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ESTIMATION OF TEMPERATURE CHANGES DUE TO PERTURBATION OF LOCAL GEOMETRY OF BOUNDARY USING IMPLICIT APPROACH OF SHAPE SENSITIVITY ANALYSIS

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Abstract. In the paper, a 2D domain in which the temperature field is described by the Laplace equation and the assumed boundary conditions is considered. To estimate the changes of temperature due to the change of the boundary local geometry, the implicit approach of shape sensitivity analysis is used. In the final part of paper, examples of numerical computations are shown and conclusions are formulated.

Introduction

The steady state temperature field $T(x, y)$ in a 2D domain is described by the Laplace equation

$$(x,y)\in\Omega:\ \lambda\frac{\partial^2 T(x,y)}{\partial x^2}+\lambda\frac{\partial^2 T(x,y)}{\partial y^2}=0 \tag{1}$$

where λ [W/(mK)] is the thermal conductivity, T is the temperature and x, y are the geometrical co-ordinates. Equation (1) is supplemented by boundary conditions, in particular:

$$\begin{aligned}
&(x,y)\in\Gamma_1: && T(x,y)=T_b\\
&(x,y)\in\Gamma_2: && q(x,y)=-\lambda\,\mathbf{n}\cdot\nabla T(x,y)=q_b\\
&(x,y)\in\Gamma_3: && q(x,y)=-\lambda\,\mathbf{n}\cdot\nabla T(x,y)=\alpha[T(x,y)-T_\infty]
\end{aligned} \tag{2}$$

where T_b is the known boundary temperature, q_b is the known boundary heat flux, α [W/(m^{2}K)] is the heat transfer coefficient and T_∞ is the ambient temperature.

To estimate the changes of temperature due to the perturbation of local geometry of the boundary, shape sensitivity analysis is applied [1-5]. Here the implicit variant of sensitivity analysis is used. At first, the boundary element method for the Laplace equation with linear boundary elements is considered and then the system of algebraic equations is created. Next, this system of equations is differentiated with respect to the shape parameter.

1. Boundary element method

Application of the boundary element method for the Laplace equation leads to the following system of algebraic equations [6-8]:

$$\mathbf{Gq} = \mathbf{HT} \tag{3}$$

where **G** and **H** are so-called influence matrices, **T** is the vector of boundary temperatures and **q** is the vector of boundary heat fluxes.

For the linear boundary elements, the system of equations (3) can be written in the form of

$$\sum_{r=1}^{R} G_{i\,r} q_r = \sum_{r=1}^{R} H_{i\,r} T_r, \quad i = 1, 2, ..., R \tag{4}$$

where R is the number of boundary nodes.

For single node r being the end of boundary element Γ_j and being the beginning of boundary element Γ_{j+1}, one has

$$G_{i\,r} = G_{i\,j}^{k} + G_{i\,j+1}^{p}, \qquad \hat{H}_{i\,r} = \hat{H}_{i\,j}^{k} + \hat{H}_{i\,j+1}^{p} \tag{5}$$

while for double node r, $r + 1$

$$\begin{aligned} G_{i\,r} &= G_{i\,j}^{k}, \quad G_{i\,r+1} = G_{i\,j+1}^{p} \\ \hat{H}_{i\,r} &= \hat{H}_{i\,j}^{k}, \quad \hat{H}_{i\,r+1} = \hat{H}_{i\,j+1}^{p} \end{aligned} \tag{6}$$

where

$$H_{i\,r} = \begin{cases} \hat{H}_{i\,r} & i \neq r \\ -\sum\limits_{\substack{r=1 \\ r \neq i}}^{R} H_{i\,r} & i = r \end{cases} \tag{7}$$

