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ON RIGHT HEREDITARY SPSD-RINGS OF BOUNDED REPRESENTATION TYPE I

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Abstract. The structure of right hereditary semiperfect semidistributive rings of bounded representation type is described in terms of Dynkin diagrams and diagrams with weights. We describe it using a reduction to mixed matrix problems.

Introduction

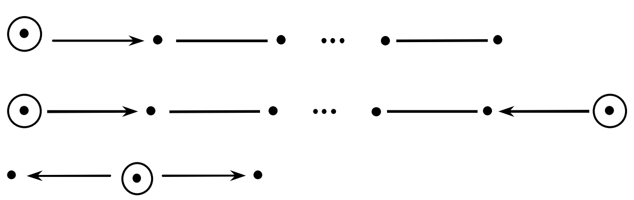
This paper is devoted to the study of boundedness of right hereditary semiperfect semidistributive rings (SPSD-rings, in short) considered in [1] and is a continuation of it. These rings were first described in [2].

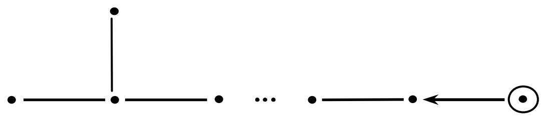
We use notation and definitions of articles [1, 3, 4] and books [5, 6].

Recall that ring A has a **bounded representation type** if there is an upper bound on the number of generators required for indecomposable finitely presented A-modules. Otherwise it is of the unbounded representation type.

In this paper we prove the necessity of the following main theorem which gives the structure of right hereditary SPSD-rings of bounded representation type in terms of Dynkin diagrams and diagrams with weights:

Theorem 1. *Let* $\{O_i\}$ *be a family of discrete valuation rings with a common skew field of fractions* D, *and let* $S = S_0 \cup S_1$ *be a disjoint union of subposets. A right hereditary SPSD-ring A is of bounded representation type if and only if* $A = A(S,O)$ *and the undirected graph* $\overline{\square(S)}$ *of the Hasse diagram* $\square(S)$ *of the poset* S *is a finite disjoint union of Dynkin diagrams of the type* A_n, D_n, E_6, E_7, E_8 *and the following diagrams with weights*:





where all vertices with weight 1 correspond to the minimal elements of the poset S.

Note that this theorem was first formulated in [7], where it is given without proof, and it can be considered as a simple generalization of [8, Theorem I]. In this paper we give two different proofs of the necessity of this theorem using the results of [3, 8, 9].

All rings considered in this paper are assumed to be associative (but not necessary commutative) with $1 \neq 0$, and all modules are assumed to be unital.

1. Preliminaries

According Gabriel [10] and Dlab and Ringel [11], a hereditary finite dimensional algebra is of finite representation type if and only if the corresponding diagram is a Dynkin diagram of the type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2. From this fundamental result [1, Theorem 3] and [1, Proposition 4] we immediately obtain the following statement:

Proposition 2. *If diagram* $\Gamma(S)$ *of a poset* S *is not a disjoint union of the Dynkin diagrams of the the type* A_n, D_n, E_6, E_7, E_8, *then the right hereditary SPSD-ring* $A = A(S,O)$ *is a ring of the unbounded representation type.*

Proposition 3. *If ring A is of the bounded representation type then each of its minors is of bounded representation type too.*

Proof. Let P be a finitely generated projective A-module, $B = \mathrm{End}_A(P)$, and let M be a finitely presented B-module. Then there are the exact sequences:

$$0 \to X \to B^n \to M \to 0 \tag{1}$$

$$0 \to Y \to B^m \to X \to 0 \tag{2}$$

Denote by $C(P)$ the full subcategory of the category of all A-modules consisting of A-modules M such that there exists an exact sequence

$$P^J \to P^I \to M \to 0 \tag{3}$$

where P^I denotes a direct sum of modules isomorphic to P. By the Morita theorem [5, Theorem 10.7.2], it follows that the categories B-mod and $C(P)$ are equivalent, hence there is an A-module $M' \in C(P)$ such that $M = F(M') = \mathrm{Hom}_A(P, M')$. Therefore there is a sequence

$$0 \to X \to B^n \to F(M') \to 0 \qquad (4)$$

Applying the exact functor $G = * \otimes_B P$ to the exact sequences (2) and (4), we get

$$0 \to G(X) \to P^n \to M' \to 0$$
$$0 \to G(Y) \to P^m \to G(X) \to 0 \qquad (5)$$

Hence $\mu_A(M') = \mu_A(P^n) - \mu_A(G(X)) = ns - \mu_A(G(X))$, where $s = \mu_A(P)$, and $\mu_A(U)$ is the minimum number of generators of an A-module U.

