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STRUCTURE OF FINITELY GENERATED MODULES OVER RIGHT HEREDITARY SPSD-RINGS

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Abstract. In this article we consider the structure of finitely generated modules over right hereditary semiperfect semidistributive rings. We show in what way a description of finitely generated modules over these rings of bounded representation type can be reduced to a mixed matrix problem. Some special mixed matrix problems over a discrete valuation ring and its skew field of fractions are also considered.

Introduction

The structure of right hereditary semiperfect semidistributed rings (SPSD-rings, in short) was described in [1]. This paper is devoted to the study of finitely generated modules (f.g. modules, in short) over such rings. Section 1 gives the general structure of these modules. In Section 2 we show in what way we can obtain the full description of special classes of f.g. modules using a reduction to some mixed matrix problems over discrete valuation rings and their skew field of fractions. In Section 3 we consider some special mixed matrix problems which are important in the proof of theorem 6 [2]. These matrix problems were first considered in [3] in connection with the study of right Noetherian right hereditary semiperfect rings of bounded representation type. Later a flat matrix problem of mixed type which is a generalization of such mixed matrix problems was considered in [4]. These matrix problems are naturally arisen in the theory of integral representations and the theory of rings. Note that first the matrix problems over fields were considered by L.A. Nazarova and A.V. Roiter in [5].

All rings considered in this paper are assumed to be associative with $1 \neq 0$, and all modules are assumed to be unital. We write O for a discrete valuation ring, and D for its skew field of fractions. We refer to [6] for general material on theory of rings and modules. All necessary information about discrete valuation rings can be found, for example, in [6] and [7].

1. Finitely generated modules over a right hereditary SPSD-ring

The full structure of right hereditary SPSD-rings was obtained in [8] in terms of discrete valuation rings and special valuated posets.

Recall that a ring A is semiperfect if any of its finitely generated modules has a projective cover. A ring A is right (left) hereditary if each right ideal of A is projective. This is equivalent to the condition that any submodule of a projective right A-module is projective. A right and left hereditary ring is hereditary.

A module M is distributive if $K\cap(L+N) = K\cap L+K\cap N$ for all submodules K, L, N. A module is semidistributive if it is a direct sum of distributive modules. A ring A is right (left) semidistributive if the right (left) regular module A_A ($_AA$) is semidistributive. A right and left semidistributive ring is semidistributive.

Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical R. Denote

$$H_n(O) = \begin{pmatrix} O & O & \cdots & O \\ R & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & O \end{pmatrix},$$

which is a subring in the matrix ring $M_n(D)$.

Let $\{O_i\}$ be a family of discrete valuation rings (not necessarily commutative) with Jacobson radicals R_i and a common skew field of fractions D, for $i = 1, 2,\ldots,k$; $S = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ a finite poset with a partial order $\le$; $S_0 = \{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ a subset of minimal points of S ($k \le n$), and $S = S_0 \cup S_1$, where $S_1 = \{\alpha_{k+1}, \alpha_{k+2},\ldots,\alpha_n\}$. According with this partition of S consider the poset S with weights so that the point i has the weight $H_{n_i}(O_i)$, $i = 1,2,\ldots,k$; $n_i \in \mathbf{N}$; and all other points j have the weight D.

Assume that A is an indecomposable right hereditary SPSD-ring. Then by [1, Theorem 2.9] there is a decomposition of the identity of A into a sum of pairwise orthogonal idempotents

$$1 = f_1 + f_2 + \ldots + f_n$$

such that A is isomorphic to a ring $A = A(S, O)$ with the two-sided Pierce decomposition of the following form:

$$A = \bigoplus_{i,j=1}^{n} f_i A f_j$$

where $f_i A f_i = H_{n_i}(O_i)$ for $i = 1,2,\ldots,k$; $fAf = T(S_1)$ for $f = f_{k+1} + \ldots + f_n$; and $A_{ij} = f_iAf_j$ is an (A_{ii}, A_{jj}) - bimodule, for $i, j = 1,2,\ldots,n$. Moreover, $A_{ij} = 0$ if $\alpha_i \le \alpha_j$ in S. If $\alpha_i \le \alpha_j$ in S and $\alpha_i \in S_0$, $\alpha_j \in S_1$, then $eAf_j = D$ for any $e \in f_i$. Therefore

$$A = \begin{pmatrix} H_{n_1}(O_1) & \cdots & O & M_1 \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & H_{n_k}(O_k) & M_k \\ O & \cdots & O & T(\mathrm{S}_1) \end{pmatrix} \qquad (1)$$

where M_i is a $(H_{n_i}(O_i), T(\mathrm{S}_1))$ - bimodule for $i = 1, 2,\ldots,k$; $T(\mathrm{S}_1)$ is the incidence ring of a poset S_1 over a division ring D. Note, that these rings were first considered in [3].

