FINITE DIFFERENCE METHOD IN FOURIER EQUATION INTERNAL CASE - DIRECT FORMULAS

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Abstract. In the paper we present the method of calculating the matrix block determinant which characterizes internal heat conduction in the (x,y,t) case.

Introduction

The paper refers to the Finite Difference Method (FDM) in the heat conduction Fourier equation in the two-dimensional case (cf. [1]) and also to work [2], which considered the (x,t) case. We consider only the internal heat conduction in FDM.

1. Mathematics preliminary

Let f be a polynomial of degree k

$$f(x)=x^k-r_1x^{k-1}+\ldots+(-1)^{k-1}r_{k-1}x+(-1)^k r_k=(x-p_1)\cdot\ldots\cdot(x-p_k) \tag{1}$$

where $p_1,\ldots,p_k$ are zeros of polynomial f. Let $\boldsymbol{A}$ be a square matrix of degree n with eigenvalues $\lambda_1,\ldots,\lambda_n$. The following method allows us to calculate the matrix determinant

$$f(\boldsymbol{A})=\boldsymbol{A}^k-r_1\boldsymbol{A}^{k-1}+\ldots+(-1)^{k-1}r_{k-1}\boldsymbol{A}+(-1)^k r_k\boldsymbol{I}=(\boldsymbol{A}-p_1\boldsymbol{I})\cdot\ldots\cdot(\boldsymbol{A}-p_k\boldsymbol{I}) \tag{2}$$

as a sum of powers of the fundamental symmetric polynomials:

$$\begin{aligned}
&\tau_1(\lambda_1,\ldots,\lambda_n)=\lambda_1+\lambda_2+\ldots+\lambda_n\\
&\tau_2(\lambda_1,\ldots,\lambda_n)=\lambda_1\lambda_2+\lambda_2\lambda_3+\ldots+\lambda_{n-1}\lambda_n\\
&\vdots\\
&\tau_n(\lambda_1,\ldots,\lambda_n)=\lambda_1\lambda_2\cdot\ldots\cdot\lambda_n
\end{aligned} \tag{3}$$

and

$$\begin{aligned} &\omega_1(p_1,\ldots,p_k) = p_1 + p_2 + \ldots + p_k \\ &\omega_2(p_1,\ldots,p_k) = p_1p_2 + p_2p_3 + \ldots + p_{k-1}p_k \\ &\vdots \\ &\omega_k(p_1,\ldots,p_k) = p_1p_2 \cdot \ldots \cdot p_k \end{aligned} \tag{4}$$

with integer coefficients.

Indeed, let

$$w_{\boldsymbol{A}}(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = (-1)^n (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_n) \tag{5}$$

be the characteristic polynomial of matrix $\boldsymbol{A}$. Then

$$\begin{aligned} &\det f(\boldsymbol{A}) = \det(\boldsymbol{A} - \boldsymbol{I}p_1) \cdot \ldots \cdot \det(\boldsymbol{A} - \boldsymbol{I}p_k) = w_{\boldsymbol{A}}(p_1) \cdot \ldots \cdot w_{\boldsymbol{A}}(p_k) \\ &= (-1)^{kn} (p_1 - \lambda_1) \cdot \ldots \cdot (p_1 - \lambda_n) \cdot (p_2 - \lambda_1) \cdot \ldots \cdot (p_2 - \lambda_n) \cdot \ldots \cdot (p_k - \lambda_1) \cdot \ldots \cdot (p_k - \lambda_n) \end{aligned} \tag{6}$$

We can assume that $p_j \neq 0$ for $j = 1,\ldots,k$. Continuing this, we have

$$\begin{aligned} &\det f(\boldsymbol{A}) = (-1)^{kn} p_1^{\,n} \cdot p_2^{\,n} \cdot \ldots \cdot p_k^{\,n} \cdot \\ &\left(1 - \frac{\lambda_1}{p_1}\right) \cdot \ldots \cdot \left(1 - \frac{\lambda_n}{p_1}\right) \cdot \left(1 - \frac{\lambda_1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{\lambda_n}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{\lambda_1}{p_k}\right) \cdot \ldots \cdot \left(1 - \frac{\lambda_n}{p_k}\right) \\ &= (-1)^{kn} \omega_k^{\,n} \left[1 - s_1\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) + \ldots \right. \\ &\quad + (-1)^l s_l\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) + \ldots \\ &\quad \left. + (-1)^{kn} s_{kn}\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) \right] \end{aligned} \tag{7}$$

where

$$s_l\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) = \left(\frac{1}{p_1^{\,l}} + \frac{1}{p_2^{\,l}} + \ldots + \frac{1}{p_k^{\,l}}\right) \cdot \left(\lambda_1^{\,l} + \lambda_2^{\,l} + \ldots + \lambda_n^{\,l}\right) \tag{8}$$

