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DETERMINATION OF A FUNDAMENTAL SOLUTION FOR DIFFUSION EQUATION IN CYLINDRICAL CO-ORDINATE SYSTEM WITH TEMPERATURE-DEPENDENT SOURCE TERM

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Abstract. In the paper the diffusion equation with temperature - dependent source function describing the heat conduction problem in axisymmetrical domain is considered. The fundamental solution is determined and the results obtained by means of the boundary element method are compared with the analytical solution.

1. Formulation of the problem

The following ordinary differential equation is considered

$$\frac{\lambda}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) - KT + Q = 0 \tag{1}$$

where r is the spatial co-ordinate, $r \in [R_1, R_2]$, λ is the thermal conductivity, K is the material constant, Q is the constant heat source.
Equation (1) can be written in the form

$$\frac{\lambda}{K}\frac{\mathrm{d}^2T}{\mathrm{d}r^2} + \frac{\lambda}{Kr}\frac{\mathrm{d}T}{\mathrm{d}r} - T = -\frac{Q}{K} \tag{2}$$

The solution of equation (2) is found under the assumption that the following boundary conditions must be fulfilled

$$\begin{cases} r = R_1: \quad q(r) = -\lambda\dfrac{\mathrm{d}T}{\mathrm{d}r} = \alpha\left[T(r) - T_a\right] \\ r = R_2: \quad T(r) = T_b \end{cases} \tag{3}$$

where α is the heat transfer coefficient, T_a is the ambient temperature, T_b is the constant boundary temperature.

2. Analytical solution

It can be observed that equation (2) is the second order nonhomogeneous linear ordinary equation. Hence its solution can be written in the form

$$T = P.S.N.E. + G.S.H.E. \tag{4}$$

where $G.S.H.E.$ is the general solution of the corresponding homogeneous equation and $P.S.N.E.$ is the particular solution of the nonhomogeneous equation (2). It can be observed that

$$P.S.N.E. = \frac{Q}{K} \tag{5}$$

The homogeneous equation corresponding to the equation (2) has the form

$$\frac{\lambda}{K}\ \frac{\mathrm{d}^2 T}{\mathrm{d} r^2} + \frac{\lambda}{K r}\ \frac{\mathrm{d} T}{\mathrm{d} r} - T = 0 \tag{6}$$

To obtain the general solution of equation (6) the following substitution is introduced

$$r = \sqrt{\frac{\lambda}{K}}\, u \tag{7}$$

Hence unknown temperature T is the composite function of the form

$$T(r) = T(r(u)) \tag{8}$$

The first order total derivative of function T has the form

$$\frac{\mathrm{d}T}{\mathrm{d}u} = \frac{\mathrm{d}T}{\mathrm{d}r}\ \frac{\mathrm{d}r}{\mathrm{d}u} \tag{9}$$

From formula (7) one obtains the derivative $\frac{\mathrm{d}r}{\mathrm{d}u}$ in the form

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \sqrt{\frac{\lambda}{K}} \tag{10}$$

Introduction of dependence (10) to the formula (9) leads to

$$\frac{\mathrm{d}T}{\mathrm{d}u} = \frac{\mathrm{d}T}{\mathrm{d}r}\ \sqrt{\frac{\lambda}{K}} \tag{11}$$

or

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \sqrt{\frac{K}{\lambda}} \frac{\mathrm{d}T}{\mathrm{d}u} \tag{12}$$

In a similar way one has

$$\frac{\mathrm{d}^2T}{\mathrm{d}u^2} = \frac{\mathrm{d}^2T}{\mathrm{d}r^2} \frac{\mathrm{d}r}{\mathrm{d}u} \sqrt{\frac{\lambda}{K}} \tag{13}$$

Taking into account the dependence (10) the above formula has the form

$$\frac{\mathrm{d}^2T}{\mathrm{d}u^2} = \frac{\mathrm{d}^2T}{\mathrm{d}r^2} \frac{\lambda}{K} \tag{14}$$

Hence

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2} = \frac{K}{\lambda} \frac{\mathrm{d}^2T}{\mathrm{d}u^2} \tag{15}$$

Introduction of (7), (12) and (15) into equation (6) leads to the homogeneous linear equation of the form

$$\frac{\mathrm{d}^2T}{\mathrm{d}u^2} + \frac{1}{u} \frac{\mathrm{d}T}{\mathrm{d}u} - T = 0 \tag{16}$$

