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SOLUTION OF OPTIMAL CONTROL PROBLEM FOR THE THREE-LEVEL HM-NETWORK - I

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Abstract. We studied the three-level exponential HM queueing network with one type of requests and incomes, which is a stochastic model for goods transport in a logistics transport system. We obtained a system of difference equations for the expected income of the central system with and without a reduction of future income to the current time in the case of a finite and infinite control horizon maximizing the expected income of the central system. These problems are proposed to be solved by the method of complete enumeration control strategies.

Introduction

Markov queueing networks (QN) with incomes or HM (Howard-Matalytski)-networks were introduced in [1, 2]. In [3] the task of optimal control for HM-networks of arbitrary structure was set. In this paper, it has been solved for a closed three-level HM-network with the method of complete enumeration control strategies.

Let us consider a closed Markov HM-network with requests of the same type, that consists of $M = n + m_1 + ... + m_{n-1}$ queueing systems (QS) S_i, $i = 1,...n, 1_1,...,1_{m_1},...,(n-1)_1,...,(n-1)_{m_{(n-1)}}$, as shown in Figure 1, which is a model of certain goods transportation. In this model, system S_n is a «plant» that produces goods, systems $S_1, S_2, \ldots, S_{n-1}$ are «warehouses» where the goods are stored; $S_{i_1}, S_{i_2}, \ldots, S_{i_{m_i}}$ are «shops» (places of goods sale), goods which come from the warehouse S_i, $i = \overline{1,(n-1)}$. Here the application is seen as goods shipment in a logistical system «plants - warehouses - shops».

Let us introduce some sets: $X_i = \{i_1, i_2, ..., i_{m_i}\}$, $i = \overline{1,(n-1)}$; $X_0 = \bigcup_{i=1}^{n-1} X_i$; $X = \{1,2,...,n\} \bigcup X_0$. The state of the network at time t we see as vector $(k,t) = (k_1, k_2, \ldots, k_n, ..., k_{1_1}, ..., k_{1_{m_1}}, ..., k_{(n-1)_1}, ..., k_{(n-1)_{m_{(n-1)}}}, t)$, where k_i is the number of requests in system S_i at time moment t, $i \in X$. The number of network states equals $L = C_{M+K-1}^{M-1}$, where K is the number of requests in the network.

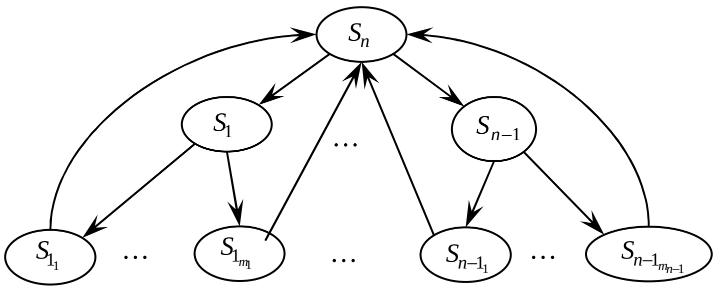

Fig. 1. Structure of three-level network

Let us consider $v_i(k,t)$ as a full expected income, obtained by system S_i during time t, when at the initial time moment the network is in state k; p_{ij} is the probability of requests transitions from system S_i to system S_j, $\sum_{j \in X} p_{ij} = 1$; $r_{ij}(k,t)$ is the income of system S_i, and accordingly the expenditure or waste of system S_j during time t if at the initial moment of time the network is in state k; μ_i is the intensity of requests service in system S_i; $r_i(k)$ is the system S_i income at a unit time when the network is in state k, $i,j \in X$.

The matrix of requests transitions probabilities between QS of the given HM-network can be written as:

$$P = \left(\begin{array}{ccccccccccccccccc}
0 & 0 & \ldots & 0 & p_{11_1} & p_{11_2} & \ldots & p_{11_{m_1}} & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots\ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & p_{22_1} & p_{22_2} & \ldots & p_{22_{m_2}} & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & p_{(n-1)(n-1)_1} & p_{(n-1)(n-1)_2} & \ldots & p_{(n-1)(n-1)_{m_{(n-1)}}} \\
p_{n1} & p_{n2} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & p_{1_1 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & p_{1_2 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & p_{1_{m_1} n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & p_{2_1 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & p_{2_2 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & p_{2_{m_2} n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & p_{(n-1)_1 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & p_{(n-1)_2 n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & p_{(n-1)_{m_{(n-1)}} n} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0
\end{array}\right).$$

Let us first get the system of equations for the expected incomes of the central system.

