

CALCULATE WAVELETS SPECTRAL ESTIMATE

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Abstract. The main task of this research work is applying techniques of wavelet analysis in spectral analysis of stationary random processes. The algorithm of calculation obtained for the spectral estimate of stationary random processes with discrete time via Douthies scaling function is studied.

Introduction

One of the main problems in spectral analysis of time series is consistent estimate formation of the second order spectral density via finite realization of stationary process. In numerous research dealing with the mentioned task periodogram methods based on inverse Fourier transform are used.

It's essential to mention that various research questions for statistics of consistent estimate obtained by periodogram smoothing via spectral windows are published, for example, in monographs [1-3] and publications [4-10].

Recent application of wavelet-analysis methods in time series study is quite relevant, as the results obtained from this method are frequently more informative and can directly deal with such input data peculiarities which are difficult to handle with the traditional approach.

Instead of using a deterministic approach scientists usually use a stochastic approach to model the data and to estimate the energy distribution (e.g. in electrical engineering, geophysics, economics or neurophysiology). One reason is that in a stochastic setup certain fluctuations of the Fourier-transform of the data can be interpreted more naturally.

1. Scaling Douthies function

Consider scaling Douthies function $\varphi(x) \in L_2(\mathbf{R})$ of order $L \in \mathbf{N}$. The scaling function $\varphi(x)$ is continuous and has the following properties (see [4]):

$$\int_{\mathbf{R}} \varphi(x) dx = 1,$$

$$\varphi(x) = \sqrt{2} \sum_{k \in \mathbf{Z}} h_k \varphi(2x - k), \quad x \in \mathbf{R}, \quad \text{supp}\varphi(x) = [0; 2L - 1],$$

where $h_k, k \in \mathbf{Z}$ - filter of scaling function, $\text{supp}\varphi(x)$ - support of scaling function.

Table 1

Example of Doubeshies scaling function (order $L = 2, L = 3$)

$L = 2$	
h_0	0.48296291314453410
h_1	0.836516337378077
h_2	0.2241438680420134
h_3	-0.1294095225512603
$L = 3$	
h_0	0.3326705529500825
h_1	0.8068915093110924
h_2	0.4598775021184914
h_3	-0.1350110200102546
h_4	-0.0854412738820267
h_5	0.0352262918857095

On Figures 1 and 2 some examples of the scaling Doubeshies function with compact support are illustrated.

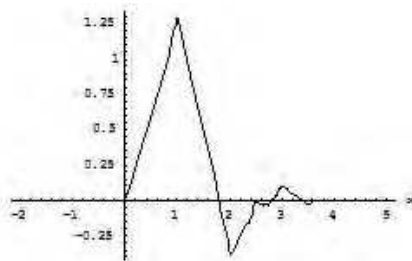


Fig. 1. Scaling Doubeshies function $L = 3$,
 $\max_{x \in \mathbf{R}} |\varphi(x)| = 1.28634$
 $\text{supp}\varphi(x) \subset [0, 5]$

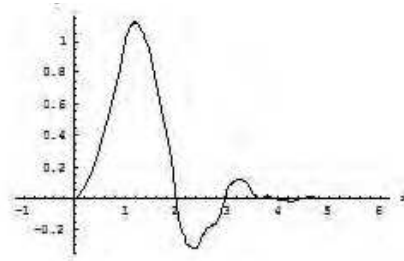


Fig. 2. Scaling Doubeshies function
 $L = 4$, $\max_{x \in \mathbf{R}} |\varphi(x)| = 1.12165$
 $\text{supp}\varphi(x) \subset [0, 7]$

We can construct 2π -periodical scaling, using scaling Doubeshies function via formula (1):

$$\tilde{\varphi}_{j,k}(\lambda) = \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}} \sum_{n=-1}^{(2L-1)2^{-j}+1} \varphi\left(\frac{2^j \lambda}{2\pi} + 2^j n - k\right), \lambda \in \Pi. \quad (1)$$

Function (1) is formed orthonormal basis of space $L_2(\Pi)$.

On Figure 3 some examples of the 2π -periodical scaling function are illustrated.