Equation (16) can be written in the equivalent form

$$u^2 \frac{\mathrm{d}^2T}{\mathrm{d}u^2} + \frac{\mathrm{d}t}{\mathrm{d}u} - u^2 T = 0 \tag{17}$$

It can be observed that equation (17) is the particular case ($n = 0$) of the modified Bessel equation of the form

$$u^2 \frac{\mathrm{d}^2T}{\mathrm{d}u^2} + \frac{\mathrm{d}t}{\mathrm{d}u} - \left(u^2 - n^2\right) T = 0 \tag{18}$$

Hence, it can be proved [1] that the general solution of equation (17) has the form

$$T(u) = C_1 \cdot I_0(u) + C_2 \cdot K_0(u) \tag{19}$$

where $I_0(u)$, $K_0(u)$ are the modified Bessel functions of the first and the second kind, respectively. Symbols C_1, C_2 stand for arbitrary constants.
Functions $I_0(u)$, $K_0(u)$ have the form [1]

$$I_0(u)=\sum_{k=0}^{\infty}\frac{1}{4^k(k!)^2}u^{2k}\ ,\ K_0(u)=-I_0(u)\cdot\ln\frac{u}{2}+\sum_{m=0}^{\infty}\frac{\Psi(m+1)}{4^m(m!)^2}u^{2m} \tag{20}$$

where $\Psi(m+1)=-\gamma+\sum_{l=1}^{m}\frac{1}{n}$ and γ is the Euler-Mascheroni constant, $\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)$, $\gamma\approx 0{,}577$.

From equation (7) one obtains

$$u=r\cdot\sqrt{\frac{K}{\lambda}} \tag{21}$$

Bearing in mind solution (19) and formula (21) one can lean that the general solution of homogeneous equation (6) has the following form

$$T(r)=C_1\tilde{I}_0(r)+C_2\tilde{K}_0(r) \tag{22}$$

where

$$\tilde{I}_0(r)=I_0(u)=I_0\left(\sqrt{\frac{K}{\lambda}}\cdot r\right)=\sum_{k=0}^{\infty}\frac{K^k}{(4\lambda)^k(k!)^2}r^{2k} \tag{23}$$

$$\tilde{K}_0(r)=K_0\left(\sqrt{\frac{K}{\lambda}}\cdot r\right)=-\tilde{I}_0(r)\cdot\ln\left(\sqrt{\frac{K}{4\lambda}}\cdot r\right)+\sum_{m=0}^{\infty}\frac{K^m\cdot\Psi(m+1)}{(4\lambda)^m(m!)^2}r^{2m} \tag{24}$$

Using the dependences (4), (5) and (22) the general solution of equation (2) has the form

$$T(r)=\frac{Q}{K}+C_1\tilde{I}_0(r)+C_2\tilde{K}_0(r) \tag{25}$$

where functions $\tilde{I}_0(r)$, $\tilde{K}_0(r)$ have the form (23) and (24), respectively.

Taking into account the boundary conditions (3) one obtains the following system of equations with C_1, C_2 as unknowns

$$\begin{cases}-\lambda\left(C_1\tilde{I}_1(R_1)-C_2\tilde{K}_1(R_1)\right)=\alpha\left(\dfrac{Q}{K}+C_1\tilde{I}_0(R_1)+C_2\tilde{K}_0(R_1)-T_a\right)\\[2mm] \dfrac{Q}{K}+C_1\tilde{I}_0(R_2)+C_2\tilde{K}_0(R_2)=T_b\end{cases} \tag{26}$$

where

$$\tilde{I}_1(r) = \frac{\mathrm{d}}{\mathrm{d}r}\tilde{I}_0(r) = \sum_{k=0}^{\infty} \frac{2(k+1)K^{k+1}}{(4\lambda)^{k+1}\left((k+1)!\right)^2} r^{2k+1}$$

$$\tilde{K}_1(r) = \frac{\mathrm{d}}{\mathrm{d}r}\tilde{K}_0(r) = -\ln\left(\sqrt{\frac{K}{4\lambda}}\cdot r\right)\cdot \sum_{k=0}^{\infty} \frac{2(k+1)K^{k+1}}{(4\lambda)^{k+1}\left((k+1)!\right)^2} r^{2k+1} - \frac{\tilde{I}_0(r)}{r} + \tag{27}$$

$$+\sum_{m=0}^{\infty} \frac{2(m+1)K^{m+1}\Psi(m+2)}{(4\lambda)^{m+1}\left((m+1)!\right)^2} r^{2m+1}$$