Suppose M is a f.g. right A-module. Then

$$M = Mf_1 \oplus \ldots \oplus Mf_k \oplus Mf_{k+1} \oplus \ldots \oplus Mf_n,$$

(as Abelian groups) where Mf_i is a f.g. right $H_{n_i}(O_i)$-module ($i = 1,2,\ldots,k$), and Mf_j is a finite dimensional left vector space over D ($j = k+1,k+2,\ldots,n$). Denote $M = (Mf_1, Mf_2,\ldots,Mf_n)$. Since $H_{n_i}(O_i)$ is a Noetherian serial ring, any f.g. module over it is serial. Let $P_{i1},\ldots, P_{in_i}$ be principal right $H_{n_i}(O_i)$-modules. Then $H_{n_i}(O_i)$-module Mf_i is decomposed into a direct sum of modules $P_{i1},\ldots, P_{in_i}$ and their quotient modules.

Theorem 1. *If a finitely generated A-module M is indecomposable, then either $M = Mf_i$ for some $i = 1,2,\ldots,k$ or all $H_{n_i}(O_i)$-modules Mf_i ($i = 1,2,\ldots,k$) are direct sums of indecomposable projective $H_{n_i}(O_i)$-modules.*

Proof. Suppose $Mf_1 = X \oplus Y$, where X is a quotient module of P_{i1}, and Y is a direct sum of projective modules P_{i1} ($i = 1,2,\ldots,n_1$). Show that in this case $M = N_1 \oplus N_2$, where $N_1 = Xf_1$, and $N_2 = Yf_1 \oplus Mf_2 \oplus \ldots \oplus Mf_r$ is a direct sum of Abelian groups.

An $H_{n_i}(O_i)$-module X has the form:

$$P_{1i}/P_{1t} = (X_1,\ldots,X_t, 0,\ldots,0)$$

($0 \leq t \leq n_1$), where X_j is a torsion O_j-module ($j = 1,2,\ldots,t$). To prove that $XM_1 = 0$ it is sufficient to show that $X_iD = 0$ for each $i = 1,\ldots,t$.

Any element $y_i \in X_iD$ has the form $y_i = \sum_{j=1}^{s} x_{ij}q_j$, where $x_{ij} \in X_i$, $q_j \in D$ ($i = 1, \ldots,t$; $j = 1,\ldots,s$). Since X_i is a torsion O_i-module, for any $x_{ij} \in X$ there is $0 \neq \alpha_{ij} \in O_i$ such that $x_{ij}\alpha_{ij} = 0$ ($j = 1,\ldots,s$). Then

$$y_i = \sum_{j=1}^{s} x_{ij}q_j = \sum_{j=1}^{s} x_{ij}\alpha_{ij}\alpha_{ij}^{-1}q_j = \sum_{j=1}^{s} (x_{ij}\alpha_{ij})(\alpha_{ij}^{-1}q_j) = 0.$$

So $X_iD = 0$ ($i = 1,\dots,t$), and therefore $XM_1 = 0$. Show that N_1 is a right A-submodule in M. Really,

$$N_1A = (X,0,\dots,0)\begin{pmatrix} H_{n_1}(O_1) & \cdots & O & M_1 \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & H_{n_k}(O_k) & M_k \\ O & \cdots & O & T(S_1) \end{pmatrix} =$$

$$= (XH_{n_1}(O_1), 0,\dots,0, XM_1) \subseteq (X, 0, \dots, 0) = N_1,$$

since X is an $H_{n_1}(O_1)$-module, and $XM_1 = 0$.

Since N_2 is also an A-submodule of M and $N_1 \cap N_2 = 0$, $M = N_1 \oplus N_2$.

Assume that M is an indecomposable A-module, then either $M = N_1$ or $M = N_2$. In the second case Mf_1 is a direct sum of indecomposable projective $H_{n_1}(O_1)$-modules.

As the saying above is true for any $i = 1,\dots,k$, the theorem is proved.

2. Reduction to a mixed matrix problem

Further we reduce the description of f.g. A(S, O)-modules to some flat matrix problem of mixed type (over discrete valuation rings and their skew field of fractions).