1. Expected income of the central system

We denote by $v_n(k,t)$ the full expected income, which central system S_n receives during time t, if initially the network was in state k. Let consider the HM-network to be in state (k,t). Assuming that system S_n receives the income of $r_n(k)$ c.u. (conventional units) at a unit time during the whole period of the network staying in state k. If it remains in state (k,t) during time interval Δt, then the expected income of system S_n is $r_n(k)\Delta t$ plus the expected income $v_n(k,t)$ which it will get during the remaining time units t. The probability of this event is $1-\sum_{j\in X}\mu_j u(k_j)\Delta t + o(\Delta t)$, where $u(x)=\begin{cases}1, & x>0,\\ 0, & x\le 0,\end{cases}$ is the Heaviside function. When the network during time Δt makes a transition from state (k,t) to state $(k-I_j+I_n,t+\Delta t)$ with probability $\mu_j u(k_j)\Delta t+o(\Delta t)$, system S_n receives the income in the amount of $-r_{jn}(k-I_j+I_n,t)$ and the expected income of system S_n make up $-r_{jn}(k-I_j+I_n,t)$ plus the expected income $v_n(k-I_j+I_n,t)$ that will be obtained for the remaining time, if the initial state of the network was $(k-I_j+I_n)$, $j\in X_0$. Under this transition, either a return of goods from the system «shop» happens, or the transport comes empty, and this in its turn means that the system suffers a loss of the size of $r_{jn}(k-I_j+I_n,t)$, $j\in X_0$.

Table 1

Determination of S_n system income during transition between network states

Transitions between network states	Transition probability	System S_n income from state transition
$(k,t)\to(k,t+\Delta t)$	$1-\sum_{j\in X}\mu_j u(k_j)\Delta t+o(\Delta t)$	$r_n(k)\Delta t+v_n(k,t)$
$(k,t)\to(k-I_j+I_n,t+\Delta t)$, $j\in X_0$	$\mu_j u(k_j)\Delta t+o(\Delta t)$	$v_n(k-I_j+I_n,t)-$ $-r_{jn}(k-I_j+I_n,t)$
$(k,t)\to(k+I_j-I_n,t+\Delta t)$, $j=\overline{1,n-1}$	$\mu_n u(k_n)p_{nj}\Delta t+o(\Delta t)$	$v_n(k+I_j-I_n,t)+$ $+r_{nj}(k-I_n+I_j,t)$
$(k,t)\to(k-I_s+I_c,t+\Delta t)$, $s=\overline{1,n-1}$, $c=s_1,s_2,\dots,s_{m_s}$	$\mu_s u(k_s)p_{sc}\Delta t+o(\Delta t)$	$r_n(k)\Delta t+v_n(k-I_s+I_c,t)$

Similarly, when the network makes a transition from state (k,t) to state $(k+I_j-I_n,t+\Delta t)$ with probability $\mu_n u(k_n)p_{nj}\Delta t+o(\Delta t)$, it brings income to system S_n in the amount of $r_{nj}(k-I_n+I_j,t)$, that is the products sale of the «plant» system to the «warehouse» system occurs and system S_n receives the income. The expected income of system S_n is $r_{nj}(k-I_n+I_j,t)$ plus the expected network income for the remaining time, if the initial network state is $(k+I_j-I_n)$, $j=\overline{1,n-1}$. Let us put the described ideas in Table 1.