After determining C_1, C_2 from (26), one gets the solution of equation (2) fulfilling the boundary conditions (3) in the form

$$T(r) = \frac{Q}{K} + \frac{\alpha\tilde{K}_0(R_2)(T_aK - Q) + (Q - T_bK)\left(\alpha\tilde{K}_0(R_1) - \lambda\tilde{K}_1(R_1)\right)}{K\left[\tilde{K}_0(R_2)\left(\lambda\tilde{I}_1(R_1) + \alpha\tilde{I}_0(R_1)\right) - \tilde{I}_0(R_2)\left(\alpha\tilde{K}_0(R_1) - \lambda\tilde{K}_1(R_1)\right)\right]}\tilde{I}_0(r) +$$

$$+\frac{-\alpha\tilde{I}_0(R_2)(T_aK - Q) - (Q - T_bK)\left(\alpha\tilde{I}_0(R_1) + \lambda\tilde{I}_1(R_1)\right)}{K\left[\tilde{K}_0(R_2)\left(\lambda\tilde{I}_1(R_1) + \alpha\tilde{I}_0(R_1)\right) - \tilde{I}_0(R_2)\left(\alpha\tilde{K}_0(R_1) - \lambda\tilde{K}_1(R_1)\right)\right]}\tilde{K}_0(r) \tag{28}$$

3. Boundary element method

To formulate the boundary element method [2-5] equation (2) can be written in the form

$$\lambda r\frac{\mathrm{d}^2T}{\mathrm{d}r^2} + \lambda\frac{\mathrm{d}T}{\mathrm{d}r} - KrT + Qr = 0 \tag{29}$$

The weighted residual criterion for equation (29) is of the form

$$\int_{R_1}^{R_2}\left[\lambda r\frac{\mathrm{d}^2T}{\mathrm{d}r^2} + \lambda\frac{\mathrm{d}T}{\mathrm{d}r} - KrT + Qr\right]\cdot T^*(\xi,r)\,dr = 0 \tag{30}$$

where ξ is the observation point, $T^*(\xi,r)$ is the fundamental solution and it is function of the form

$$T^*(\xi,r) = \frac{\operatorname{sgn}(r-\xi)}{2\lambda}\left[I_0\left(\sqrt{\frac{K}{\lambda}}\xi\right)K_0\left(\sqrt{\frac{K}{\lambda}}r\right) - I_0\left(\sqrt{\frac{K}{\lambda}}r\right)K_0\left(\sqrt{\frac{K}{\lambda}}\xi\right)\right] \tag{31}$$

It can be checked (using for example the computer package Maple) that this function fulfils the equation

$$\lambda r\frac{\partial^2 T^*(\xi,r)}{\partial r^2}+\lambda\frac{\partial T^*(\xi,r)}{\partial r}-KrT^*(\xi,r)=-\delta(\xi,r) \tag{32}$$

where $\delta(\xi,r)$ is the Dirac function.
Heat flux resulting from the fundamental solution is of the form

$$q^*(\xi,r)=-\lambda\frac{\partial T^*(\xi,r)}{\partial r} \tag{33}$$

this means

$$q^*(\xi,r)=\frac{\sqrt{K}\,\mathrm{sgn}(r-\xi)}{2\sqrt{\lambda}}\left[K_0\left(\sqrt{\frac{K}{\lambda}}\xi\right)I_1\left(\sqrt{\frac{K}{\lambda}}r\right)+K_1\left(\sqrt{\frac{K}{\lambda}}r\right)I_0\left(\sqrt{\frac{K}{\lambda}}\xi\right)\right] \tag{34}$$

where $\mathrm{sgn}(r-\xi)$ is the sign function.
To apply the property (32) equation (30) can be expressed as follows

$$\begin{aligned}&\left[rq^*(\xi,r)T(r)-rT^*(\xi,r)q(r)\right]_{R_1}^{R_2}+Q\int_{R_1}^{R_2}rT^*(\xi,r)\,dr+\\&+\int_{R_1}^{R_2}\left(\lambda r\frac{\partial^2 T^*(\xi,r)}{\partial r^2}+\lambda\frac{\partial T^*(\xi,r)}{\partial r}-KrT^*(\xi,r)\right)T(r)\,dr=0\end{aligned} \tag{35}$$