$l = 2,\ldots,kn-1$, are successive symmetric polynomials which depend on the set of variables indicated in parentheses. Of course

$$\begin{aligned} &s_1\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) = \left(\frac{1}{p_1} + \ldots + \frac{1}{p_k}\right)(\lambda_1 + \ldots + \lambda_n) = \frac{\omega_{k-1}}{\omega_k} \cdot \tau_1 \\ &s_{kn}\left(\frac{\lambda_1}{p_1},\ldots,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},\ldots,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},\ldots,\frac{\lambda_n}{p_k}\right) = \frac{\lambda_1^{\,k} \cdot \ldots \cdot \lambda_n^{\,k}}{p_1^{\,n} \cdot \ldots \cdot p_k^{\,n}} = \frac{\tau_n^{\,k}}{\omega_k^{\,n}} \end{aligned} \tag{9}$$

Moreover, at most n-ths power of zeros, $p_1,...,p_k$ appear in the denominators of these fractions. It follows from Newton's formulas (cf. [3], [4]) that every sum s_l is the determinant

$$s_l = \frac{1}{l!}\begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \dots & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \dots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \sigma_{l-1} & \sigma_{l-2} & \dots & \sigma_2 & \sigma_1 & l-1 \\ \sigma_l & \sigma_{l-1} & \dots & \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix} \tag{10}$$

where

$$\begin{aligned} &\sigma_l\left(\frac{\lambda_1}{p_1},...,\frac{\lambda_n}{p_1},\frac{\lambda_1}{p_2},...,\frac{\lambda_n}{p_2},\frac{\lambda_1}{p_k},...,\frac{\lambda_n}{p_k}\right) \\ &= \frac{\lambda_1{}^l}{p_1{}^l}+...+\frac{\lambda_n{}^l}{p_1{}^l}+\frac{\lambda_1{}^l}{p_2{}^l}+...+\frac{\lambda_n{}^l}{p_2{}^l}+\frac{\lambda_1{}^l}{p_k{}^l}+...+\frac{\lambda_n{}^l}{p_k{}^l} \\ &= \left(\frac{1}{p_1{}^l}+\frac{1}{p_2{}^l}+...+\frac{1}{p_k{}^l}\right)\left(\lambda_1{}^l+\lambda_2{}^l+...\lambda_n{}^l\right) \end{aligned} \tag{11}$$

Using Newton's formulas again, cf. [3], both for case $l \le n$ as well as in case $l > n$, we express factor $\frac{1}{p_1{}^l}+...+\frac{1}{p_k{}^l}$ by the power of fundamental symmetric polynomials ω_j, and factor $\lambda_1{}^l+...+\lambda_n{}^l$ by the power of fundamental symmetric polynomials τ_i. Moreover, in each component s_l, $l > n$, terms with powers ω_j^l in the denominators are reduced. After multiplying by $\omega_k{}^n = p_1{}^n \cdot p_2{}^n \cdot ... \cdot p_k{}^n$, the procedure ends.

2. Fourier equation in (x,y,t) case

We consider equation

$$\lambda\left(\frac{\Delta^2 T(x,y,t)}{\Delta x^2}+\frac{\Delta^2 T(x,y,t)}{\Delta y^2}\right) = \rho c \frac{\Delta T(x,y,t)}{\Delta t} \tag{12}$$

where $T = T(x,y,t)$ - temperature function, (x,y) - point of two-dimensional plate, t - time, λ - heat conductivity of the plate, ρ - density of the plate, c - specific heat.

As usual, we assume that

$$
\begin{aligned}
&\frac{\Delta^2 T}{\Delta x^2} = \frac{T_{i+1jl} - 2T_{ijl} + T_{i-1jl}}{\Delta x^2} \ \text{ for } 1 \le i \le m-1 \\
&\frac{\Delta^2 T}{\Delta y^2} = \frac{T_{ij+1l} - 2T_{ijl} + T_{ij-1l}}{\Delta y^2} \ \text{ for } 1 \le j \le n-1 \\
&\frac{\Delta T}{\Delta t} = \frac{T_{ijl} - T_{ijl-1}}{\Delta t} \ \text{ for } 1 \le l \le q
\end{aligned}
\tag{13}
$$

The FDM leads to the system of equations

$$
\begin{aligned}
\frac{\lambda}{\Delta x^2} T_{i-1jl} - \frac{2\lambda}{\Delta x^2} T_{ijl} + \frac{\lambda}{\Delta x^2} T_{i+1jl} + \frac{\lambda}{\Delta y^2} T_{ij-1l} - \frac{2\lambda}{\Delta y^2} T_{ijl} + \frac{\lambda}{\Delta y^2} T_{ij+1l} \\
= \frac{\rho c}{\Delta t} T_{ijl} - \frac{\rho c}{\Delta t} T_{ijl-1}
\end{aligned}
\tag{14}
$$

at each time step l.