Additionally

$$G_{i\,j}^{p} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_p \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{8}$$

$$G_{i\,j}^{k} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_k \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{9}$$

and

$$\hat{H}_{ij}^{p} = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\,\mathrm{d}\theta \tag{10}$$

$$\hat{H}_{ij}^{k} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\,\mathrm{d}\theta \tag{11}$$

where

$$r_{ij} = \sqrt{(N_p x_j^p + N_k x_j^k - \xi_i)^2 + (N_p y_j^p + N_k y_j^k - \eta_i)^2} = \sqrt{(r_x^j)^2 + (r_y^j)^2} \tag{12}$$

$$l_j = \sqrt{(x_j^k - x_j^p)^2 + (y_j^k - y_j^p)^2} = \sqrt{(l_x^j)^2 + (l_y^j)^2} \tag{13}$$

In the above formulas, $N_p = (1-\theta)/2$, $N_k = (1+\theta)/2$, $\theta \in [-1,1]$ are the shape functions, (x_j^p, y_j^p), (x_j^k, y_j^k) are the co-ordinates of the beginning and end of element Γ_j. It should be pointed out that the solution to system (4) allows one to determine the "missing" boundary temperatures and heat fluxes. Next, the temperatures in an optional set of internal nodes can be calculated using the formula

$$T_i = \sum_{r=1}^{R} H_{ir} T_r - \sum_{r=1}^{R} G_{ir} q_r \tag{14}$$

2. Shape sensitivity analysis - implicit differentiation method

It is assumed that b is the shape parameter, this means b corresponds to the x or y coordinate of one boundary node. The implicit differentiation method [3, 9-11] consists in the differentiation of the algebraic system of equations (3) and then

$$\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} + \mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b} = \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} + \mathbf{H}\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b} \tag{15}$$

or

$$\mathbf{GW} = \mathbf{HU} + \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} - \frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} \tag{16}$$

where: $\mathbf{U} = \mathrm{D}\mathbf{T}/\mathrm{D}b$, $\mathbf{W} = \mathrm{D}\mathbf{q}/\mathrm{D}b$.

The differentiation of boundary conditions (2) gives:

$$\begin{aligned}
&(x,y)\in\Gamma_1: \quad U = \frac{\mathrm{D}T_b}{\mathrm{D}b} = 0\\
&(x,y)\in\Gamma_2: \quad W = \frac{\mathrm{D}q_b}{\mathrm{D}b} = 0\\
&(x,y)\in\Gamma_3: \quad \frac{\mathrm{D}q}{\mathrm{D}b} = \alpha\frac{\mathrm{DT}}{\mathrm{D}b} \;\rightarrow\; W = \alpha U
\end{aligned} \tag{17}$$

Hence, this approach of shape sensitivity analysis is connected with the differentiation of the elements of matrices **G** and **H** [8, 12].

3. Examples of computations

A square of dimensions 0.05×0.05 m is considered. The thermal conductivity equals λ = 35 W/(mK). On the bottom and left side of the domain, the Dirichlet condition T = 400°C is assumed, on the remaining parts of the boundary, the Robin condition q = 30(T-20) is accepted. In successive variants of computations, the boundary has been divided into 8, 16, 40, 60 and 80 linear boundary elements respectively. The following set of internal points has been taken into account: A = = (0.0125, 0.0125), B = (0.0375, 0.0125), C = (0.025, 0.025), D = (0.0125, 0.0375), E = (0.0375, 0.0375).

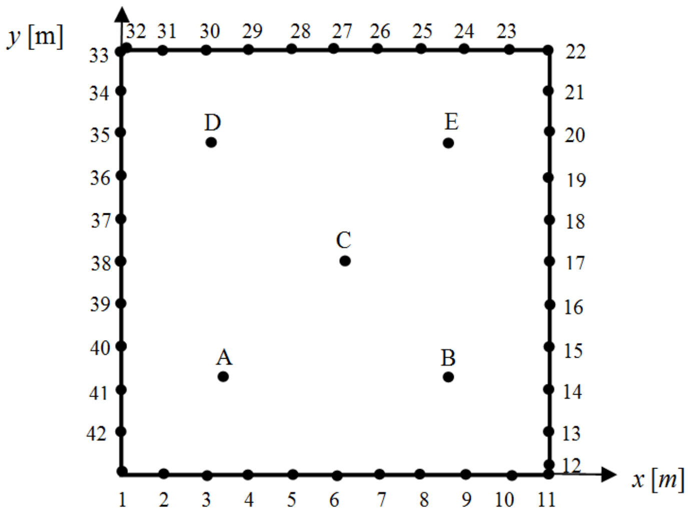

Fig. 1. Discretization (N = 40) and position of internal nodes

In Table 1, the temperatures at the internal points for different variants of discretization are presented.

