APPLICATION OF GREEN'S MATRIX METHOD IN VIBRATION PROBLEMS OF TIMOSHENKO BEAMS

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Abstract. In the present paper, a Green's matrix used to solve vibration problems of Timoshenko beams is determined. The problem formulation includes beam vibrations which are described by differential equations with variable parameters. To determine the Green's matrix, the power series method was used.

Introduction

The analysis of the vibration problem of a non-uniform Timoshenko beam was the subject of papers [1-6]. In general, assuming that the coefficients of differential equations are variables of spatial coordinates, a solution in an analytical form is not available. The exact solution by the dynamic stiffness method for the free vibration problem was given in [1] and [2]. The Authors of paper [1] used the method of Frobenius to obtain the exact fundamental solutions of differential equations. The same method was proposed in [3] and [4], but in these papers the authors reduced two characteristic equations into one four-order ordinary differential equation. The solution to the vibration problem of the non prismatic Timoshenko beam, by using Chebyshev series approximation was presented in [5]. The Green's function method (GFM) for a stepped beam was shown in [6]. In this paper a functional matrix needed in GFM applied to solving the vibration problems of Timoshenko beams is derived.

1. Formulation of the problem

Let us consider a set of two differential equations:

$$\begin{cases} \dfrac{d}{dx}\left[Q(x)\left(\dfrac{dY}{dx}-\psi\right)\right]+m(x)\omega^2 Y=\bar{\mathbf{M}}_1[Y,\psi] \\ \dfrac{d}{dx}\left[R(x)\dfrac{d\psi}{dx}\right]+Q(x)\left(\dfrac{dY}{dx}-\psi\right)+\omega^2 J(x)\psi=\bar{\mathbf{M}}_2[Y,\psi] \end{cases} \tag{1}$$

The presented equations describe the motion of a Timoshenko beam for time harmonic vibrations with angular frequency ω. The meaning of functions Y, ψ is, respectively, beam lateral displacement and angle of rotation due to bending, Q is shear rigidity, R - bending rigidity of a beam, m - mass per unit length and J is the mass moment of inertia of the beam per unit length. The form of operators $\bar{\mathbf{M}}_1$, $\bar{\mathbf{M}}_2$ depend on the nature of the attached discrete systems. At points x = 0 and $x = L$ (ends of the beam), functions Y, ψ satisfy the boundary conditions, which can be written in a symbolical form as:

$$\bar{\mathbf{B}}_0\left[Y,\psi\right]\Big|_{x=0}=0\,,\quad \bar{\mathbf{B}}_1\left[Y,\psi\right]\Big|_{x=L}=0 \tag{2}$$

Introducing dimensionless coordinate $\xi=\frac{x}{L}$ (L - length of the beam) into equations (1), (2) we obtain:

$$\begin{cases} \dfrac{d}{d\xi}\left[\dfrac{q(\xi)}{\Delta}\left(\dfrac{dy(\xi)}{d\xi}-\psi(\xi)\right)\right]+s(\xi)\Omega^2 y(\xi)=\mathbf{M}_1\left[y(\xi),\psi(\xi)\right] \\ \dfrac{d}{d\xi}\left[r(\xi)\dfrac{d\psi(\xi)}{d\xi}\right]+\dfrac{q(\xi)}{\Delta}\left(\dfrac{dy(\xi)}{d\xi}-\psi(\xi)\right)+\gamma(\xi)\eta\Omega^2\psi(\xi)=\mathbf{M}_2\left[y(\xi),\psi(\xi)\right] \end{cases} \tag{3}$$

and boundary conditions

$$\mathbf{B}_0\left[y,\psi\right]\Big|_{\xi=0}=0\,,\quad \mathbf{B}_1\left[y,\psi\right]\Big|_{\xi=1}=0 \tag{4}$$

where $y=\dfrac{Y}{L}$, $q(\xi)=\dfrac{Q(\xi)}{Q(0)}$, $s(\xi)=\dfrac{m(\xi)}{m(0)}$, $r(\xi)=\dfrac{R(\xi)}{R(0)}$, $\gamma(\xi)=\dfrac{J(\xi)}{J(0)}$

$$\Delta=\frac{R(0)}{Q(0)L^2}\,,\ \eta=\frac{J(0)}{m(0)L^2}\ \text{and}\ \Omega^2=\frac{m(0)L^4\omega^2}{R(0)}$$