Then for the expected income of system S_n: the following system of difference equations can be written

$$v_n(k,t+\Delta t)=\left(1-\sum_{j\in X}\mu_j u(k_j)\Delta t\right)\left(r_n(k)\Delta t+v_n(k,t)\right)+$$

$$+\sum_{j=1}^{n-1}\mu_n u(k_n)p_{nj}\Delta t\left(r_{nj}(k-I_n+I_j,t)+v_n(k-I_n+I_j,t)\right)+$$

$$+\sum_{j\in X_0}\mu_j u(k_j)\Delta t\left(v_n(k-I_j+I_n,t)-r_{jn}(k-I_j+I_n,t)\right)+$$

$$+\sum_{\substack{s=\overline{1,n-1}\\ c=s_1,s_2,\ldots,s_{m_S}}}\mu_s u(k_s)p_{sc}\Delta t\left(r_n(k)\Delta t+v_n(k-I_s+I_c,t)\right),$$

or in matrix form

$$V_n(k,t+\Delta t)=\hat{Q}_n(k,t,\Delta t)+\hat{A}_n(k,\Delta t)V_n(k,t), \tag{1}$$

where $V_n(k,t)$ is a column vector of the expected incomes for the central system S_n in time t, if initially the network was in state k consisting of components $v_n(k,t)$ recorded over the states of the network; $\hat{A}_n(k,\Delta t)=\left\|\hat{a}_{ij}(k,\Delta t)\right\|_{L\times L}$ is a matrix of transition probabilities between the states of the network over time Δt, if the initial state of the network was state (k,t), and $\hat{Q}_n(k,t,\Delta t)$ is a column vector of the average single-step incomes obtained by system S_n during time Δt, if at time moment t the state of the network was (k,t). Matrix $\hat{A}_n$ and vector $\hat{Q}_n$ can be found using matrix P, the intensity of service applications and single-step incomes.

In a particular case, if $r_n(k)$, $r_{nj}(k,t)$, $r_{jn}(k,t)$ are not dependent on the network conditions and are respectively r_n, $r_{nj}(t)$, $r_{jn}(t)$, then

$$v_n(k,t+\Delta t)=\left(1-\sum_{j\in X}\mu_j u(k_j)\Delta t\right)\left(r_n\Delta t+v_n(k,t)\right)+$$
$$+\sum_{j=1}^{n-1}\mu_n u(k_n)p_{nj}\Delta t\left(r_{nj}(t)+v_n(k-I_n+I_j,t)\right)+$$
$$+\sum_{j\in X_0}\mu_j u(k_j)\Delta t\left(v_n(k-I_j+I_n,t)-r_{jn}(t)\right)+$$
$$+\sum_{\substack{s=1,n-1\\ c=s_1,s_2,\ldots,s_{m_s}}}\mu_s u(k_s)p_{sc}\Delta t\left(r_n\Delta t+v_n(k-I_s+I_c,t)\right). \qquad (2)$$

Thus, we have a system of difference equations (1) for system S_n expected income without revaluation, i.e. without reducing future incomes to the current time moment. Let us further obtain its counterpart, taking into account that the amount in S c.u. obtained through step Δt, is equivalent to βS at the present moment, the amount in S c.u. obtained in n years, is equivalent to $\beta^n S$ c.u. at the moment. Coefficient $\beta\in(0,1]$ is called the reevaluation (reduction) of the future income coefficient. Obviously, if $S=1$ the reduction coefficient is equal to the value of the capital, that bears a single income in a single step.

Let us explain this idea with the following example. Each c.u. that is put into the business makes profit as much as $t\%$ a year. This quantity is called the profit rate, interest rate, etc. Each year amount x is capitalized to the amount $x+xt=x(1+t)$, in 2 years - up to the amount $x(1+t)+xt(1+t)=x(1+t)^2$, in n years - to sum $x(1+t)^n$. From equation $x(1+t)=1$, we get $\beta=x=\frac{1}{1+t}=(1+t)^{-1}$ and therefore $\beta^n=\left(\frac{1}{1+t}\right)^n=(1+t)^{-n}$.