Table 1

Temperature at internal points for different variants of discretization

N	T_A	T_B	T_C	T_D	T_E
8	398.437	395.296	392.922	395.296	386.904
16	398.388	394.911	393.672	394.911	386.696
40	398.370	394.817	393.620	394.817	386.653
60	398.369	394.809	393.616	394.809	386.652
80	398.368	394.807	393.615	394.807	386.652

It can be seen that the internal temperatures for 40, 60 and 80 linear boundary elements are close. The temperature distribution in the domain considered is shown in Figure 2.

Next, the shape sensitivity analysis for 40 linear boundary elements has been done. Three shape parameters b_1, b_2, b_3 have been taken into account, namely $b_1 = x_{22}$, $b_2 = y_{22}$, $b_3 = x_{17}$ as shown in Figure 1.

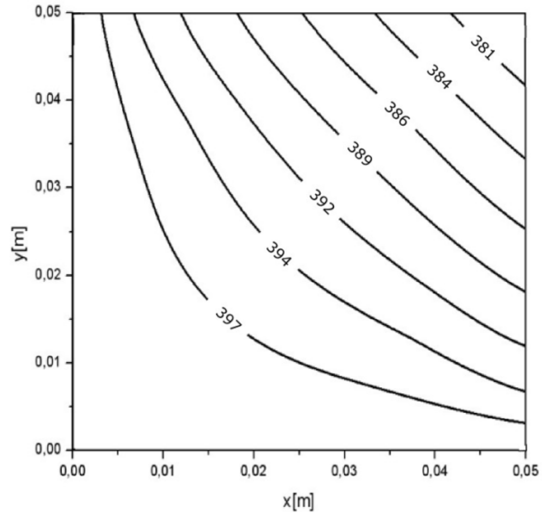

Fig. 2. Temperature distribution

The distributions of sensitivity functions DT/Db_1, DT/Db_2 and DT/Db_3 are shown in Figures 3-5, respectively.