The set of equations (3) we can rewrite in a simply form as a matrix equation [1]:

$$\left[\mathbf{A_2}\right]\left\{\mathbf{V}''\right\}+\left[\mathbf{A_1}\right]\left\{\mathbf{V}'\right\}+\left[\mathbf{A_0}\right]\left\{\mathbf{V}\right\}=\left[\mathbf{M}\right] \tag{5}$$

where

$$\left[\mathbf{A_2}\right]=\begin{bmatrix} q & 0 \\ 0 & \Delta r \end{bmatrix},\ \left[\mathbf{A_1}\right]=\begin{bmatrix} q & -q \\ q & \Delta r' \end{bmatrix},\ \left[\mathbf{A_0}\right]=\begin{bmatrix} \Delta\Omega^2 s & -q' \\ 0 & \Delta\eta\Omega^2\gamma r \end{bmatrix}$$

$$\left\{\mathbf{V}''\right\}=\begin{bmatrix} y'' \\ \psi'' \end{bmatrix},\ \left\{\mathbf{V}'\right\}=\begin{bmatrix} y' \\ \psi' \end{bmatrix},\ \left\{\mathbf{V}\right\}=\begin{bmatrix} y \\ \psi \end{bmatrix},\ \left[\mathbf{M}\right]=\begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix}$$

and $q=q(\xi)$, $s=s(\xi)$, $\gamma=\gamma(\xi)$, $r=r(\xi)$, $y=y(\xi)$, $\psi=\psi(\xi)$. The form of operator $\mathbf{M}$ depends on the attached discrete elements [6].

To find a solution to boundary problem (4), (5), Green's function (matrix) method can be used.

2. Functional Green's matrix

Let us suppose that the Green's matrix of the matrix differential operator $\mathfrak{L}$

$$\mathfrak{L}=\begin{bmatrix} q\dfrac{d^2}{d\xi^2}+\dfrac{dq}{d\xi}\dfrac{d}{d\xi}+\square s\Omega^2 & -q\dfrac{d}{d\xi}-\dfrac{dq}{d\xi}\dfrac{d}{d\xi} \\ q\dfrac{d}{d\xi} & \square r\dfrac{d^2}{d\xi^2}+\dfrac{dr}{d\xi}\dfrac{d}{d\xi}+\square\eta\Omega^2 r\gamma \end{bmatrix} \tag{6}$$

is known, then the solution of (4), (5) may be written as:

$$\mathbf{V}(\xi)=\int_0^1 \mathbf{G}^{\mathrm{T}}(\xi,\zeta)\mathbf{M}(\zeta)\,d\zeta \tag{7}$$

In the vibration analysis of the considered Timoshenko beam, solution (7) is used to obtain the frequency equation, which is then solved numerically with respect to eigenfrequencies ω_n. Matrix **G**, occurring in equation (7), has the form:

$$\begin{aligned}\mathbf{G}(\xi,\zeta)&=\mathbf{G}_0(\xi,\zeta)+\mathbf{G}_1(\xi,\zeta)H(\xi-\zeta)=\\ &=\begin{bmatrix} g_{01}(\xi,\zeta)+g_{11}(\xi-\zeta)H(\xi-\zeta) & g_{02}(\xi,\zeta)+g_{12}(\xi-\zeta)H(\xi-\zeta) \\ g_{03}(\xi,\zeta)+g_{13}(\xi-\zeta)H(\xi-\zeta) & g_{04}(\xi,\zeta)+g_{14}(\xi-\zeta)H(\xi-\zeta)\end{bmatrix}\end{aligned} \tag{8}$$

($H(\cdot)$ is a unit step function) and satisfies the equation:

$$\mathfrak{L}\{\mathbf{G}\}=\mathbf{I}\delta(\xi-\zeta) \tag{9}$$

where **I** is an identity matrix and δ $(\cdot)$ is a delta function. Matrices $\mathbf{G_0}$, $\mathbf{G_1}$ satisfy the homogeneous equation

$$\mathfrak{L}\{\mathbf{U}\}=0. \tag{10}$$

Moreover, $\mathbf{G_1}$ fulfills the additional conditions presented in [6]. The solution of (10) we want find in the form of a vector of the power series:

$$\{\mathbf{V}\}=\begin{bmatrix} y(\xi) \\ \psi(\xi)\end{bmatrix}=\begin{bmatrix} \sum_{i=0}^{\infty}\dfrac{y_i}{i!}\xi^i \\ \sum_{i=0}^{\infty}\dfrac{p_i}{i!}\xi^i \end{bmatrix} \tag{11}$$

Let us assume that functions q, s, γ, r are also given as a sum:

$$q(\xi)=s(\xi)=\sum_{i=0}^{\infty}\frac{a_i}{i!}\xi^i\ ,\ \gamma(\xi)=r(\xi)=\sum_{i=0}^{\infty}\frac{b_i}{i!}\xi^i \tag{12}$$

Substituting (11) and (12) into a homogeneous equation, we can write it as follows:

$$\begin{bmatrix}\sum_{i=0}^{\infty}\frac{z_i}{i!}\xi^i\\ \sum_{i=0}^{\infty}\frac{\overline{z}_i}{i!}\xi^i\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{13}$$

where:

$$z_i=\sum_{j=0}^{i}\binom{i}{j}\left[a_j\left(\square\Omega^2 y_{i-j}+y_{i+1-j}+y_{i+2-j}-p_{i+1-j}\right)-a_{j+1}p_{i-j}\right]$$

$$\overline{z}_i=\sum_{j=0}^{i}\binom{i}{j}\left[a_j\left(y_{i+1-j}-p_{i-j}\right)+b_j\left(\square\eta\Omega^2 p_{i-j}+p_{i+2-j}\right)+b_{j+1}\square\, p_{i+1-j}\right] \tag{14}$$