The matrix of this system has the three-band block form

$$
\boldsymbol{P} = \begin{bmatrix}
\boldsymbol{A} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\
\boldsymbol{D} & \boldsymbol{A} & \boldsymbol{D} & \boldsymbol{0} & \dots & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{D} & \boldsymbol{A} & \boldsymbol{D} & \dots & \boldsymbol{0} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{D} & \boldsymbol{A} & \boldsymbol{D} \\
\boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{A}
\end{bmatrix}_{n \times n}
\tag{15}
$$

for

$$
\boldsymbol{A} = \begin{bmatrix}
\delta & -\frac{\lambda}{\Delta x^2} & 0 & 0 & \dots & 0 \\
-\frac{\lambda}{\Delta x^2} & \delta & -\frac{\lambda}{\Delta x^2} & 0 & \dots & 0 \\
0 & -\frac{\lambda}{\Delta x^2} & \delta & -\frac{\lambda}{\Delta x^2} & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \dots & -\frac{\lambda}{\Delta x^2} & \delta & -\frac{\lambda}{\Delta x^2} \\
0 & 0 & \dots & 0 & -\frac{\lambda}{\Delta x^2} & \delta
\end{bmatrix}_{m \times m}
\tag{16}
$$

where $\delta = \frac{2\lambda}{\Delta x^2} + \frac{2\lambda}{\Delta y^2} + \frac{\rho c}{\Delta t}$ (compare [5]). However

$$\boldsymbol{D}=\begin{bmatrix} -\frac{\lambda}{\Delta y^2} & 0 & 0 & 0 & \dots & 0 \\ 0 & -\frac{\lambda}{\Delta y^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & -\frac{\lambda}{\Delta y^2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -\frac{\lambda}{\Delta y^2} & 0 \\ 0 & 0 & \dots & 0 & 0 & -\frac{\lambda}{\Delta y^2} \end{bmatrix}_{m\times m} = -\frac{\lambda}{\Delta y^2}\cdot\boldsymbol{I} \tag{17}$$

According to the method given in Section 1, the determinant of matrix $\boldsymbol{P}$, after removing factor $\left(-\frac{\lambda}{\Delta y^2}\right)^{mn}$, we count by the determinant of the matrix

$$\boldsymbol{P}'=\begin{bmatrix} \boldsymbol{A}' & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{A}' & \boldsymbol{I} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{A}' & \boldsymbol{I} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{I} & \boldsymbol{A}' & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{A}' \end{bmatrix}_{n\times n} \tag{18}$$

where

$$\boldsymbol{A}'=\begin{bmatrix} \delta' & \frac{\Delta y^2}{\Delta x^2} & 0 & 0 & \dots & 0 \\ \frac{\Delta y^2}{\Delta x^2} & \delta' & \frac{\Delta y^2}{\Delta x^2} & 0 & \dots & 0 \\ 0 & \frac{\Delta y^2}{\Delta x^2} & \delta' & \frac{\Delta y^2}{\Delta x^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\Delta y^2}{\Delta x^2} & \delta' & \frac{\Delta y^2}{\Delta x^2} \\ 0 & 0 & \dots & 0 & \frac{\Delta y^2}{\Delta x^2} & \delta' \end{bmatrix}_{m\times m} \tag{19}$$

$\delta'=-\frac{2\Delta y^2}{\Delta x^2}-2-\frac{\rho c\Delta y^2}{\lambda\Delta t}$, as follows (cf. [5])

$$\det \boldsymbol{P'} = \det\left(\boldsymbol{A'}^n - \binom{n-1}{1}\boldsymbol{A'}^{n-2} + \binom{n-2}{2}\boldsymbol{A'}^{n-4} - \ldots \right) = \det f\left(\boldsymbol{A'}\right) \tag{20}$$

where

$$f\left(x\right) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \ldots \tag{21}$$

Consequently, the determinant of the three-band block matrix $\boldsymbol{P}$ is expressed by the power of the coefficients of polynomial f and power sums of the principal minors of matrix $\boldsymbol{A'}$.

References

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