Since A is a ring of the bounded representation type, there exists a number N such that $\mu_A(U) \leq N$ for any A-module U. Therefore $\mu_A(M') \leq N$ and $\mu_A(G(X)) \leq N$, that is, $ns = \mu_A(M') + \mu_A(G(X)) \leq 2N$, i.e. $n \leq 2N/s$. Writing $2N/s = N_1$ we obtain that $\mu_B(M) \leq n \leq N_1$ and this is true for any finitely presented B-module M. Therefore B is a ring of bounded representation type.

2. Mixed matrix problems and posets

Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$.

By left O-elementary transformations of rows of a matrix $\mathbf{X}$ we mean the transformations of the following three types:

(a) interchanging of two rows;
(b) multiplications of a row on the left by an invertible element of O;
(c) addition of a row multiplied on the left by an arbitrary element of O to another row.

In a similar way we define left D-elementary transformations of rows and, by symmetry, right O-elementary and right D-elementary transformations of columns.

Let $\mathbf{T} = (\mathbf{T}_{ij})$ be a rectangular matrix with entries in D partitioned into n horizontal strips $\{\mathbf{T}_i\}_{i=1,\dots,n}$ and m vertical strips $\{\mathbf{T}^j\}_{j=1,\dots,m}$ so that a block $\mathbf{T}_{ij}$ is the intersection of the i-th horizontal strip $\mathbf{T}_i$ and the j-th vertical strip $\mathbf{T}^j$.

Let $\mathrm{M}_n(D)$ be the ring of $n \times n$ matrices over D with matrix units e_{ij}. Following [8] we consider a **transforming algebra** $X = \bigoplus_{i,j=1}^{n} e_{ij} X_{ij} \subseteq \mathrm{M}_n(D)$ such that:

(a) $X_{ii} = O$ or $X_{ii} = D$;
(b) $X_{ij} X_{jk} \subseteq X_{ik}$
(c) $X_{ij} X_{ji} \neq X_{ii}$ for $i \neq j$.
for each $i,j,k = 1,2,\dots,n$.

Obviously, $X_{ij} = D$ or $X_{ij} = \pi^{\alpha_{ij}} O$, where $\alpha_{ij} \in \mathbf{Z}$. We set $D = \pi^{-\infty} O$ and $0 = \pi^{+\infty} O$.

A family of elementary transformations over the row strips of a rectangular matrix $\mathbf{T} = (\mathbf{T}_{ij})$ of the following form:

(i) left X_{ii}-elementary transformations of rows in the strip $\mathbf{T}_i$;

(ii) addition of rows in a strip $\mathbf{T}_i$ multiplied on the left by elements of X_{ij} to rows of a strip $\mathbf{T}_j$

will be called **admissible transformations** with respect to an algebra X.

In a similar way one can define admissible transformations over the column strips of a matrix $\mathbf{T}$ with respect to an algebra $Y = \bigoplus_{i,j=1}^{m} e_{ij}Y_{ij} \subseteq \mathrm{M}_m(D)$.

The **dimension** of a stripped matrix $\mathbf{T}$ is the vector

$$\mathbf{d} = \mathrm{d}(\mathbf{T}) = (d_1, d_2, \ldots, d_n; d^1, d^2, \ldots, d^m), \tag{6}$$

where d_i is the number of rows of the i-th horizontal strip $\mathbf{T}_i$ and d^j is the number of columns of the j-th vertical strip $\mathbf{T}^j$ for $j = 1, \ldots, m$. We set

$$\dim(\mathbf{T}) = \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} d^j \tag{7}$$

According to [9], a mixed matrix problem has a **bounded representation type**, if there is a constant C such that $\dim(\mathbf{T}) < C$ for all indecomposable matrices $\mathbf{T}$.