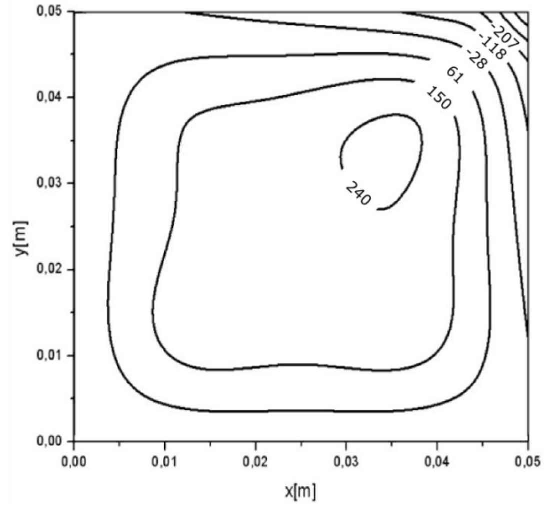

Fig. 3. Distribution of DT/Db_1

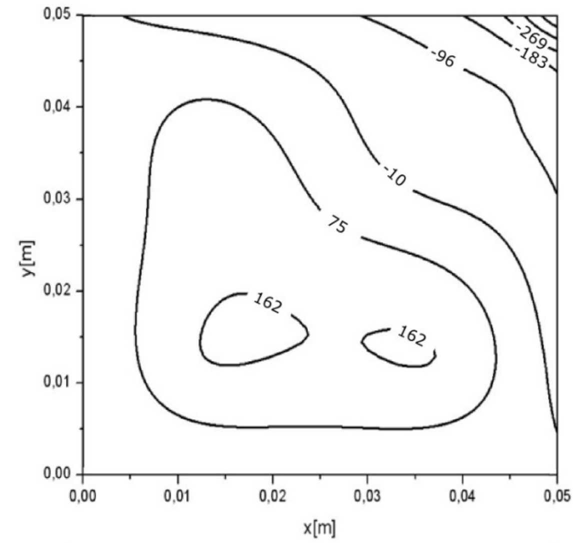

Fig. 4. Distribution of DT/Db_2

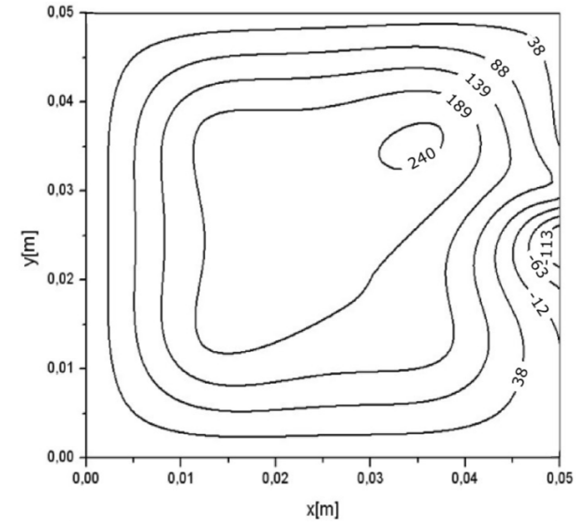

Fig. 5. Distribution of DT/Db_3

Using the expansions of function T into the Taylor series

$$\begin{aligned} T(x,y,b_1+\Delta b_1,b_2+\Delta b_2) &= T(x,y,b_1,b_2)+\frac{DT}{Db_1}\Delta b_1+\frac{DT}{Db_2}\Delta b_2 \\ T(x,y,b_1-\Delta b_1,b_2-\Delta b_2) &= T(x,y,b_1,b_2)-\frac{DT}{Db_1}\Delta b_1-\frac{DT}{Db_2}\Delta b_2 \end{aligned} \tag{18}$$

one has

$$\Delta T = 2\frac{\mathrm{D}T}{\mathrm{D}b_1}\Delta b_1 + 2\frac{\mathrm{D}T}{\mathrm{D}b_2}\Delta b_2 \tag{19}$$

where Δb_1 is the perturbation of parameter b_1, while Δb_2 is the perturbation of parameter b_2. For b_1, the value of the sensitivity function equals $U_C = \mathrm{D}T_C/\mathrm{D}b_1 = 186.46$; for b_2 one has $U_C = \mathrm{D}T_C/\mathrm{D}b_2 = 100.15$, while for b_3: $U_C = \mathrm{D}T_C/\mathrm{D}b_3 = 193.52$. Therefore, under the assumption that $\Delta b_1 = \Delta b_2 = \Delta b_3 = 0.0005$ m, the change of temperature at node C due to the change of shape parameter b_1 and b_2 is equal to 0.3°C (c. f. formula (19)), while the change of temperature at point C due to the change of parameter b_3 equals 0.2°C.

The next example concerns the fragment of a heating panel. The thermal conductivity is equal to $\lambda = 30$ W/(mK). The boundary conditions and discretization are shown in Figure 6. In Figure 7, the temperature distribution in the domain is presented.