If we return to conclusion (1), but take into account the reevaluation of future incomes then

$$v_{n,\beta}(k,t+\Delta t)=\left(1-\sum_{j\in X}\mu_j u(k_j)\Delta t\right)\left(r_n(k)\Delta t+\beta v_{n,\beta}(k,t)\right)+$$
$$+\sum_{j=1}^{n-1}\mu_n u(k_n)p_{nj}\Delta t\left(r_{nj}(k-I_n+I_j,t)+\beta v_{n,\beta}(k-I_n+I_j,t)\right)+$$
$$+\sum_{j\in X_0}\mu_j u(k_j)\Delta t\left(\beta v_{n,\beta}(k-I_j+I_n,t)-r_{jn}(k-I_j+I_n,t)\right)+$$
$$+\sum_{\substack{s=1,n-1\\ c=s_1,s_2,\ldots,s_{m_s}}}\mu_s u(k_s)p_{sc}\Delta t\left(r_n(k)\Delta t+\beta v_{n,\beta}(k-I_s+I_c,t)\right),$$
$$V_{n,\beta}(k,t+\Delta t)=\hat{Q}_n(k,\Delta t)+\beta\hat{A}_n(k,\Delta t)V_{n,\beta}(k,t). \qquad (3)$$

Let us consider the behavior of the total expected income for $t \to \infty$. Above we have formula (3) for the fully expected income in a matrix form.

If $\left[\frac{t}{\Delta t}\right] = \varphi$ and $\left\{\frac{t}{\Delta t}\right\} = \psi$ respectively are the integer and the fractional part of number $\frac{t}{\Delta t}$. Then for $t \to \infty$ we get

$$V_{n,\beta}(k,t) = \hat{Q}_n(k,\Delta t) + \beta\hat{A}_n(k,\Delta t)V_{n,\beta}(k,t-\Delta t) =$$
$$= \hat{Q}_n(k,\Delta t) + \beta\hat{A}_n(k,\Delta t)\left(\hat{Q}_n(k,\Delta t) + \beta\hat{A}_n(k,\Delta t)V_{n,\beta}(k,t-2\Delta t)\right) = \ldots =$$
$$= \hat{Q}_n(k,\Delta t) + \hat{Q}_n(k,\Delta t)\beta\hat{A}_n(k,\Delta t) + \hat{Q}_n(k,\Delta t)\left(\beta\hat{A}_n(k,\Delta t)\right)^2 + \ldots +$$
$$+ \hat{Q}_n(k,\Delta t)\left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi-1} + \left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi}V_{n,\beta}(k,\psi) = \hat{Q}_n(k,\Delta t)\sum_{m=0}^{\varphi-1}\left(\beta\hat{A}_n(k,\Delta t)\right)^m +$$
$$+ \left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi}\left[\hat{Q}_n(k,\psi) + \beta\hat{A}_n(k,\psi)V_{n,\beta}(k,0)\right] = \hat{Q}_n(k,\Delta t)\sum_{m=0}^{\varphi-1}\left(\beta\hat{A}_n(k,\Delta t)\right)^m +$$
$$+ \left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi}\hat{Q}_n(k,\psi) + \beta^{\varphi+1}\left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi}\hat{A}_n(k,\psi)V_{n,\beta}(k,0) \qquad (4)$$

Let us find limit income $V_{n,\beta,\infty}(k,t) = \lim\limits_{t\to\infty} V_{n,\beta}(k,t)$. Let us consider that $0 < \beta < 1$ and the network income at time moment $t = 0$ equals zero. Then, $\left(\beta\hat{A}(k,\Delta t)\right)^{\varphi} \underset{\varphi\to\infty}{\to} O$ where O is a zero matrix, and according to the fact that if $t \to \infty$, then $\varphi \to \infty$ we see that $\sum_{m=0}^{\varphi-1}\left(\beta\hat{A}_n(k,\Delta t)\right)^m \underset{\varphi\to\infty}{\to} \left(1 - \beta\hat{A}_n(k,\Delta t)\right)^{-1}$, therefore

$$V_{n,\beta,\infty}(k,\Delta t) = \hat{Q}_n(k,\Delta t)\left(1 - \beta\hat{A}_n(k,\Delta t)\right)^{-1}$$

Suppose that $\beta = 1$, then $\left(\hat{A}_n(k,\Delta t)\right)^{\varphi} \underset{\varphi\to\infty}{\to} \Gamma(k,\Delta t)$, $\Gamma(k,\Delta t) \neq 0$, and $V_{n,\beta,\infty}(k,\Delta t) = \pm\infty$. Hence the limit income equals

$$V_{n,\beta,\infty}(k,\Delta t) = \begin{cases} \pm\infty, \text{ when } \beta = 1; \\ \hat{Q}_n(k,\Delta t)\left(1 - \beta\hat{A}_n(k,\Delta t)\right)^{-1}, \text{ when } 0 < \beta < 1 \end{cases} \qquad (5)$$