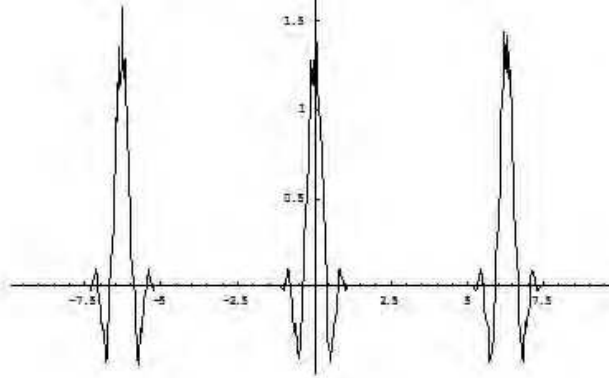


Fig. 3. 2π -periodical scaling function

2. Spectral estimate

Let $X(t)$, $t \in \mathbf{Z}$, be a wide-sense stationary stochastic process with $EX(t) = 0$, $t \in \mathbf{Z}$, belonging to a set of random processes $\chi(\lambda, f, \alpha, L, C_2)$.

The set $\chi(\lambda, f, \alpha, L, C_2)$ is defined as the set of wide-sense stationary processes $X(t)$, $t \in \mathbf{Z}$, whose spectral density is $f(\lambda)$, $\lambda \in \Pi = [-\pi, \pi]$, having a fourth-order semi-invariant spectral density $f_4(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_j \in \Pi$, $j = \overline{1, 3}$, and such that for fixed $\lambda \in \Pi$ the spectral density f satisfies $f \in Lip_\alpha(L)$ and the fourth-order semi-invariant spectral density is bounded by a constant $C_2 > 0$.

The definition of the class $\chi(\lambda, f, \alpha, L, C_2)$ can be found in Zhurbenko [9]. It contains processes with spectral densities whose peaks and troughs increase with T , for example AR -process with peaks [5].

For a process $X \in \chi(\lambda, f, \alpha, L, C_2)$, the rate of convergence of the mean-square deviation of a linear wavelet estimate of the spectral density is studied in [6].

The coefficients of the asymptotically dominant term, which depend on the smoothness of the spectral density, are calculated for some scaling functions and data tapering windows you can find in [5].

Thus the information on value α according to the aprioristic information on spectral density for investigated stochastic process. Such information, as a rule, is undertaken on the basis of supervision over several realizations for the concrete phenomenon.

As spectral estimate $f(\lambda)$ let's consider statistics:

$$\hat{f}(\lambda) = \sum_{k=1}^{2^J} \hat{\alpha}_{J,k} \tilde{\varphi}_{J,k}(\lambda) \quad (2)$$

where

$$\hat{\alpha}_{J,k} = \int_{\Pi} I_T(\alpha) \tilde{\varphi}_{J,k}(\alpha) d\alpha, \quad (3)$$

are wavelet-coefficients estimates in (2), and $I_T(\lambda)$ – modified periodogram:

$$I_T(\lambda) = \frac{1}{2\pi H_2^{(T)}(0)} d_T(\lambda) d_T(-\lambda) \quad (4)$$

$$d_T(\lambda) = \sum_{t=0}^{T-1} h_T(t) X(t) e^{-i\lambda t} \quad (5)$$

$$H_k^{(T)}(\lambda) = \sum_{t=0}^{T-1} (h_T(t))^k e^{-i\lambda t} \quad (6)$$

$k \in \mathbf{N}$, $T \in \mathbf{N}$, and function $h_T(t) = h\left(\frac{t}{T}\right)$, $h: [0,1] \rightarrow \mathbf{R}$ - data taper, its behavior is studied sufficiently in [2, 6].

$$\tilde{\varphi}_{J,k}(\lambda) = \sum_{n \in \mathbf{Z}} (2\pi)^{-1/2} \varphi_{J,k}((2\pi)^{-1}\lambda + n) \quad (7)$$

– 2π - periodical scaling function;

$$\varphi_{J,k}(x) = 2^{J/2} \varphi(2^J x - k), \quad (8)$$

$J \in \mathbf{N}_0 = \{0,1,2,\dots\}$, $\lambda \in \Pi$, $\varphi(x)$ - scaling function $x \in \mathbf{R}$. We have to mention that the results cited in this article are obtained with condition of data tapers variation restriction.

In the spectral analysis we can use the following data taper:

1. Function $h(x)$, $x \in [-1,1]$, bounded variation, $h(x)=1$ in point $x=0$, and $h(x)=0$ for $|x| \geq 1$, and $h(x)=0$ for $x < 0$.

2. Function $h(x)$, bounded variation, with $h(x)=1$ in point $x = \frac{1}{2}$, and $h(x)=0$ for $x \leq 0$ and $x \geq 1$.