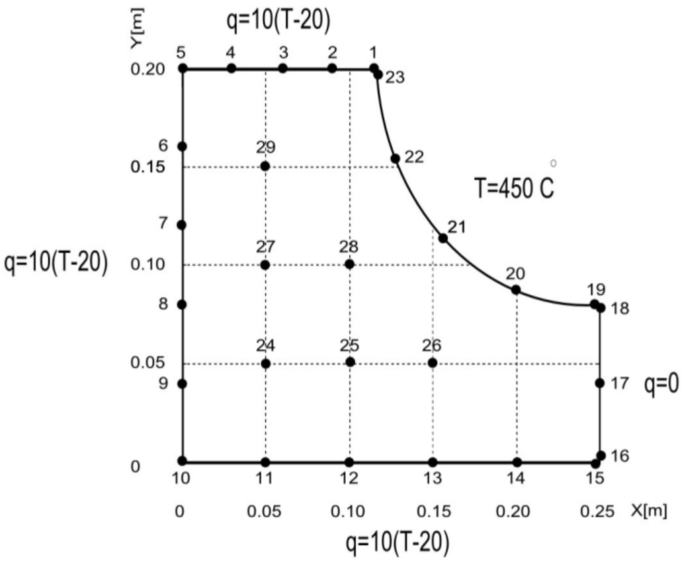

Fig. 6. Discretization and boundary conditions

It is assumed that shape parameters b_1, b_2 correspond to the co-ordinates of boundary node 21 (c. f. Figure 6) and then for $b_1 = x_{21}$, one has $U_{28} = 49.42$, while for $b_2 = y_{21}$: $U_{28} = 306.91$.

Let $\Delta b_1 = \Delta b_2 = 0.0025$ m. Thus, the change of temperature at internal node 28 due to the change of shape parameters b_1 and b_2 is equal to 1.78°C (c.f. formula (19)).

Figures 8 and 9 illustrate the distribution of sensitivity functions $\mathrm{D}T/\mathrm{D}b_1$ and $\mathrm{D}T/\mathrm{D}b_2$, respectively.

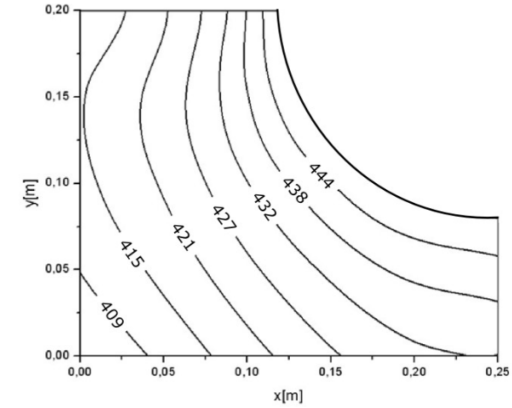

Fig. 7. Temperature distribution

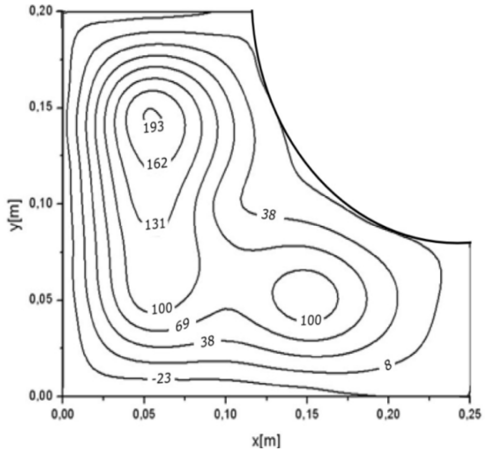

Fig. 8. Distribution of DT/Db_1

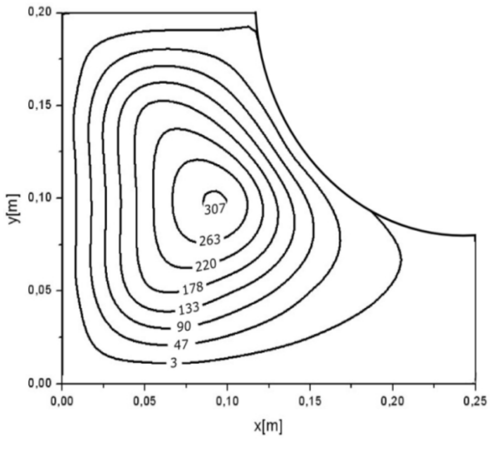

Fig. 9. Distribution of DT/D

Summing up, the implicit approach of shape sensitivity analysis, coupled with the boundary element method is a good tool to estimate the change of temperature due to the perturbation of boundary local geometry.

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