We introduce a new quantity, called the HM-network profit with the incomes:

$$G_{\beta}(k,\Delta t) = \lim_{t\to\infty}\left(V_{\beta}(k,t+\Delta t) - V_{\beta}(k,t)\right)$$

that equals the limit increment of income over the expected step, i.e. profit is an average one-step income «in the long run». According to (4)

$$G_{n,\beta}(k,\Delta t)=\lim_{t\to\infty}\left(V_{n,\beta}(k,t+\Delta t)-V_{n,\beta}(k,t)\right)=\hat{Q}_n(k,\Delta t)\left(\beta\hat{A}_n(k,\Delta t)\right)^{\varphi}$$

If $\beta<1$ this quantity tends to 0 and $t\to\infty$, consequently, $\varphi\to\infty$. And when $\beta=1$ it tends to $\hat{Q}_n(k,\Delta t)\Gamma(k,\Delta t)$. Thus,

$$G_{n,\beta}(k,t)=\begin{cases}0, \text{ when } \beta=1;\\ \hat{Q}_n(k,\Delta t)\Gamma(k,\Delta t), \text{ when } \beta<1\end{cases} \tag{6}$$

2. Formulation of optimal control problem

The income amount of each network system is determined by the control selected at each step and every state. Let us consider the problem of control choosing with or without future income reduction to the current time in cases of a finite and infinite control horizon.

We call a QN controlled, if at every time t and in every state $l=1,2,...,L$ we can choose the row of matrix $\hat{A}_n$:

$$\hat{a}_l^{\theta_l}=(\hat{a}_{l1}^{\theta_l},\hat{a}_{l2}^{\theta_l},...,\hat{a}_{lL}^{\theta_l}) \tag{7}$$

and the vector one-step income $\hat{Q}_n$:

$$\hat{Q}_n^{(\theta_l)}=(\hat{q}_1^{(\theta_l)},\hat{q}_2^{(\theta_l)},...,\hat{q}_L^{(\theta_l)}) \tag{8}$$

that determine further behavior of the network.

Value θ_l is called a control strategy in the l-th state, and $\Theta_l=\{\theta_l\}$ is the control strategy set in the l-th state. The vector of strategies $\overline{\theta}=(\overline{\theta}_1,\overline{\theta}_2,...,\overline{\theta}_L)\in\Theta_1\times\Theta_2\times...\times\Theta_L$ is called a policy. If a strategy θ_l or policy $\overline{\theta}$ is chosen at time t, then we write $\theta_l(t)$ or $\overline{\theta}(t)=(\overline{\theta}_1(t),\overline{\theta}_2(t),...,\overline{\theta}_L(t))$. The sequence of the policies chosen at every moment of time forms the control $\overline{\overline{\theta}}=(\overline{\theta}(t),\overline{\theta}(t+\Delta t),...,\overline{\theta}(T_{\max}))$ that uniquely determines the network evolution. If $T_{\max}<\infty$, we speak of a finite control horizon, otherwise - of an infinite one.

Let $E=E(\overline{\overline{\theta}})$ be the efficiency of the network behavior on a given control interval; then control $\overline{\overline{\theta}}^*$ which maximizes efficiency is called optimal. The optimal control problem for an HM-network is to find the optimal control: $E(\bar{\bar{\theta}}^*) = \max_{\bar{\bar{\theta}}} E(\bar{\bar{\theta}})$. As $E(\bar{\bar{\theta}})$ we can take the overall expected income of an HM--network that can be found by using the expected income systems of the network that satisfy the obtained systems of difference-differential equations (DDE), or the income of central system S_n, found using relations (1), (3).

3. Solution of optimal control problem by total exhaustion method of control strategies

Let us consider two cases clarifying this statement.