Let $A = A(\mathrm{S}, O)$ and M a f.g. A-module. Since Mf_j ($j = k+1,\dots,n$) is a finite dimensional vector space over a skew field D, $Mf_j \cong D^{l_j}$. Then

$$M = N_1 \oplus N_2 \oplus \dots \oplus N_k \oplus L, \tag{2}$$

where $N_i = (0,\dots, 0, X_i, 0,\dots,0)$ (X_i is at the i-th place, $i = 1,\dots,k$), and X_i is a direct sum of quotient modules of indecomposable projective $H_{n_i}(O_i)$-modules ($i = 1,\dots,k$);

$$L = (L_1,\dots,L_k, D^{l_1},\dots, D^{l_m}), \tag{3}$$

where $m = n - k$ and L_i is a direct sum of indecomposable projective $H_{n_i}(O_i)$-modules ($i = 1,\dots,k$).

Since any module N_i is a direct sum of cyclic modules, the boundedness of ring A depends only on the modules of the form L. Therefore the description of rings $A = A(\mathrm{S}, O)$ of bounded representation type is reduced to the description of f.g. modules of the form L. Therefore in the following we will consider only f.g. modules L.

Let M be a f.g. A-module and

$$M = Mf_1 \oplus \dots \oplus Mf_k \oplus D^{l_1} \oplus \dots \oplus D^{l_m}, \tag{4}$$

where Mf_i is a direct sum of indecomposable projective right $H_{n_i}(O_i)$-modules ($i = 1,\dots,k$). If $Mf_i = P_{i1}^{t_{i1}} \oplus \dots \oplus P_{in_i}^{t_{in_i}}$, ($i = 1,\dots,k$), then M is uniquely defined, as an Abelian group, by means of the set $\{t_{i1},\dots,t_{in_i}$ $(i = 1,\dots,k);\ l_1,\dots,l_m\}$.

Let e_{ij} be matrix units of the ring $M_r(D)$, where $r = n_1 + \dots + n_k + m$. Denote

$$e'_{ij} = \begin{cases} e_{ij} & \text{if } e_{ij} \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i, j = 1,2,\dots,r \ \text{ and}$$

$$f_{ii} = f_i, \quad (i = 1,\dots,k)$$

$$f_{i,k+j} = \sum_{s=1}^{n_i} e'_{n_1+\dots+n_{i-1}+s,\,r-m+j}, \ (i = 1,\dots,k;\ j = 1,\dots,m;\ n_0 = 0)$$

$$f_{k+i,k+j} = e'_{r-m+i,\,r-m+j} \ (i,j = 1,\dots,m)$$

To define M as a right A-module it is sufficient to give the actions of elements f_{ij} on M (where i,j are such indexes that $\alpha_i \le \alpha_j$ in S and there are no $\alpha_k \in$ S such that $\alpha_k \ne \alpha_i$, α_j and $\alpha_i \le \alpha_k \le \alpha_j$ in S).

Let $f_{ij} \ne 0$, then multiplication of elements of M on the right side by f_{ij} gives a homomorphism of Abelian groups:

$$\overline{f_{ij}} : Mf_i \to Mf_j \ (i,j = 1,\dots,n). \tag{5}$$

Fixing bases in Mf_i and Mf_j, we can give an operator of multiplication of M on the right side by f_{ij} by means of a $k_i \times k_j$ matrix $\mathbf{T}_{ij}$ with entries from f_iAf_j, where

$$k_i = \sum_{j=1}^{n_i} t_{ij} \text{ for } i = 1,\dots,k \ \text{ and } k_i = l_i \text{ for } i = k+1,\dots,n.$$

It is convenient to write the action of elements f_{ij} on M in the form of the following rectangular matrix $\mathbf{T}$:

$\mathbf{T}_{11}$	$\cdots$	$\mathbf{T}_{1j}$	$\cdots$	$\mathbf{T}_{1n}$
$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{T}_{i1}$	$\cdots$	$\mathbf{T}_{ij}$	$\cdots$	$\mathbf{T}_{in}$
$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{T}_{n1}$	$\cdots$	$\mathbf{T}_{nl}$	$\cdots$	$\mathbf{T}_{nn}$

partitioned into n horizontal strips and n vertical strips so that a block $\mathbf{T}_{ij}$ is the intersection of the j-th vertical strip and the i-th horizontal strip.

As A is a generalized rectangular matrix ring, $\mathbf{T}_{ij} = 0$ if $i > j$ or α_i and α_j is not in relation in S. So $\mathbf{T}$ is a block upper triangular matrix.