Flat mixed matrix problem:

Given a triangular matrix $\mathbf{T} = (\mathbf{T}_{ij})$ *with entries in D partitioned into n horizontal strips* $\{\mathbf{T}_i\}_{i=1,\ldots,n}$ *and m vertical strips* $\{\mathbf{T}^j\}_{j=1,\ldots,m}$, *two transforming algebras* $X \subseteq \mathrm{M}_n(D)$ *and* $Y \subseteq \mathrm{M}_m(D)$. *One performs admissible transformations over row strips with respect to X and admissible transformations over column strips with respect to Y. Define the boundedness type of this matrix problem.*

This matrix problem was solved in [9] in terms of critical pairs of sets in the sense of Kleiner [12].

Recall that a totally ordered set consisted of n elements is called a **chain** and denoted by (n). A cardinal sum of k chains consisting of $n_1, n_2, \ldots, n_k$ elements is denoted by $(n_1, n_2, \ldots, n_k)$. A cardinal sum of posets P and Q is denoted by $P \leq Q$. Denote by N the poset $\{a < b > c < d\}$.

Associate with a transforming algebra X a poset $P(X) = \sum_{i=1}^{n} P_i$, which is a cardinal sum of posets P_i, where P_i is a chain of the following type:

(a) $P_i = \{ p_i^0 \}$ is a one-point chain if $X_{ii} = D$;

(b) $P_i = \{ p_i^k \}_{k \in \mathbf{Z}}$ is an infinite chain if $X_{ii} = O$.

The order relation in $P(X)$ is defined as follows:

$$p_i^k \leq p_i^l \iff k - l \geq \alpha_{ij} \text{ if } X_{ij} = \pi^{\alpha_{ij}} O. \tag{8}$$

Definition 1.
A pair (P,Q) of posets is called a **critical pair of sets** (in the sense of Kleiner) if one of the following conditions is satisfied up to the transposition of P and Q:

- $P = (1)$; $Q = (1,1,1,1) \vee (2,2,2) \vee (1,3,3) \vee (1,2,5) \vee N \leq 4$;
- $P = (2)$; $Q = (1,1,1) \vee (3,3) \vee (2,5)$;
- $P = (3)$; $Q = (2,2) \vee (1,5)$;
- $P = (4)$; $Q = (1,3)$; (9)
- $P = (5)$; $Q = N$;
- $P = (6)$; $Q = (1,2)$;
- $P = (1,1)$; $Q = (1,1)$.

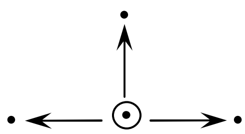
Theorem 4 [9]. *A flat matrix problem defined by a pair of transforming algebras* (X,Y) *of the above type is of bounded representation type if and only if the pair of partially ordered sets* (P(X), P(Y)) *contains no critical pairs of sets in the sense of Kleiner.*

3. Proof of the necessity in Theorem 1

Lemma 5. *Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring*

$$A = \begin{pmatrix} O & D & D & D \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{10}$$

corresponding to the diagram



is a ring of unbounded representation type.

Proof. Let M be a finitely generated right A-module that is given by the set $\{t; l_1, l_2, l_3; \mathbf{T}\}$, in which the matrix $\mathbf{T}$ has the following form:

E	$\mathbf{T}_{12}$	$\mathbf{T}_{13}$	$\mathbf{T}_{14}$
O	**E**	**O**	**O**
O	**O**	**E**	**O**
O	**O**	**O**	**E**

where $\mathbf{T}_{1i} \in \mathrm{M}_{t\times l_i}(D)$ (i = 2,3,4) are matrices over D. The matrix of transformations $\mathbf{U}$ has the following form:

$\mathbf{U}_{11}$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{O}$
$\mathbf{O}$	$\mathbf{U}_{22}$	$\mathbf{O}$	$\mathbf{O}$
$\mathbf{O}$	$\mathbf{O}$	$\mathbf{U}_{33}$	$\mathbf{O}$
$\mathbf{O}$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{U}_{44}$

where $\mathbf{U}_{11}$ is an invertible matrix with entries in O, and $\mathbf{U}_{ii}$ (i = 2,3,4) are invertible matrices with entries in D. Reducing the matrix $\mathbf{T}$ by the matrix $\mathbf{U}$ leads to the following matrix problem, given by a matrix $\mathbf{T}_1$

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_1$

and the following admissible transformations:
(a) left O-elementary transformations of rows of the matrix $\mathbf{T}_1$;
(b) right D-elementary transformations of columns inside any vertical strip $\mathbf{A}_i$ (i = 1,2,3).