The last integral in the above equation is equal to

$$\int_{R_1}^{R_2}\left(\lambda r\frac{\partial^2 T^*(\xi,r)}{\partial r^2}+\lambda\frac{\partial T^*(\xi,r)}{\partial r}-KrT^*(\xi,r)\right)T(r)\,dr=-\int_{R_1}^{R_2}\delta(\xi,r)T(r)\,dr=-T(\xi) \tag{36}$$

and then equation (35) takes the form

$$\left[rq^*(\xi,r)T(r)-rT^*(\xi,r)q(r)\right]_{R_1}^{R_2}+Q\int_{R_1}^{R_2}rT^*(\xi,r)\,dr-T(\xi)=0 \tag{37}$$

or

$$\begin{aligned}T(\xi)=&R_2q^*(\xi,R_2)T(R_2)-R_2T^*(\xi,R_2)q(R_2)-R_1q^*(\xi,R_1)T(R_1)+\\&+R_1T^*(\xi,R_1)q(R_1)+Q\int_{R_1}^{R_2}rT^*(\xi,r)\,dr\end{aligned} \tag{38}$$

For $\xi \to R_1^+$ and $\xi \to R_2^-$ one obtains

$$\begin{gathered}\frac{1}{2}T(R_1) = R_2 q^*(R_1,R_2)T(R_2) - R_2 T^*(R_1,R_2)q(R_2) - R_1 q^*(R_1,R_1)T(R_1) + \\ + R_1 T^*(R_1,R_1)q(R_1) + Q\int_{R_1}^{R_2} rT^*(R_1,r)\,dr\end{gathered} \tag{39}$$

and

$$\begin{gathered}\frac{1}{2}T(R_2) = R_2 q^*(R_2,R_2)T(R_2) - R_2 T^*(R_2,R_2)q(R_2) - R_1 q^*(R_2,R_1)T(R_1) + \\ + R_1 T^*(R_2,R_1)q(R_1) + Q\int_{R_1}^{R_2} rT^*(R_2,r)\,dr\end{gathered} \tag{40}$$

The system of equations (39), (40) can be written in the matrix form

$$\mathbf{G}\cdot\mathbf{q} = \mathbf{H}\cdot\mathbf{T} - Q\mathbf{Z} \tag{41}$$

where

$$\mathbf{G} = \begin{bmatrix} R_1 T^*(R_1,R_1) & -R_2 T^*(R_1,R_2) \\ R_1 T^*(R_2,R_1) & -R_2 T^*(R_2,R_2) \end{bmatrix} \tag{42}$$

$$\mathbf{q} = \begin{bmatrix} q(R_1) \\ q(R_2) \end{bmatrix} \tag{43}$$

$$\mathbf{H} = \begin{bmatrix} \dfrac{1}{2} + R_1 q^*(R_1,R_1) & -R_2 q^*(R_1,R_2) \\ R_1 q^*(R_2,R_1) & \dfrac{1}{2} - R_2 q^*(R_2,R_2) \end{bmatrix} \tag{44}$$

$$\mathbf{T} = \begin{bmatrix} T(R_1) \\ T(R_2) \end{bmatrix} \tag{45}$$

$$\mathbf{Z} = \begin{bmatrix} \displaystyle\int_{R_1}^{R_2} rT^*(R_1,r)\,dr \\ \displaystyle\int_{R_1}^{R_2} rT^*(R_2,r)\,dr \end{bmatrix} \tag{46}$$

In the matrix equation (41) two of four values $T(R_1)$, $T(R_2)$, $q(R_1)$, $q(R_2)$ are known from the boundary conditions, the remaining two should be determined. Introduction of values $T(R_1)$, $T(R_2)$, $q(R_1)$, $q(R_2)$ to formula (38) allows one to determine the temperature at the optional point ξ from interior of domain.