Since an operator $\overline{f_{ii}}$ is identical (i = 1,…,n), its matrix is an identical matrix. Therefore all diagonal blocks of **T** are identical matrices.

Changing bases in Mf_i and changing the numbering of Abelian groups Mf_i (i = 1,…,n), there is a possibility to perform the following admissible transformations:

1) simultaneous interchanging of the same vertical and horizontal strips;
2) the transformation of similarity $\mathbf{T} \to \mathbf{UTU}^{-1}$ by means of an unsingular matrix **U** of the form:

$$\begin{array}{|c|c|c|c|}\hline \mathbf{U}_{11} & \mathbf{O} & \dots & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{U}_{22} & \dots & \mathbf{O} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{O} & \mathbf{O} & \dots & \mathbf{U}_{nn} \\ \hline \end{array}$$

where $\mathbf{U}_{ii}$ are invertible matrices with entries from f_iAf_i (i = 1,2,…,n).

Thus, any f.g. right A-module M is uniquely (up to isomorphism) given by the following set:

$$\{ t_{i1},\dots,t_{in_i} \ (i = 1,\dots,k);\ l_1,\dots,l_m;\ \mathbf{T}\}.$$

Let M' be an A-module which is isomorphic to M and given by the following set:

$$\{ t'_{i1},\dots,t'_{in_i} \ (i = 1,\dots,k);\ l'_1,\dots,l'_m;\ \mathbf{T}'\}.$$

Then it is not difficult to show that M and M' are isomorphic if and only if there exist permutations p,q such that

$$t_{i1} = t'_{p(i)1},\dots,t_{in_i} = t'_{p(i)n_{p(i)}} \ (i = 1,\dots,k)$$

$$l_j = l'_{q(j)} \ (j = 1,\dots,m)$$

and there is a matrix **U** such that after simultaneous interchanging of the same vertical and horizontal strips of the matrix **T′** in accordance with permutations p,q we have $\mathbf{T}' = \mathbf{UTU}^{-1}$.

A matrix **T** is said to be indecomposable, if it is impossible to reduce **T** to the form:

$$\begin{array}{|c|c|}\hline \mathbf{T}_1 & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{T}_2 \\ \hline \end{array}$$

using simultaneous interchanging of the same vertical and horizontal strips and transformations 1) and 2).

Clearly, a matrix **T** is indecomposable if and only if the A-module M is indecomposable.

So, the problem of description of all (up to isomorphism) indecomposable f.g. right A-modules of the form L may be considered as the following:

Mixed matrix problem:

Given an upper triangular block matrix **T** *and admissible transformations* 1) *and* 2) *mentioned above. Find all indecomposable matrices* **T** *up to equivalence.*

By the dimension of a stripped matrix **T** we shall mean the vector

$$\mathbf{d} = \mathrm{d}(\mathbf{T}) = (d_1,d_2,\ldots,d_n;\ d^1,\ d^2,\ldots,d^n), \tag{6}$$

where d_i is the number of rows of the i-th horizontal strip of **T** and d^i is the number of columns of the i-th vertical strip of **T** for $i = 1,\ldots,n$. We set

$$\dim(\mathbf{T}) = \sum_{j=1}^{n} d_j + \sum_{i=1}^{n} d^i \tag{7}$$

Recall that a ring A has a bounded representation type if there is an upper bound on the number of generators required for indecomposable finitely presented A-modules. Obviously, a ring A is of bounded representation type if and only if there is a constant C such that $\dim(\mathbf{T}) < C$ for all indecomposable matrices **T**. Otherwise it is of unbounded representation type.

According to [4] we shall say that a mixed matrix problem has a bounded representation type, if there is a constant C such that $\dim(\mathbf{T}) < C$ for all indecomposable matrices **T**. (Note, that in [4] such matrix problem is called of bounded-module type. But since this concept was first introduced for rings by R.R. Warfield [9] where the corresponding rings were called of bounded representation type, we will use this name.)

Thus, a right hereditary SPSD-ring A has a bounded representation type if and only if the corresponding mixed matrix problem has a bounded representation type.