1. Finite control horizon $T_{\max} < \infty$ with or without reevaluation, $\beta \in (0;1]$. It is necessary to find the control $\bar{\bar{\theta}}^*$ that maximizes the total expected income of system S_n

$$V_n(k, T_{\max}, \bar{\bar{\theta}}^*) \to \max_{\bar{\bar{\theta}}} \tag{9}$$

where $\bar{\bar{\theta}} = (\hat{\theta}(t), \hat{\theta}(t+\Delta t), ..., \hat{\theta}(T_{\max}))$, $V_n(k, T_{\max}, \bar{\bar{\theta}})$ is the vector of the expected income of system S_n with control $\bar{\bar{\theta}}$ at time $T_{\max}$.

2. Infinite control horizon $T_{max} = \infty$. In this case, we will search for the optimal control in the class of stationary controls $\bar{\bar{\theta}} = (\hat{\theta}, \hat{\theta}, ...)$, i.e. policies that do not depend on time. Then optimal policy $\bar{\theta}^*$ is defined as the solution of the following problems:

2.1) if $\beta < 1$, then

$$V_{n,\beta,\infty}(k, \bar{\theta}^*) \to \max_{\bar{\theta}} \tag{10}$$

2.2) if $\beta = 1$, then

$$G_n(k, \bar{\theta}^*) \to \max_{\bar{\theta}} \tag{11}$$

where $\bar{\theta} = (\theta_1, \theta_2, ..., \theta_L)$, $\theta_i \in \Theta_i$, $i = 1, 2, \ldots, L$; $V_{n,\beta,\infty}(k, \bar{\theta}) = \hat{Q}_n(k, \bar{\theta}) \times$ $\times \left(1 - \beta \hat{A}_n(k, \bar{\theta})\right)^{-1}$ is the vector of limit incomes; $G_n(k, \bar{\theta}) = R(k, \bar{\theta}) \left(A_n(k, \bar{\theta}) \right)^T$ is the vector of profits (or profit).

Let us consider the case where in every network state, we can apply u control strategies $\theta_{l1}, \theta_{l2}, ..., \theta_{lu}$, for the sake of simplicity we denote them as $\theta_1, \theta_2, ..., \theta_u$. Let us consider $P(\theta_s) = \left\| p_{ij}(\theta_s) \right\|_{M \times M}$ as a matrix of requests transition probabilities between the QS if we use strategy θ_s; $R(\theta_s) = \left\| r_{ij}(\theta_s) \right\|_{M \times M}$ is a matrix of one-step incomes, $r_{ij}(\theta_s)$ is the incomes of system S_i, as well as it is respectively the waste or loss of system S_j using strategy θ_s; $r(\theta_s) = \left\| r_i(\theta_s) \right\|_{1 \times M}$ is the vector of constant income by using strategy θ_s, $r_i(\theta_s)$ is the income of system S_i per unit of time if the network remains in the same state as when using θ_s; $\mu_i(\theta_s)$ is the service intensity of requests in system S_i if using strategy θ_s; $\mu(\theta_s) = (\mu_1(\theta_s), \mu_2(\theta_s), ..., \mu_n(\theta_s))$, $s = \overline{1,u}$. Then, the step by step algorithm to solve optimal control problem (10) for the infinite control horizon with the method of exhaustive strategy search looks as follows:

1) for each strategy θ_s, $s = \overline{1,u}$, depending on the considering reevaluation, we find using (3), the expression for $V_{n,\beta,\infty}(k, \overline{\theta})$ in the optimal control problem in (10);
2) by the total exhaustion method of control strategies θ_s, $s = \overline{1,u}$, we define the maximum value $V_{n,\beta,\infty}(k, \overline{\theta}^*)$ for the infinite control horizon and the corresponding control strategy;
3) the selected strategies will present the optimal control plan.

Example 1. Let us consider a network consisting of $n = 6$ service delivery systems and the number of requests $K = 15$, $\Delta t = 1$, $\beta = 0.85$. The number of states equals $L = C_{6+15-1}^{6-1} = 15504$. We consider the problem on the infinite control horizon taking into account reevaluation.

Let the transport company plan to produce a discount on freight, an advertising campaign and hiring additional transport. Consequently, the amount of goods traffic will be increased in the supply system «plants - warehouses - shops». However, other factors will be changed as well. Hence, we must consider the following strategies: 1) to make discounts on freight; 2) to conduct an advertising campaign; 3) to conduct freight discounts and a promotional campaign; 4) neither to offer any discounts for freight nor conduct a advertising campaign; 5) hire additional transport; 6) to make discounts on freight and rent additional transportation; 7) to conduct an advertising compaign and hire additional transport.