Set

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \underline{\cdots} & \underline{0} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \pi^{-2} & 0 & \cdots & 0 \\ 0 & \pi^{-4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \underline{\cdots} & \underline{\pi^{-2n}} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} \pi^{n-1} \\ \pi^{n-2} \\ \vdots \\ \underline{1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (11)$$

where $\pi \in R = \mathrm{rad}\, O$, $\pi \neq 0$. By [4, Lemma 3], the matrix $\mathbf{T}_1$ is indecomposable and therefore ring A is of unbounded representation type.

Remark 1.
A mixed matrix problem over a matrix T_1 is defined by two transforming algebras:

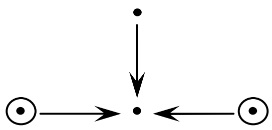
$$X = O, \quad Y = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \qquad (12)$$

Correspondingly, $P(X)$ is an infinite chain, and $P(Y) = \{1,1,1\}$. Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of sets $\{(2), (1,1,1)\}$. By theorem 4 this matrix problem is of unbounded representation type.

Lemma 6. *Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring*

$$\mathbf{A} = \begin{pmatrix} O & 0 & 0 & D \\ 0 & O & 0 & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix}, \qquad (13)$$

corresponding to the diagram



is a ring of the unbounded representation type.

Proof. Let M be a finitely generated right A-module which is given by the set $\{t_1, t_2; l_1, l_2; \mathbf{T}\}$, in which the matrix $\mathbf{T}$ has the following form:

E	**O**	**O**	$\mathbf{T}_{14}$
O	**E**	**O**	$\mathbf{T}_{24}$
O	**O**	**E**	$\mathbf{T}_{34}$
O	**O**	**O**	**E**

where $\mathbf{T}_{i4} \in M_{t_i \times l_2}(D)$ $(i = 1,2)$ and $\mathbf{T}_{34} \in M_{l_1 \times l_2}(D)$ are matrices over D. The matrix of transformations $\mathbf{U}$ has the following form:

$\mathbf{U}_{11}$	**O**	**O**	**O**
O	$\mathbf{U}_{22}$	**O**	**O**
O	**O**	$\mathbf{U}_{33}$	**O**
O	**O**	**O**	$\mathbf{U}_{44}$

where $\mathbf{U}_{ii}$ is an invertible matrix with entries in O $(i = 1,2)$ and $\mathbf{U}_{ii}$ $(i = 3,4)$ are invertible matrices with entries in D. Reducing the matrix $\mathbf{T}$ by the matrix $\mathbf{U}$ is equivalent to the matrix problem given by a matrix $\mathbf{T}_1$

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$

and the following admissible transformations:
(a) left D-elementary transformations of rows of the matrix $\mathbf{T}_1$;
(b) right O-elementary transformations of columns inside any block $\mathbf{A}_i$ (i = 1,2);
(c) right D-elementary transformations of columns inside the block $\mathbf{A}_3$.

Set

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \pi^2 & 0 & \cdots & 0 \\ 0 & \pi^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{2n} \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 1 \\ \pi \\ \vdots \\ \pi^{n-1} \end{pmatrix}. \tag{14}$$

By [4, Lemma 3], the matrix $\mathbf{T}_1$ is indecomposable. So the corresponding module M is indecomposable and the ring A is of the unbounded representation type.

Remark 2.
A mixed matrix problem over a matrix T_1 is defined by transforming algebras:

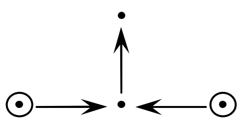
$$X = D, \qquad Y = \begin{pmatrix} O & 0 & 0 \\ 0 & O & 0 \\ 0 & 0 & D \end{pmatrix} \tag{15}$$

Correspondingly, we have two posets: $P(X)$ = (1) is a one-point chain (1), and $P(Y)$ is a cardinal sum of two infinite chains and a one-point chain (1). Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of sets {(1), (3,3,1}}. By theorem 4 this matrix problem is of the unbounded representation type.

Lemma 7. *Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring*

$$\mathbf{A} = \begin{pmatrix} O & 0 & D & D \\ 0 & O & D & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{16}$$

corresponding to the diagram



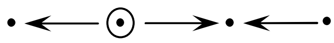
is a ring of the unbounded representation type.

