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# THE DETERMINANTS OF THE BLOCK MATRICES IN THE 3D FOURIER EQUATION

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**Abstract.** In the paper we present the method of calculating the determinant of the block matrix which characterizes internal heat conduction in the 3D case. The Finite Difference Method can be used.

## Introduction

The determinants of the block matrices are applicable among others in problems of heat transfer. For example, the Finite Differences Method (FDM) for the Fourier equation (in the case of the internal nodes) leads to the seven-band internal equations in each time step and then the matrix of the system equations is a five-band block matrix.

## 1. Solution of the problem

The heat conduction equation for 3D problem is as follows

$$\lambda\left(\frac{\partial^2 T(x,y,z,t)}{\partial x^2}+\frac{\partial^2 T(x,y,z,t)}{\partial y^2}+\frac{\partial^2 T(x,y,z,t)}{\partial z^2}\right)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \tag{1}$$

where $\lambda$ is a thermal conductivity, $c$ is a specific heat, $\rho$ is a mass density and $T$, $x$, $y$, $z$, $t$ denote the temperature, geometrical co-ordinates and time.

Assuming the following difference quotients we obtain the differential approximation of the second derivatives appearing in the equation (1) [1]

$$\frac{\Delta^2 T}{\Delta x^2}=\frac{T_{i-1,j,k,l}-2T_{i,j,k,l}+T_{i+1,j,k,l}}{(\Delta x)^2},\quad 1\le i\le m-1 \tag{2}$$

$$\frac{\Delta^2 T}{\Delta y^2} = \frac{T_{i,j-1,k,l} - 2T_{i,j,k,l} + T_{i,j+1,k,l}}{(\Delta y)^2}, \quad 1 \le j \le n-1$$

$$\frac{\Delta^2 T}{\Delta z^2} = \frac{T_{i,j,k-1,l} - 2T_{i,j,k,l} + T_{i,j,k+1,l}}{(\Delta z)^2}, \quad 1 \le k \le p-1$$

and the approximation of the first derivative of the time

$$\frac{\Delta T}{\Delta t} = \frac{T_{i,j,k,l} - T_{i,j,k,l-1}}{\Delta t}, \quad 1 \le l \le q \tag{3}$$

So, the internal iterations takes the following differential form

$$\lambda \left( \frac{\Delta^2 T}{\Delta x^2} + \frac{\Delta^2 T}{\Delta y^2} + \frac{\Delta^2 T}{\Delta z^2} \right) = \rho c \frac{\Delta T}{\Delta t} \tag{4}$$

and the Finite Difference Method leads to the seven-band of the internal system of equations

$$\begin{aligned} &\frac{\lambda}{(\Delta x)^2} T_{i-1,j,k,l} - \frac{2\lambda}{(\Delta x)^2} T_{i,j,k,l} + \frac{\lambda}{(\Delta x)^2} T_{i+1,j,k,l} + \\ &+ \frac{\lambda}{(\Delta y)^2} T_{i,j-1,k,l} - \frac{2\lambda}{(\Delta y)^2} T_{i,j,k,l} + \frac{\lambda}{(\Delta y)^2} T_{i,j+1,k,l} + \\ &+ \frac{\lambda}{(\Delta z)^2} T_{i,j,k-1,l} - \frac{2\lambda}{(\Delta z)^2} T_{i,j,k,l} + \frac{\lambda}{(\Delta z)^2} T_{i,j,k+1,l} = \\ &= \frac{\rho c}{\Delta t} T_{i,j,k,l} - \frac{\rho c}{\Delta t} T_{i,j,k,l-1} \end{aligned} \tag{5}$$

in each time step *l*.

For example, we present the first three equations of the system for $1 \le i \le 3$, $1 \le j \le 3$, $1 \le k \le 3$.

So, the first equation of the system takes the form

$$\begin{aligned} &\left( \frac{2\lambda}{(\Delta x)^2} + \frac{2\lambda}{(\Delta y)^2} + \frac{2\lambda}{(\Delta z)^2} + \frac{\rho c}{\Delta t} \right) T_{1,1,1,l} - \frac{\lambda}{(\Delta x)^2} T_{2,1,1,l} + 0 \cdot T_{3,1,1,l} + \\ &- \frac{\lambda}{(\Delta y)^2} T_{1,2,1,l} + 0 \cdot T_{2,2,1,l} + 0 \cdot T_{3,2,1,l} + 0 \cdot T_{1,3,1,l} + 0 \cdot T_{2,3,1,l} + 0 \cdot T_{3,3,1,l} + \\ &- \frac{\lambda}{(\Delta z)^2} T_{1,1,2,l} + 0 \cdot T_{2,1,2,l} + 0 \cdot T_{3,1,2,l} + 0 \cdot T_{1,2,2,l} + 0 \cdot T_{2,2,2,l} + 0 \cdot T_{3,2,2,l} + \end{aligned} \tag{6}$$

