b$ is similar to matrices G_{ir} and H_{ir} (c.f. equations (27), (28)) and when, for example, $b = x_r$, where (x_r, y_r) is a single boundary node, the non-zero elements of these matrices appear in columns $r-1$, r, $r+1$ and in row r.

Additionally (c.f. equation (32))

$$\frac{\partial H_{ii}}{\partial b} = -\sum_{\substack{r=1 \\ r\neq i}}^{R} \frac{\partial H_{ir}}{\partial b}, \quad i = 1, 2, \ldots, R \tag{45}$$

After solving the system of equations (35), the values of function U at optional internal points can be calculated using formula

$$U_i = \sum_{r=1}^{R} H_{ir} U_r - \sum_{r=1}^{R} G_{ir} W_r + \sum_{r=1}^{R} \frac{\partial H_{ir}}{\partial b} T_r - \sum_{r=1}^{R} \frac{\partial G_{ir}}{\partial b} q_r \tag{46}$$

It should be pointed out that using Taylor expansion

$$\begin{aligned} T(x,y,b+\Delta b) &= T(x,y,b) + U(x,y,b)\Delta b \\ T(x,y,b-\Delta b) &= T(x,y,b) - U(x,y,b)\Delta b \end{aligned} \tag{47}$$

one has

$$\Delta T(x,y) = T(x,y,b+\Delta b) - T(x,y,b-\Delta b) = 2U(x,y,b)\Delta b \tag{48}$$

where Δb is the perturbation of parameter b. Hence, on the basis of formula (48) the change of temperature due to the change of parameter b can be estimated.

4. Example of computations

A square of dimensions 0.05×0.05 m has been considered. Thermal conductivity equals $\lambda = 1$ W/(mK). On the bottom boundary, Neumann condition $q_b = -10^4$ W/m^2 has been assumed, on the remaining parts of the boundary Dirichlet condition T_b = 600°C has been accepted. The boundary has been divided into 8 linear boundary elements (Fig. 2) and 10 boundary nodes have been distinguished (two double boundary nodes).

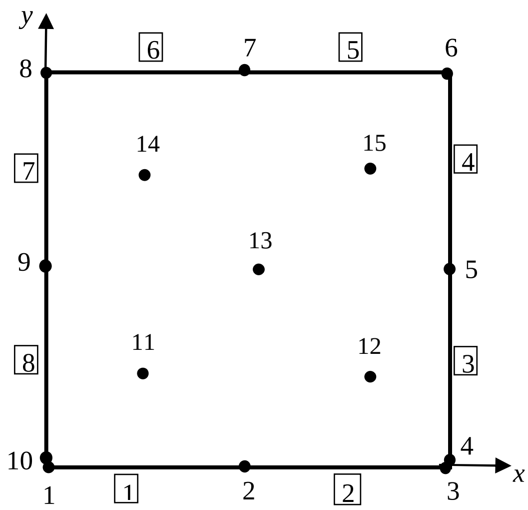

Fig. 3. Discretization

The solution of the basic problem (equations (1), (2)) is the following:

$$\mathbf{T} = \begin{bmatrix} 600 \\ 779.28 \\ 600 \\ 600 \\ 600 \\ 600 \\ 600 \\ 600 \\ 600 \\ 600 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -10000 \\ -10000 \\ -10000 \\ 18951.74 \\ -236.43 \\ 246.19 \\ 937.35 \\ 246.19 \\ -236.43 \\ 18951.74 \end{bmatrix} \tag{49}$$

For shape parameter $b_1 = x_6$, one has $U_2 = 1.02$ and $U_{13} = 4.67$, while for $b_2 = y_6$ $U_2 = 11.89$ and $U_{13} = 29.77$. Therefore, under the assumption that $\Delta b = 0.01$ m, the change of temperature at node 2 due to the change of parameter b_1 is equal to 0.02°C, while the change of temperature at node 2 due to the change of parameter b_2 is equal to 0.24°C (c.f. Expression (48)).

The temperature at point 13 changes from 630.96 to 631.01 for parameter b_1 while for parameter b_2 from 630.96 to 631.26.

The temperature distribution is shown in Figure 4. Figures 5, 6 illustrate the distributions of sensitivity functions DT/Db_1 and DT/Db_2, respectively.

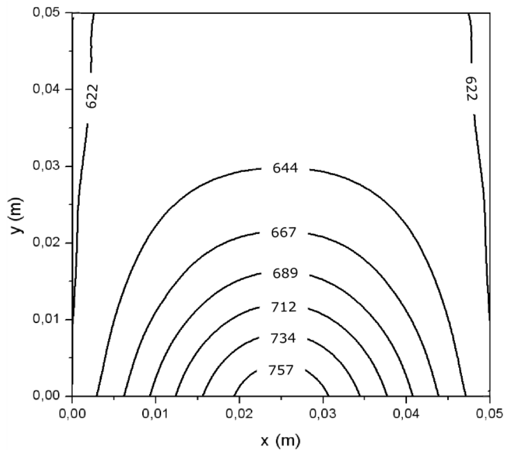

Fig. 4. Temperature distribution

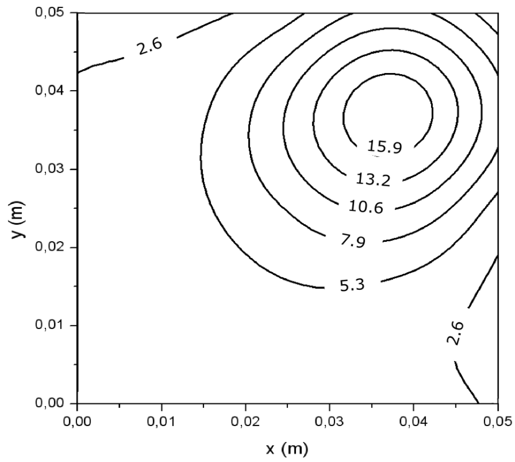

Fig. 5. Distribution of DT/Db_1

Conclusion

An implicit approach of shape sensitivity analysis coupled with the boundary element method has been discussed. Linear boundary elements have been used and then it is possible, in a simple way, to change the local geometry of the boundary. To estimate the change of temperature due to the perturbation of a shape parameter, the Taylor series containing the sensitivity function has been applied.

In this paper the steady state has been considered but in future a similar approach will be used for transient heat transfer.

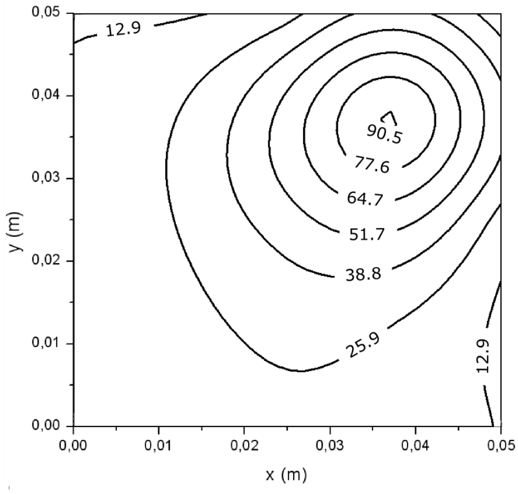

Fig. 6. Distribution of DT/Db_2

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