The proof of this lemma is the same as for Lemma 6.

Lemma 8. *Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring*

$$\mathbf{A} = \begin{pmatrix} O & 0 & D & D \\ 0 & D & 0 & D \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{17}$$

corresponding to the diagram



is a ring of the unbounded representation type.

Proof. Let M be a finitely generated right A-module that is given by the set $\{t; l_1, l_2, l_3; \mathbf{T}\}$, in which the matrix $\mathbf{T}$ has the following form:

E	**O**	$\mathbf{T}_{13}$	$\mathbf{T}_{14}$
O	**E**	**O**	$\mathbf{T}_{24}$
O	**O**	**E**	**O**
O	**O**	**O**	**E**

where $\mathbf{T}_{13} \in \mathrm{M}_{t\times l_1}(D)$, $\mathbf{T}_{14} \in \mathrm{M}_{t\times l_2}(D)$, and $\mathbf{T}_{24} \in \mathrm{M}_{l_3\times l_2}(D)$. The matrix of transformations $\mathbf{U}$ has the following form:

$\mathbf{U}_{11}$	**O**	**O**	**O**
O	$\mathbf{U}_{22}$	**O**	**O**
O	**O**	$\mathbf{U}_{33}$	**O**
O	**O**	**O**	$\mathbf{U}_{44}$

where $\mathbf{U}_{11}$ is an invertible matrix with entries from O, and $\mathbf{U}_{ii}$ $(i = 2,3,4)$ are invertible matrices with entries from D. Reducing the matrix $\mathbf{T}$ by the matrix $\mathbf{U}$ is equivalent to the matrix problem given by a matrix $\mathbf{T}_1$

$\mathbf{A}_1$	$\mathbf{A}_2$
O	$\mathbf{A}_3$

and the following admissible transformations:

(a) left O-elementary (D-elementary) transformations of rows inside the first (second) horizontal strip of the matrix $\mathbf{T}_1$;

(b) right D-elementary transformations of columns inside each vertical strip of the matrix $\mathbf{T}_1$.

Reducing the matrix $\mathbf{A}_3$ to the form

E	**O**
O	**O**

we get that the matrix $\mathbf{A}_2$ has the form:

$\mathbf{B}_1$	$\mathbf{B}_2$

We can add any column of $\mathbf{B}_2$ multiplied on the right by elements of D to any column of $\mathbf{B}_1$. Thus the matrices $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{B}_2$ form the matrix problem considered in [4, Problem II]. By [4, Lemma 4] the ring A is of the unbounded representation type.

Remark 3.

A mixed matrix problem which forms matrices $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{B}_2$ is defined by transforming algebras:

$$X = O, \qquad Y = \begin{pmatrix} D & D & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \tag{18}$$

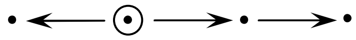
Correspondingly, we have two posets: $P(X)$ is an infinite chain and $P(Y)$ is a cardinal sum (1, 2). Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of sets $\{(6), (1, 2)\}$. By Theorem 4 this matrix problem is of unbounded representation type.

Analogously, one can prove the following lemma:

Lemma 9. *Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring*

$$\mathbf{A} = \begin{pmatrix} O & D & D & D \\ 0 & D & 0 & 0 \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{19}$$

corresponding to the diagram

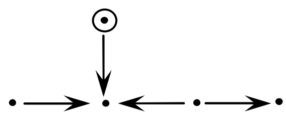


is a ring of the unbounded representation type.

Lemma 10. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$. Then the ring*

$$\mathbf{A} = \begin{pmatrix} O & 0 & 0 & 0 & D \\ 0 & D & 0 & 0 & D \\ 0 & 0 & D & D & D \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & D \end{pmatrix} \qquad (20)$$

corresponding to the diagram



is a ring of the unbounded representation type.