Depending on the strategies applied, the parameters listed below will differ.

Let the matrixes of requests transition probabilities between QS by using each strategy be as follows:

$$P(\theta_1)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.45 & 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P(\theta_2)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.45 & 0.55 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P(\theta_3)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P(\theta_4)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0.55 & 0.45 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P(\theta_5)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.35 & 0.65 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P(\theta_6)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.38 & 0.62 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P(\theta_7)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0.56 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrixes of one-step incomes under the request transition between QS are respectively:

$$R(\theta_1)=\begin{pmatrix} 0 & 0 & 0 & 150 & 0 & 0 \\ 0 & 0 & 0 & 0 & 230 & 215 \\ 241 & 199 & 0 & 0 & 0 & 0 \\ 0 & 0 & 260 & 0 & 0 & 0 \\ 0 & 0 & 180 & 0 & 0 & 0 \\ 0 & 0 & 160 & 0 & 0 & 0 \end{pmatrix}, \quad R(\theta_2)=\begin{pmatrix} 0 & 0 & 0 & 231 & 0 & 0 \\ 0 & 0 & 0 & 0 & 280 & 213 \\ 262 & 261 & 0 & 0 & 0 & 0 \\ 0 & 0 & 290 & 0 & 0 & 0 \\ 0 & 0 & 280 & 0 & 0 & 0 \\ 0 & 0 & 195 & 0 & 0 & 0 \end{pmatrix},$$

$$R(\theta_3)=\begin{pmatrix} 0 & 0 & 0 & 150 & 0 & 0 \\ 0 & 0 & 0 & 0 & 265 & 275 \\ 219 & 206 & 0 & 0 & 0 & 0 \\ 0 & 0 & 259 & 0 & 0 & 0 \\ 0 & 0 & 205 & 0 & 0 & 0 \\ 0 & 0 & 218 & 0 & 0 & 0 \end{pmatrix}, \quad R(\theta_4)=\begin{pmatrix} 0 & 0 & 0 & 231 & 0 & 0 \\ 0 & 0 & 0 & 0 & 264 & 243 \\ 230 & 242 & 0 & 0 & 0 & 0 \\ 0 & 0 & 232 & 0 & 0 & 0 \\ 0 & 0 & 221 & 0 & 0 & 0 \\ 0 & 0 & 230 & 0 & 0 & 0 \end{pmatrix},$$

$$R(\theta_5)=\begin{pmatrix} 0 & 0 & 0 & 225 & 0 & 0 \\ 0 & 0 & 0 & 0 & 260 & 240 \\ 218 & 196 & 0 & 0 & 0 & 0 \\ 0 & 0 & 215 & 0 & 0 & 0 \\ 0 & 0 & 205 & 0 & 0 & 0 \\ 0 & 0 & 212 & 0 & 0 & 0 \end{pmatrix},\quad R(\theta_6)=\begin{pmatrix} 0 & 0 & 0 & 240 & 0 & 0 \\ 0 & 0 & 0 & 0 & 230 & 213 \\ 217 & 255 & 0 & 0 & 0 & 0 \\ 0 & 0 & 232 & 0 & 0 & 0 \\ 0 & 0 & 223 & 0 & 0 & 0 \\ 0 & 0 & 204 & 0 & 0 & 0 \end{pmatrix},$$

$$R(\theta_7)=\begin{pmatrix} 0 & 0 & 0 & 242 & 0 & 0 \\ 0 & 0 & 0 & 0 & 290 & 204 \\ 245 & 261 & 0 & 0 & 0 & 0 \\ 0 & 0 & 230 & 0 & 0 & 0 \\ 0 & 0 & 281 & 0 & 0 & 0 \\ 0 & 0 & 215 & 0 & 0 & 0 \end{pmatrix}.$$