A heuristic explanation is the following. Straightforward calculation gives for the expectation of the wavelet estimation:

$$E\hat{f}(\lambda) = \iint_{\Pi^2} f(\lambda + x + \gamma) \Phi_2^{(T)}(\gamma) K_J(\lambda, \lambda + x) dx d\gamma \tag{9}$$

where

$$K_J(\lambda, \alpha) = \sum_{k=1}^{2^J} \tilde{\varphi}_{J,k}(\lambda) \tilde{\varphi}_{J,k}(\alpha) \tag{10}$$

$$\Phi_2^{(T)}(x) = \frac{H_1^{(T)}(x) H_1^{(T)}(-x)}{2\pi H_2^{(T)}(0)} \tag{11}$$

are two kernel functions. One is dependent on the scaling function, the second on data taper.

In paper [6] it is proved that for first moment's estimate (2) we have:

$$E\hat{f}(\lambda) = f(\lambda) + O\left(\frac{1}{2^{J\alpha}}\right) + R_T, \tag{12}$$

where

$$R_T = \begin{cases} O\left(\frac{1}{T^\alpha}\right), & \text{if } 0 < \alpha < 1, \\ O\left(\frac{\ln(\pi T)}{T}\right), & \text{if } \alpha = 1. \end{cases} \tag{13}$$

In fact, one can prove that

$$E\hat{f}(\lambda) = f(\lambda) + O(T^{-2}) \tag{14}$$

for a sufficiently smooth data taper and for a scaling function of higher order. For dispersion it is valid:

$$D(\hat{f}(\lambda)) = \frac{2\pi H_4^{(T)}(0)}{(H_2^{(T)}(0))^2} \sum_{k_1=1}^{2^J} \sum_{k_2=1}^{2^J} \tilde{\varphi}_{J,k_1}(\lambda) \tilde{\varphi}_{J,k_2}(\lambda) \int_{\Pi} \tilde{\varphi}_{J,k_1}(\alpha) (\tilde{\varphi}_{J,k_2}(-\alpha) + \tilde{\varphi}_{J,k_2}(\alpha)) f^2(\alpha) d\alpha + O\left(\frac{1}{T}\right) + O\left(\frac{2^J \ln^5(\pi T)}{T^{1+\alpha}}\right) + O\left(\frac{2^{J(\mu+1)} \ln^5(\pi T)}{T^{1+\mu}}\right) + O\left(\frac{2^{2J} \ln^3(\pi T)}{T^2}\right). \tag{15}$$

Wavelet estimate $\hat{f}(\lambda)$, defined by formula (1), is consistent in the mean-square sense estimate of spectral density $f(\lambda)$, $\lambda \in \Pi$ for all $2^J \leq RT^{1-\rho}$, where $0 < \rho < 1$, $0 < R < \infty$, is some fixed constant.

3. Calculation

Step 1. Choose data taper $h_r(t)$. Data taper $h_r(t) = h\left(\frac{t}{T}\right)$ can be found from condition of minimization of value:

$$\Delta = \int_{-\pi}^{\pi} |x|^\alpha |\Phi_2^{(T)}(x)| dx + \frac{2}{T} \sum_{k=0}^{\left[\frac{T}{2}\right]-1} \left(\frac{H_2^{(T)}\left(\frac{2\pi k}{T}\right)}{H_2^{(T)}(0)} \right)^2 \rightarrow \min_H \quad (16)$$

where H - same bounded set of data taper $h(x)$, $x \in \mathbf{R}$.

The integral

$$\int_{-\pi}^{\pi} |x|^\alpha |\Phi_2^{(T)}(x)| dx \quad (17)$$

and the sum

$$\frac{2}{T} \sum_{k=0}^{\left[\frac{T}{2}\right]-1} \left(\frac{H_2^{(T)}\left(\frac{2\pi k}{T}\right)}{H_2^{(T)}(0)} \right)^2 \quad (18)$$

can be calculated, using standard numerical methods. More examples of data taper functions are given below:

1. Hemming's window:

$$h(x) = 0.54 + 0.46 \cos(\pi x), \quad x \in [-1, 1]. \quad (19)$$

2. A triangular window:

$$h(x) = 1 - |x|, \quad x \in [-1, 1]. \quad (20)$$

3. Rice's, Bohner's, Parzen's window:

$$h(x) = 1 - x^2, \quad x \in [-1, 1]. \quad (21)$$

The data taper, chosen via this method is optimal in the sense of minimum mean-square deviation of the modified periodogram.

Thus, the advantages of data tapers could also be established theoretically. The choice of the data taper is an important problem, for which no rigorous results exist. It is obvious that the choice depends on the true (unknown) spectral density, in particular on the relation of the peaks and troughs to each other.

Step 2. Calculate modified periodogram $I_T(\lambda)$, via formulas (3) and (4), using data taper $h(x)$, $x \in [-1,1]$, which we find on step 1.