$$+0\cdot T_{1,3,2,l}+0\cdot T_{2,3,2,l}+0\cdot T_{3,3,2,l}+0\cdot T_{1,1,3,l}+0\cdot T_{2,1,3,l}+0\cdot T_{3,1,3,l}+$$
$$+0\cdot T_{1,2,3,l}+0\cdot T_{2,2,3,l}+0\cdot T_{3,2,3,l}+0\cdot T_{1,3,3,l}+0\cdot T_{2,3,3,l}+0\cdot T_{3,3,3,l}=$$
$$=\frac{\lambda}{(\Delta x)^2}T_{0,1,1,l}+\frac{\lambda}{(\Delta y)^2}T_{1,0,1,l}+\frac{\lambda}{(\Delta z)^2}T_{1,1,0,l}+\frac{\rho c}{\Delta t}T_{1,1,1,l-1}$$

the second equation of the system is as follows

$$-\frac{\lambda}{(\Delta x)^2}T_{1,1,1,l}+\left(\frac{2\lambda}{(\Delta x)^2}+\frac{2\lambda}{(\Delta y)^2}+\frac{2\lambda}{(\Delta z)^2}+\frac{\rho c}{\Delta t}\right)T_{2,1,1,l}-\frac{\lambda}{(\Delta x)^2}T_{3,1,1,l}+$$
$$+0\cdot T_{1,2,1,l}-\frac{\lambda}{(\Delta y)^2}T_{2,2,1,l}+0\cdot T_{3,2,1,l}+0\cdot T_{1,3,1,l}+0\cdot T_{2,3,1,l}+0\cdot T_{3,3,1,l}+$$
$$+0\cdot T_{1,1,2,l}-\frac{\lambda}{(\Delta z)^2}T_{2,1,2,l}+0\cdot T_{3,1,2,l}+0\cdot T_{1,2,2,l}+0\cdot T_{2,2,2,l}+0\cdot T_{3,2,2,l}+\qquad(7)$$
$$+0\cdot T_{1,3,2,l}+0\cdot T_{2,3,2,l}+0\cdot T_{3,3,2,l}+0\cdot T_{1,1,3,l}+0\cdot T_{2,1,3,l}+0\cdot T_{3,1,3,l}+$$
$$+0\cdot T_{1,2,3,l}+0\cdot T_{2,2,3,l}+0\cdot T_{3,2,3,l}+0\cdot T_{1,3,3,l}+0\cdot T_{2,3,3,l}+0\cdot T_{3,3,3,l}=$$
$$=\frac{\lambda}{(\Delta y)^2}T_{2,0,1,l}+\frac{\lambda}{(\Delta z)^2}T_{2,1,0,l}+\frac{\rho c}{\Delta t}T_{2,1,1,l-1}$$

and the third equation of the system is present below

$$0\cdot T_{1,1,1,l}-\frac{\lambda}{(\Delta x)^2}T_{2,1,1,l}+\left(\frac{2\lambda}{(\Delta x)^2}+\frac{2\lambda}{(\Delta y)^2}+\frac{2\lambda}{(\Delta z)^2}+\frac{\rho c}{\Delta t}\right)T_{3,1,1,l}+$$
$$+0\cdot T_{1,2,1,l}+0\cdot T_{2,2,1,l}-\frac{\lambda}{(\Delta y)^2}T_{3,2,1,l}+0\cdot T_{1,3,1,l}+0\cdot T_{2,3,1,l}+0\cdot T_{3,3,1,l}+$$
$$+0\cdot T_{1,1,2,l}+0\cdot T_{2,1,2,l}-\frac{\lambda}{(\Delta z)^2}T_{3,1,2,l}+0\cdot T_{1,2,2,l}+0\cdot T_{2,2,2,l}+0\cdot T_{3,2,2,l}+\qquad(8)$$
$$+0\cdot T_{1,3,2,l}+0\cdot T_{2,3,2,l}+0\cdot T_{3,3,2,l}+0\cdot T_{1,1,3,l}+0\cdot T_{2,1,3,l}+0\cdot T_{3,1,3,l}+$$
$$+0\cdot T_{1,2,3,l}+0\cdot T_{2,2,3,l}+0\cdot T_{3,2,3,l}+0\cdot T_{1,3,3,l}+0\cdot T_{2,3,3,l}+0\cdot T_{3,3,3,l}=$$
$$=\frac{\lambda}{(\Delta y)^2}T_{3,0,1,l}+\frac{\lambda}{(\Delta z)^2}T_{3,1,0,l}+\frac{\rho c}{\Delta t}T_{3,1,1,l-1}$$

The matrix of the system has the form

$$A_3 = \left[\begin{array}{ccccccccccccccccccccccccccc}
a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & a & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & b & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & b & a & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & 0 & b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & a & b & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & b & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & a & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c & 0 & b & a
\end{array}\right] \tag{9}$$

where

$$a = \frac{2\lambda}{(\Delta x)^2} + \frac{2\lambda}{(\Delta y)^2} + \frac{2\lambda}{(\Delta z)^2} + \frac{\rho c}{\Delta t}, \quad b = -\frac{\lambda}{(\Delta x)^2}, \quad c = -\frac{\lambda}{(\Delta y)^2}, \quad d = -\frac{\lambda}{(\Delta z)^2} \tag{10}$$