The vectors of systems incomes per unit of time:

$$r(\theta_1)=\begin{pmatrix}250 & 342 & 251 & 295 & 282 & 260\end{pmatrix},\quad r(\theta_2)=\begin{pmatrix}243 & 264 & 272 & 257 & 291 & 290\end{pmatrix},$$
$$r(\theta_3)=\begin{pmatrix}250 & 272 & 279 & 260 & 285 & 260\end{pmatrix},\quad r(\theta_4)=\begin{pmatrix}204 & 224 & 256 & 240 & 234 & 245\end{pmatrix},$$
$$r(\theta_5)=\begin{pmatrix}249 & 260 & 261 & 254 & 267 & 260\end{pmatrix},\quad r(\theta_6)=\begin{pmatrix}220 & 224 & 268 & 240 & 237 & 245\end{pmatrix},$$
$$r(\theta_7)=\begin{pmatrix}236 & 224 & 248 & 240 & 235 & 243\end{pmatrix}.$$

Intensity of requests service:

$$\mu(\theta_1)=\begin{pmatrix}0{,}53 & 0{,}62 & 0{,}58 & 0{,}4 & 0{,}7 & 0{,}61\end{pmatrix},\quad \mu(\theta_2)=\begin{pmatrix}0{,}4 & 0{,}36 & 0{,}8 & 0{,}75 & 0{,}62 & 0{,}48\end{pmatrix},$$
$$\mu(\theta_3)=\begin{pmatrix}0{,}52 & 0{,}43 & 0{,}6 & 0{,}29 & 0{,}7 & 0{,}43\end{pmatrix},\quad \mu(\theta_4)=\begin{pmatrix}0{,}28 & 0{,}39 & 0{,}5 & 0{,}46 & 0{,}6 & 0{,}51\end{pmatrix},$$
$$\mu(\theta_5)=\begin{pmatrix}0{,}4 & 0{,}47 & 0{,}53 & 0{,}5 & 0{,}45 & 0{,}64\end{pmatrix},\quad \mu(\theta_6)=\begin{pmatrix}0{,}54 & 0{,}61 & 0{,}33 & 0{,}5 & 0{,}62 & 0{,}7\end{pmatrix},$$
$$\mu(\theta_7)=\begin{pmatrix}0{,}39 & 0{,}56 & 0{,}8 & 0{,}65 & 0{,}7 & 0{,}34\end{pmatrix},$$

for example $\mu_1(\theta_1)=0{,}53$ is the intensity of requests service in system S_1 using strategy θ_1.

An optimal control problem was solved by the total exhaustion method of strategies $\theta_s,\ s=\overline{1,u}$. A computer program that allows one to find the optimal functioning strategy on the interval needed is worked out.

The need to use strategy in every state can be characterized by Table 2, where 1 means that the discount for freight should be made, 2 - to conduct an advertising campaign, 3 - to conduct freight discounts and to conduct an advertising campaign, 4 - neither offer discounts for freight, nor an conduct advertising campaign; 5 - rent additional transportation, 6 - to make discounts for freight and to rent additional transport, 7 - to conduct an advertising campaign and hire additional transport.

Table 2

Choosing strategy for system «plant - producer»

Network states	Result of strategy choice	Network states	Result of strategy choice
…	…	…	…
(12,0,0,2,1,0)	3	(2,1,0,9,2,1)	3
(12,0,0,2,0,1)	3	(2,1,0,9,1,2)	3
(12,0,0,0,3,0)	3	(2,1,0,9,3,0)	3
(12,0,0,0,2,1)	2	(2,1,0,9,2,1)	4
(12,0,0,0,1,2)	2	(2,1,0,9,1,2)	4
(12,0,0,0,0,3)	2	(2,1,0,9,0,3)	4
(11,4,0,0,0,0)	3	(2,1,0,8,4,0)	5
(11,3,1,0,0,0)	3	(2,1,0,8,3,1)	4
(11,3,0,1,0,0)	1	(2,1,0,8,2,2)	4

Table 2 prolongation

Network states	Result of strategy choice	Network states	Result of strategy choice
(11,3,0,0,1,0)	3	(2,1,0,8,1,3)	4
(11,3,0,0,0,1)	1	(2,1,0,8,0,4)	4
(11,3,0,0,0,1)	1	(2,1,0,7,5,0)	7
(11,2,2,0,0,0)	2	(2,1,0,7,4,1)	7

For example, if the initial state is $(2,1,0,8,4,0)$ we have found that the optimal strategy for system S_n is θ_5, which should involve additional transport.

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