Proof. Let M be a finitely generated right A-module which is given by the set $\{t; l_1, l_2, l_3, l_4; \mathbf{T}\}$, in which the matrix $\mathbf{T}$ has the following form:

E	**O**	**O**	**O**	$\mathbf{T}_{15}$
O	**E**	**O**	**O**	$\mathbf{T}_{25}$
O	**O**	**E**	$\mathbf{T}_{34}$	$\mathbf{T}_{35}$
O	**O**	**O**	**E**	**O**
O	**O**	**O**	**O**	**E**

where $\mathbf{T}_{15} \in M_{t\times l_4}(D)$, $\mathbf{T}_{i5} \in M_{l_{i-1}\times l_4}(D)$, $(i = 2,3)$ and $\mathbf{T}_{34} \in M_{l_2\times l_3}(D)$ are matrices over D. The matrix of transformations $\mathbf{U}$ has the following form:

$\mathbf{U}_{11}$	**O**	**O**	**O**	**O**
O	$\mathbf{U}_{22}$	**O**	**O**	**O**
O	**O**	$\mathbf{U}_{33}$	**O**	**O**
O	**O**	**O**	$\mathbf{U}_{44}$	**O**

O	**O**	**O**	**O**	$\mathbf{U}_{55}$

where $\mathbf{U}_{11}$ is an invertible matrix with entries in O, and $\mathbf{U}_{ii}$ (i = 2,3,4,5) are invertible matrices with entries in D. Reducing the matrix **T** by the matrix **U** is leads to the matrix problem given by a matrix $\mathbf{T}_1$ partitioned into 3 horizontal strips and 2 vertical strips:

O	$\mathbf{A}_1$
O	$\mathbf{A}_2$
$\mathbf{A}_4$	$\mathbf{A}_3$

and the following admissible transformations:
(a) left O-elementary transformations with rows of the first horizontal strip of $\mathbf{T}_1$;
(b) left D-elementary transformations of rows of the second horizontal strip and third horizontal strip of $\mathbf{T}_1$;
(c) right D-elementary transformations of columns inside each vertical strip of $\mathbf{T}_1$.

Then one can reduce the second horizontal strip to the form:

0	0	0	0	0	0	0	0	0	0	E	0
0	0	0	0	0	0	0	0	E	0	0	0
0	0	0	0	0	0	E	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

and the third horizontal strip to the form:

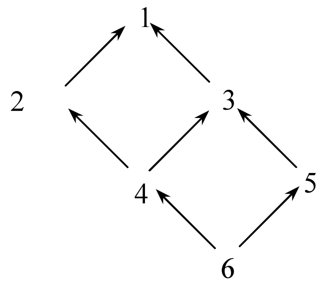
E	0	0	0	0	0	0	0	E	0	0	0
0	E	0	0	0	0	0	0	0	E	0	0
0	0	E	0	0	0	0	0	0	0	0	0
0	0	0	E	0	0	0	0	0	0	0	0
0	0	0	0	0	0	E	0	0	0	0	0
0	0	0	0	0	0	0	E	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

This process leads to the matrix problem given by a matrix **B** partitioned into 6 vertical strips

$\mathbf{B}_1$	$\mathbf{B}_2$	$\mathbf{B}_3$	$\mathbf{B}_4$	$\mathbf{B}_5$	$\mathbf{B}_6$

and the following admissible transformations:

(a) left *O*-elementary transformations with rows of the first horizontal strip of **B**;
(b) right *D*-elementary transformations with columns inside of each vertical strip $\mathbf{B}_i$ (i = 1,2,..,6);
(c) right *D*-elementary transformations with columns of vertical strips $\mathbf{B}_i$ which are in one-to-one relation with the following poset S:



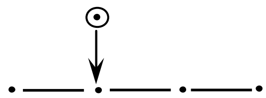
i.e., if $\alpha_i \le \alpha_j$ in the poset S then any column of the block $\mathbf{B}_i$ can be added to any column of the $\mathbf{B}_j$.

It is easy to see the blocks $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$ form the matrix problem II, considered in [4]. Therefore ring *A* is of unbounded representation type.

Remark 4.

A mixed matrix problem which forms matrices $\mathbf{B}_i$ (i = 1,...,6) is defined by two posets: $P(X)$ is an infinite chain, and $P(Y)$ = S. Therefore the pair of posets {$P(X)$, $P(Y)$} contains a critical pair of sets {(6), (1, 2)}. By Theorem 4 this matrix problem is of unbounded representation type.

Lemma 11. *Ring A corresponding to the diagram*



with arbitrary directions of arrows is a ring of unbounded representation type.

Conclusions

This paper proves the necessity in Theorem 1 for the ring A(S, O), when all discrete valuation rings corresponding to minimal elements of the poset S are the same. In this case the necessity follows from Lemmas 5-11 and Propositions 2 and 3.

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