The matrix $A_3$ can be written in the following block form

$$A_3 = \begin{bmatrix} A_2 & D_2 & \mathbf{0} \\ D_2 & A_2 & D_2 \\ \mathbf{0} & D_2 & A_2 \end{bmatrix} \tag{11}$$

where

$$A_2 = \begin{bmatrix} A_1 & D_1 & \mathbf{0} \\ D_1 & A_1 & D_1 \\ \mathbf{0} & D_1 & A_1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} C & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \end{bmatrix}, \quad C = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \tag{12}$$

and continue

$$A_1 = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}, \quad D_1 = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \tag{13}$$

The matrix from the formula (11) can be transformed in this way

$$A_3 = d \begin{bmatrix} A_2' & I_2 & \mathbf{0} \\ I_2 & A_2' & I_2 \\ \mathbf{0} & I_2 & A_2' \end{bmatrix} \tag{14}$$

where

$$A_2' = \frac{1}{d} A_2 \tag{15}$$

$$A_1' = \frac{1}{c} A_1 \tag{16}$$

The determinant of the matrix $A_3$ is calculated using the formulas presented in [2-4]

$$\begin{aligned} \det A_3 = {} & d^{27} \det A_2' \cdot \det\left(A_2' - \sqrt{2}\, I_2\right) \cdot \det\left(A_2' + \sqrt{2}\, I_2\right) = \\ & = \det A_2 \cdot \det\left(A_2 - \sqrt{2}\, d I_2\right) \cdot \det\left(A_2 + \sqrt{2}\, d I_2\right) \end{aligned} \tag{17}$$

where

$$\begin{aligned} \det A_2 = {} & c^{9} \det A_1' \cdot \det\left(A_1' - \sqrt{2}\, I_1\right) \cdot \det\left(A_1' + \sqrt{2}\, I_1\right) = \\ & = \det A_1 \cdot \det\left(A_1 - \sqrt{2}\, c I_1\right) \cdot \det\left(A_1 + \sqrt{2}\, c I_1\right) \end{aligned} \tag{18}$$

$$\begin{aligned} & \det\left(A_2 - \sqrt{2}\, d I_2\right) = \\ & = \det\left(A_1 - \sqrt{2}\, d I_1\right) \cdot \det\left[A_1 - \sqrt{2}\,(d+c) I_1\right] \cdot \det\left[A_1 - \sqrt{2}\,(d-c) I_1\right] \end{aligned} \tag{19}$$

$$\det\left(A_2+\sqrt{2}\,dI_2\right)= \\ =\det\left(A_1+\sqrt{2}\,dI_1\right)\cdot\det\left[A_1+\sqrt{2}\,(d+c)I_1\right]\cdot\det\left[A_1+\sqrt{2}\,(d-c)I_1\right] \tag{20}$$

consequently

$$\det A_3=\det A_1\cdot\det\left(A_1^2-2c^2I_1\right)\cdot\det\left(A_1^2-2d^2I_1\right)\cdot \\ \cdot\det\left(A_1^2-2(d+c)^2I_1\right)\cdot\det\left(A_1^2-2(d-c)^2I_1\right) \tag{21}$$

and finally

$$\det A_3=(a^3-2ab^2)[(2c^2)^3-(3a^2+4b^2)(2c^2)^2+2c^2(3a^4+4b^4)-(a^3-2ab^2)^2]\cdot \\ \cdot[(2d^2)^3-(3a^2+4b^2)(2d^2)^2+2d^2(3a^4+4b^4)-(a^3-2ab^2)^2]\cdot \\ \cdot\{[2(c+d)^2]^3-(3a^2+4b^2)[2(c+d)^2]^2+2(3a^4+4b^4)(c+d)^2-(a^3-2ab^2)^2\}\cdot \\ \cdot\{[2(c-d)^2]^3-(3a^2+4b^2)[2(c-d)^2]^2+2(3a^4+4b^4)(c-d)^2-(a^3-2ab^2)^2\} \tag{22}$$

The Maple program presented the above product in the following form

$$\det A_3=[a^2-2(c+d)^2]a[a^2-2(b+c-d)^2][a^2-2(b+d)^2]\cdot \\ \cdot[a^2-2(b-d)^2][a^2-2(b+c+d)^2](-2d^2+a^2)\cdot \\ \cdot[a^2-2(b+c)^2][a^2-2(b-c-d)^2][a^2-2(b-c)^2]\cdot \\ \cdot(-2b^2+a^2)[a^2-2(b-c+d)^2][a^2-2(c-d)^2](a^2-2c^2) \tag{23}$$

The procedure given above constitutes a special case of the general procedure for calculating the determinants of the matrix block in the three-dimensional tasks.

In conclusion, this paper is the introduction to an algebraic solving equations occuring in the heat flow issues.

## References


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