Step 3. Choose scaling function

$$\left(\frac{(2\pi)^\alpha 2L \max_{\Phi} |\varphi(x)|}{T^\alpha} \left[\int_{\mathbf{R}} |z|^\alpha |\varphi(z)| dz \right] \right)^2 \rightarrow \min_{\Phi} \quad (22)$$

where $\Phi = \{\varphi_L(x), x \in \mathbf{R}, L \in \mathbf{N}\}$ - the set of Doubeshies scaling function of different order.

The choice of scaling function is influenced by two characteristics by: maximum of scaling function and its support.

Step 4. Calculate level J

$$J = [\log_2(T)], \quad (23)$$

where $[\bullet]$ - the whole part of number.

Step 5. For calculation of initial factors we will put $J_0 = T$.

Step 6. Coefficients $\hat{\alpha}_{J_0,k}$ calculate, using formula of left rectangle (see step 6.1), or using quadrature formula (see step 6.2).

Step 6.1. Calculate coefficients via formula of left rectangle:

$$\hat{\alpha}_{J_0,k} = \sqrt{2^{-J_0+1}} \pi \frac{(2L-1)}{N_1} \sum_{l=0}^{N_1-1} I_T \left(2^{-J_0+1} \pi \left(k + \frac{2L-1}{N_1} l \right) \right) \varphi \left(\frac{2L-1}{N_1} l \right), \quad (24)$$

where: N_1 - quantity of parts for integration interval ($N_1 \leq T$), $k = \overline{0, 2^{J_0} - 1}$.

Step 6.2. Calculate coefficients via quadrature formulas

$$\hat{\alpha}_{J_0,k} \approx 2^{\frac{-J_0+1}{2}} \sqrt{\pi} \sum_{k=1}^r \omega_k I_T \left(\frac{(x_k + k)}{2^{J_0-1}} \right), \quad (25)$$

$k = 0, 2^{j_0} - 1$; where method of are calculated of abscissas x_k , $k = \overline{1, r}$ and weight ω_k , $k = \overline{1, r}$ is considered in paper [7].

Table 2

Example of abscissas x_k , $k = \overline{1, r}$ and weight ω_k , $k = \overline{1, r}$, of quadrature formulas for Doubeshies system of order $L = 2, L = 3$

	k	x_k	ω_k
$L = 2$ $r = 3$	1	0.565179	0.899173
	2	1.565179	0.132858
	3	2.565179	-0.032031
$L = 2$ $r = 4$	1	0.247825	0.268749
	2	0.747825	0.561228
	3	1.247825	0.298997
	4	1.747825	-0.128975
$L = 2$ $r = 5$	1	0.253425	0.276273
	2	0.753425	0.557197
	3	1.253425	0.296560
	4	1.753425	-0.130903
	5	2.253425	0.000872
$L = 3$ $r = 3$	1	0.804695	0.990491
	2	2.804695	0.012666
	3	4.804695	-0.003156
$L = 3$ $r = 4$	1	0.701350	0.817228
	2	1.701350	0.264924
	3	2.701350	-0.097581
	4	3.701350	0.015430
$L = 3$ $r = 5$	1	0.661075	0.747720
	2	1.661075	0.384507
	3	2.661075	-0.174764
	4	3.661075	0.048801
	5	4.661075	-0.006264

Step 7. Calculate coefficient $\hat{\alpha}_{J_0, k, \dots}, \hat{\alpha}_{J-1, k}$, using modified formulas

$$\hat{\alpha}_{j, k} = \frac{1}{L} \sum_{n=0}^{2L-1} h_n \hat{\alpha}_{j+1, (n+2k) \bmod 2^{j+1}}, \quad (26)$$

where h_k , $k \in \mathbf{Z}$ - filters of Doubeshies scaling function.

Step 8. Build wavelet estimate $\hat{f}(\lambda)$, using formula

$$\hat{f}(\lambda) = \sum_{k=1}^{2^J} \hat{\alpha}_{J,k} \tilde{\Phi}_{J,k}(\lambda) \quad (27)$$

on level J .

Estimate $\hat{f}(\lambda)$, constructed via the considered algorithm is optimal in minimum biases square. We note that other existing spectral estimates have similar resolution properties as the tapered periodogram. Those statistics are usually non-linear, non-quadratic and therefore very difficult to investigate theoretically.

Conclusions

Our theoretical results in this paper are also used for developing computational algorithms for wavelet estimates of the spectral density. These algorithms enable us:

- 1) to select a data tapering windows;
- 2) to choose a scaling function;
- 3) to compute the level of decomposition;
- 4) in order to construct the estimate minimizing the mean square deviation, depending on the sample length and the smoothness of the spectral density.

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