4. Example

The cylindrical layer of thickness 0.04 m ($R_1 = 0{,}01\,\text{m}, R_2 = 0.05\,\text{m}$) is considered. Thermal conductivity is equal to $\lambda = 10\,\text{W/(mK)}$, coefficient $K = 2000\ \text{W/}\left(\text{m}^3\text{K}\right)$, parameter $Q = 1000\ \text{W/m}^3$. On the internal surface $r = R_1$ the Dirichlet condition $T(R_1) = 100°\text{C}$ is assumed, while on the external surface $r = R_2$ the Robin condition $q(R_2) = \alpha[T(R_2) - T_a]$, where $\alpha = 250\ \text{W/(m}^2\text{K)}$, $T_{ot} = 20°\text{C}$, is accepted.

The problem has been solved by means of the BEM. Under the above assumptions the equation (41) takes the form

$$\mathbf{A} \cdot \mathbf{Y} = \mathbf{F} \tag{47}$$

where

$$\mathbf{A} = \begin{bmatrix} -R_1 T^*(R_1, R_1) & \alpha R_2 T^*(R_1, R_2) - R_2 q^*(R_1, R_2) \\ -R_1 T^*(R_2, R_1) & \dfrac{1}{2} + \alpha R_2 T^*(R_2, R_2) - R_2 q^*(R_2, R_2) \end{bmatrix} \tag{48}$$

$$\mathbf{Y} = \begin{bmatrix} q(R_1) \\ T(R_2) \end{bmatrix} \tag{49}$$

and

$$\mathbf{F} = T(R_1)\mathbf{C} + \alpha R_2 T_{ot} \mathbf{D} + Q\mathbf{Z} \tag{50}$$

while

$$\mathbf{C} = \begin{bmatrix} -\dfrac{1}{2} - R_1 q^*(R_1, R_1) \\ -R_1 q^*(R_2, R_1) \end{bmatrix} \tag{51}$$

$$\mathbf{D} = \begin{bmatrix} T^*(R_1, R_2) \\ T^*(R_2, R_2) \end{bmatrix} \tag{52}$$

Matrix **Z** has the form (46) and its elements, for the problem under considerations, were determined by means of the 6-point Gaussian quadrature rule [2].
From (47) one obtains $T\left(R_2\right)=41.76052°\mathrm{C}$, $q\left(R_1\right)=0.00041\ \mathrm{W/m^2}$ and then from the Robin condition $q\left(R_2\right)=0{,}00544\ \mathrm{W/m^2}$.

Introduction of the values $T\left(R_1\right)$, $T\left(R_2\right)$, $q\left(R_1\right)$, $q\left(R_2\right)$ to formula (38) leads to the determination of temperatures at the optional points from the interior of the considered cylindrical layer.
The results obtained by means of the BEM are compared with those obtained analytically (Fig. 1, Tab. 1).

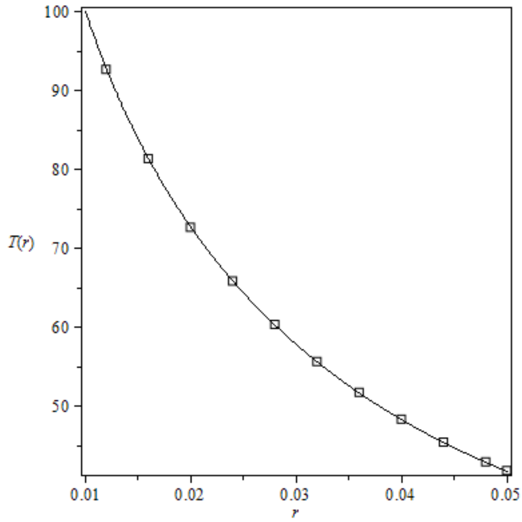

Fig. 1. Temperature distribution (solid line - analytical solution, symbols - BEM solution)

Table 1

Temperature values as the points selected from the domain considered

r	$T(r)$ - analytical solution	$T(r)$ - BEM solution
0.012	92.65582	92.65582
0.016	81.25915	81.25915
0.02	72.64995	72.64994
0.024	65.82624	65.82624
0.028	60.25028	60.25028
0.032	55.59894	55.59893
0.036	51.66234	51.66233
0.04	48.29622	48.29621
0.044	45.39701	45.397
0.048	42.88778	42.88778
0.05	41.76052	41.76052

Conclusions

The fundamental solution for diffusion equation in cylindrical co-ordinate system with temperature - dependent heat source is determined. The example of computations is presented and the results obtained by the boundary element method are compared with the analytical solution.

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