It leads to the system of equations:

$$\sum_{j=0}^{i}\binom{i}{j}\left[a_j\left(\square\Omega^2 y_{i-j}+y_{i+1-j}+y_{i+2-j}-p_{i+1-j}\right)-a_{j+1}p_{i-j}\right]=0$$

$$\sum_{j=0}^{i}\binom{i}{j}\left[a_j\left(y_{i+1-j}-p_{i-j}\right)+b_j\left(\square\eta\Omega^2 p_{i-j}+p_{i+2-j}\right)+b_{j+1}\square\, p_{i+1-j}\right]=0,\quad i=0,1,2,.. \tag{15}$$

or, in another form:

$$y_{i+2}=-\square\Omega^2 y_i-y_{i+1}+p_{i+1}+\frac{a_1}{a_0}p_i-\frac{1}{a_0}\sum_{j=1}^{i}Z_{ij}$$

$$p_{i+2}=-\frac{a_0}{b_0}y_{i+1}+\left(\frac{a_0}{b_0}-\square\eta\Omega^2\right)p_i-\frac{b_1}{b_0}\square\, p_{i+1}-\frac{1}{b_0}\sum_{j=1}^{i}\overline{Z}_{ij}\ ,\quad i=0,1,2,.. \tag{16}$$

where

$$Z_{ij}=\binom{i}{j}\left[a_j\left(\square\Omega^2 y_{i-j}+y_{i+1-j}+y_{i+2-j}-p_{i+1-j}\right)-a_{j+1}p_{i-j}\right]$$

$$\overline{Z}_{ij}=\binom{i}{j}\left[a_j\left(y_{i+1-j}-p_{i-j}\right)+b_j\left(\square\eta\Omega^2 p_{i-j}+p_{i+2-j}\right)+b_{j+1}\square\, p_{i+1-j}\right]\ ,\quad i=0,1,2,.. \tag{17}$$

The values of unknown coefficients y_{i+2} , p_{i+2} (i = 0,1,2,3,…) can be expressed by means of coefficients y_0,y_1,p_0,p_1. It is possible to prove that four sets of conditions:

a) $y_0 = 0,\ y_1 = 1,\ p_0 = 0,\ p_1 = 1$

b) $y_0 = 0,\ y_1 = 1,\ p_0 = 1,\ p_1 = 0$

c) $y_0 = 1,\ y_1 = 0,\ p_0 = 0,\ p_1 = 1$

d) $y_0 = 1,\ y_1 = 0,\ p_0 = 1,\ p_1 = 0$ (18)

give us four linear independent solutions and therefore, we may write:

$$\{\mathbf{U}\} = \begin{bmatrix} C_1 \sum_{i=0}^{\infty} \frac{y_{1,i}}{i!} \xi^i + C_2 \sum_{i=0}^{\infty} \frac{y_{2,i}}{i!} \xi^i \\ C_3 \sum_{i=0}^{\infty} \frac{p_{1,i}}{i!} \xi^i + C_4 \sum_{i=0}^{\infty} \frac{p_{2,i}}{i!} \xi^i \end{bmatrix} = \begin{bmatrix} C_1 A_1(\xi) + C_2 A_2(\xi) \\ C_3 B_1(\xi) + C_4 B_2(\xi) \end{bmatrix} \tag{19}$$

Finally, matrix $\mathbf{G}(\xi, \zeta)$ corresponding to system $\mathfrak{L}\{\mathbf{V}\} = [\mathbf{M}]$, has the form:

$$\mathbf{G}(\xi,\zeta) = \begin{bmatrix} C_1 A_1(\xi) + \bar{C}_1 A_1(\xi-\zeta) & C_2 A_2(\xi) + \bar{C}_2 A_2(\xi-\zeta) \\ C_3 B_1(\xi) + \bar{C}_3 B_1(\xi-\zeta) & C_4 B_2(\xi) + \bar{C}_4 B_2(\xi-\zeta) \end{bmatrix} \cdot H(\xi-\zeta) \tag{20}$$

Unknown coefficients C_i, i = 1,2,3,4 are determined by the use of boundary conditions (4). For example, the boundary conditions for a simply supported end of the beam are:

$$y = 0, \quad \psi' = 0 \tag{21}$$

and for the rigidly fixed end of the beam:

$$y = 0, \quad \psi = 0 \tag{22}$$

Conclusions

In this paper Green’s matrix of the linear matrix operator occurring in equations of motion of a Timoshenko beam, was found. To determine the solution of homogeneous equations, power series were proposed. The presented method can be used for numerical calculations of the vibration problem of the considered non-uniform beam with attachments.

References

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