Lemma 2. *Let O be a discrete valuation ring with a skew field of fractions D. Then D is an injective torsion-free right and left O-module.*

Proof. Since D is a skew field of fractions of O and O is a right and left PID, $Da = D$ and $aD = D$ for any non-zero element $a \in O$. So D is a divisible torsion-free right and left O-module. Therefore, by the Baer criterion, it is sufficient to prove that for any non-zero right ideal I in O and a homomorphism f: $I \to D$ there exists an element $d' \in D$ such that $f(x) = d'x$, $x \in I$. Any right ideal in O has the form $I = aO$ for some non-zero element $a \in O$. Suppose $f(a) = d$. Since D is a divisible O-module, there exists an element $d' \in D$ such that $d = d'a$. An arbitrary element of the ideal I has the form ab, where $b \in O$. Therefore

$$f(ab) = f(a)b = db = d'ab = d'(ab),$$

as required. Therefore D is an injective right O-module. Analogously D is an injective left O-module.

3. Some mixed matrix problems

Let O be a discrete valuation ring with a skew field of fractions D, R = rad $O = \pi O = O\pi$. By left O-elementary transformations of rows of a matrix $\mathbf{T}$ over D we mean the transformations of the following two types:

(a) multiplications of a row on the left by an invertible element of the ring O;

(b) addition of a row multiplied on the left by an arbitrary element of O to another row.

In a similar way one can define left D-elementary transformation of rows and, by symmetry, right O-elementary and right D-elementary transformations of columns.

In many cases, in particular to prove Theorem 6 [2], the following two mixed matrix problems are important.

Matrix problem I.

Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical R. Consider a rectangular matrix $\mathbf{T}$ with entries from D partitioned into n vertical strips $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$:

$\mathbf{A}_1$	$\mathbf{A}_2$	…	$\mathbf{A}_n$

Let $\varepsilon_1, \ldots, \varepsilon_{n-1} \in R$. Over the matrix $\mathbf{T}$ one performs the following admissible transformations:

1) *left elementary D-transformations of rows of $\mathbf{T}$;*
2) *right O-elementary transformations of columns inside any block $\mathbf{A}_i$ ($i = 1,\ldots,n$);*
3) *addition of any column of the block $\mathbf{A}_i$ multiplied on the right by an arbitrary element of O to any column of the block $\mathbf{A}_j$ if $i < j$ for $i, j = 1,\ldots,n$;*
4) *addition of any column of the block $\mathbf{A}_{i+1}$ multiplied on the right by an arbitrary element of ε_i to any column of the block $\mathbf{A}_i$ for $i = 1,\ldots,n-1$.*

Find all indecomposable matrices $\mathbf{T}$ up to equivalence.

Two matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ with k rows partitioned into n strips with $k_1, k_2, \ldots, k_n$ columns satisfying the conditions of the matrix problem I are equivalent if there exist invertible matrices $\mathbf{X} \in \mathrm{M}_k^*(D)$ and $\mathbf{Y} = (\mathbf{Y}_{ij}) \in \mathrm{M}_r^*(O)$ ($r = k_1 + \ldots + k_n$) with $\mathbf{Y}_{ii} \in \mathrm{M}_{k_i}^*(O)$, $\mathbf{Y}_{ij} \in \mathrm{M}_{k_i \times k_j}(O)$ if $i < j$, $\mathbf{Y}_{ij} \in \mathrm{M}_{k_i \times k_j}(\varepsilon_i \varepsilon_{i-1} \ldots \varepsilon_1 O)$ if $i > j$ such that

$$\mathbf{T}_2 = \mathbf{X}\,\mathbf{T}_1 \mathbf{Y}.$$

Note that indecomposable matrices in this case are defined in a natural way.

Lemma 3. *Let O be a discrete valuation ring with a skew field of fractions D, $R = rad\, O = \pi O = O\pi$. Suppose $\varepsilon_i = \pi^2$ for all i = 1,2,…,n–1. Then there exists an indecomposable matrix* $\mathbf{T}$ *satisfying the conditions of the matrix problem I such that* $\dim(\mathbf{T}) > n$.

Proof. Consider the following matrix $\mathbf{T}$ which is partitioned into n vertical strips:

π^{n-1}	π^{n-2}	…	π^2	π	1

In this case $k = k_1 = \ldots = k_n = 1$, so $r = n$.

Show that the matrix $\mathbf{T}$ is indecomposable. Otherwise there exist invertible matrices $\mathbf{X} = (x)$, where $x \in D$, $x \neq 0$ and $\mathbf{Y} \in M_n^*(O)$ of the following form:

$$\mathbf{Y} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ \pi^2 a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \pi^4 a_{31} & \pi^2 a_{32} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi^{2^{n-1}} a_{n1} & \pi^{2^{n-2}} a_{n2} & \cdots & \pi^2 a_{n,n-1} & a_{nn} \end{pmatrix},$$

where $a_{ij} \in O$ for $i, j = 1,\ldots,n$; a_{ii} are an invertible elements in O for $i = 1,\ldots,n$, such that the equivalent matrix $\mathbf{XTY}$ is decomposable. Then there is a number i such that

$$x(\pi^{n-1}a_{1i} + \ldots + \pi^{n-i}a_{ii} + \pi^{n-i-1}\pi^2 a_{i+1,i} + \pi^{n-i-2}\pi^4 a_{i+2,i} + \ldots + \pi\,\pi^{2^{n-1-i}} a_{n-i,i} + \pi^{2^{n-i}} a_{ni}) = 0.$$

Since $x \neq 0$, we obtain that $a_{ii} \in \pi O = R$, i.e., a_{ii} is not invertible element of O. This contradiction proves the lemma.

Matrix problem II.

Let O be a discrete valuation ring with a skew field of fractions D. Consider a rectangular matrix $\mathbf{T}$ *with entries from D partitioned into 3 vertical strips* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$:

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$

Over the matrix **T** *one performs the following admissible transformations:*
1) *left elementary O-transformations of rows of* **T**;
2) *right D-elementary transformations of columns inside any block* $\mathbf{A}_i$ (i = 1,2,3);
3) *addition of any column of the block* $\mathbf{A}_1$ *multiplied on the right by an arbitrary element of D to any column of the block* $\mathbf{A}_2$.

Find all indecomposable matrices **T** *up to equivalence.*

For this matrix problem two matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ with k rows partitioned into 3 strips with k_1, k_2, k_3 columns are equivalent if there exist invertible matrices $\mathbf{X} \in M_k^*(O)$ and $\mathbf{Y} = (\mathbf{Y}_{ij}) \in M_r^*(D)$ $(i,j = 1,2,3;\ r = k_1 + k_2 + k_3)$ with $\mathbf{Y}_{ii} \in M_{k_i}^*(O)$, $\mathbf{Y}_{12} \in M_{k_1 \times k_2}(D)$ and zero for all other elements, such that

$$\mathbf{T}_2 = \mathbf{X}\,\mathbf{T}_1\mathbf{Y}.$$

Note, that in this case indecomposable matrices are also defined in a natural way.

Lemma 4. *Let O be a discrete valuation ring with a skew field of fractions D,* $R = \mathrm{rad}\, O = \pi O = O\pi$. *Then the matrix problem* II *has an unbounded representation type, i.e. for any* $n \in \mathbf{N}$ *there exists an indecomposable matrix* **T** *such that* $\dim(\mathbf{T}) > n$.

Proof. Consider the triangular matrix **T** with entries from D partitioned into 3 vertical strips $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ of the form:

$$\mathbf{A}_1 = \begin{pmatrix} \underline{1} \\ 0 \\ 0 \\ \vdots \\ \underline{0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} \underline{0} & \underline{0} & \underline{\cdots} & \underline{0} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \underline{\cdots} & \underline{1} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \mathbf{A}_3 = \begin{pmatrix} \underline{a_1} & \underline{a_2} & \underline{\cdots} & \underline{a_n} \\ \pi^{-2} & 0 & \cdots & 0 \\ 0 & \pi^{-4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \underline{\cdots} & \underline{\pi^{-2n}} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $a_1, a_2, \ldots, a_n \in \pi^{-3n}O$.

Show that this matrix **T** is indecomposable. Otherwise the first row of the third strip $\mathbf{A}_3$ must be decomposable. But this row forms a matrix $\mathbf{T}_1 = [a_1|a_2|\ldots|a_n]$ satisfying the conditions of the matrix problem I. If we set

$$a_1 = \pi^{-2n-1},\ a_2 = \pi^{-2n-2},\ldots,a_n = \pi^{-3n},$$

we obtain that the stripped triangular matrix $\mathbf{T}_1$:

π^{-2n-1}	π^{-2n-2}	$\ldots$	π^{-3n}

is indecomposable, by lemma 3.

So the matrix $\mathbf{T}$ is indecomposable with $\dim(\mathbf{T}) = 4n+2 > n$, which proves that the matrix problem II has an unbounded representation type.

Conclusions

This paper gives the structure of finitely generated indecomposable modules over right hereditary SPSD-rings. Their structure may be described using a reduction to mixed matrix problems over discrete valuation rings and their skew fields of fractions. In this paper we give the solution of some basic mixed matrix problems.

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