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ON RIGHT HEREDITARY SPSD-RINGS OF BOUNDED REPRESENTATION TYPE II

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Abstract. In this paper right hereditary semiperfect semidistributive rings $A = A(\mathrm{S},O)$ of bounded representation type are described in terms of Dynkin diagrams and diagrams with weights. We describe these rings using a reduction to mixed matrix problems over a discrete valuation ring and its skew field of fractions.

Introduction

One of the main problems in the representation theory of rings and algebras is to obtain information about the possible structure of indecomposable modules and to describe the isomorphism classes of all indecomposable modules. There are different approach to study the representations of algebras and rings. One of them belongs to P. Gabriel who reduced the study of f.d. algebras to study of representation of quivers in 1972 [1]. In this paper P. Gabriel gave a full description of quivers of finite representation type over an algebraically closed field. The other approach belongs to L.A. Nazarova and A.V. Roiter [2] who reduced these representations to solving some matrix problems over fields, i.e. the reduction of some classes of matrix by an admissible set of transformations.

Recall that a quiver is an oriented graph without any restriction to the number of arrows between two vertices and possibly with loops or oriented cycles. More exactly, a quiver $Q = (Q_0, Q_1, s, e)$ is given by a set of vertices Q_0, a set of arrows Q_1 and two maps s, e: $Q_1 \to Q_0$ which associate to each arrow $\sigma \in Q_1$ the start point $s(\sigma) \in Q_0$ and the end point $e(\sigma) \in Q_0$.

Theorem 1. (P. Gabriel, [1])

A connected quiver Q is of a finite type if and only if the underlying undirected graph $\overline{Q}$ of Q (obtained from Q by deleting the orientation of the arrows) is a Dynkin diagram of the form A_n, D_n, E_6, E_7, E_8.

The number of isomorphism classes of indecomposable representations of Q is finite if and only if the corresponding quadratic form is positive definite. In this case the number of indecomposable representations equals the number of positive roots of the corresponding root systems. Moreover there is a bijection between the *isomorphic classes of indecomposable representations of Q and positive roots of the underlying Dynkin diagrams.*

This result was generalized for an arbitrary field by Bernstein, Gelfand and Ponomarev [3].

P. Gabriel also introduced in [1] the notion of K-species which consists of a finite family of skew fields $(K_i)_{i\in I}$ which are finitely dimension and central over a common commutative subfield K, together with a family of (K_i, K_j)-bimodules which are finite dimensional over K and K operates centrally on each this bimodule. In this paper he characterized K-species of finite type for the case when any $K_i = F$ is a fixed skew field. Later the results of P. Gabriel on representations of species were generalized by V. Dlab and C.M. Ringel in [4, 5].

Theorem 2. [4, Theorem B]

A K-species is of finite representation type if and only if its diagram is a finite disjoint union of Dynkin diagrams of the form A_n, D_n, E_6, E_7, E_8, F_4, G_2.

Theorem 3. [4, Theorem C]

A finite dimensional K-algebra A is a hereditary algebra of finite type if and only if A is Morita equivalent to the tensor algebra of a K-species of finite type.

The generalization of this theorem for the case of hereditary Artinian rings was stated by P. Dowbor, C.M. Ringel and D. Simson [6, 7]. Each Artinian ring A can be associated with the corresponding species $\Gamma(A)$ which is a generalization of a K-species. In the definition of species we do not assume that all skew fields are finitely generated over their common commutative subfield K.

Theorem 4. [6, Theorem 2]

The hereditary Artinian ring A is of finite representation type if and only if $\Gamma(A)$ *is a disjoin union of Coxeter diagrams* A_n, B_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ ($p = 5$ or $p \geq 7$).

Note that this theorem was only stated in [6, 7] without full proof and it was remarked that the proof is rather technical and the details were left out. The full proof of the theorem from which theorem 4 follows was obtained by S. Oppermann in [8].

From theorem 4 we obtain the following theorem:

Theorem 5. [9, Theorem 3]

A hereditary Artinian semidistributive ring $T(\mathrm{S}, D)$ *is of finite type if and only if the undirected graph* $\overline{\Gamma(\mathrm{S})}$ *of the Hasse diagram* $\Gamma(\mathrm{S})$ *is a disjoint union of the Dynkin diagrams of the form* A_n, D_n, E_6, E_7, E_8.

This paper is devoted to the description of right hereditary semiperfect semidistributive rings of bounded representation type. In this paper we prove the sufficiency of the main theorem [10, Theorem 1] that gives the structure of right hereditary SPSD-rigs of bounded representation type for the case when rings of endomorphisms of simple modules are either a skew field or the same discrete valuation ring O. This structure is given in terms of Dynkin diagrams and diagrams with weights. The paper is a continuation of [10] where the necessity of the main theorem was proved.

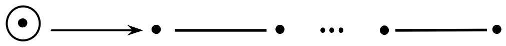
We use the notions, definitions and results of [9-14].

1. Proof of sufficiency of the main theorem

Lemma 1. Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring

$$A = \begin{pmatrix} H_m(O) & M \\ 0 & T(\mathrm{S}_1) \end{pmatrix} \tag{1}$$

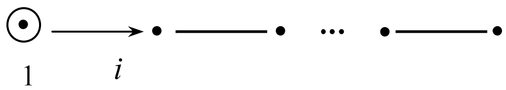
corresponding to the diagram:



and $T(S_1)$ is a ring corresponding to the diagram S_1:

where M is a $(H_m(O), T(S_1))$-bimodule, is a ring of bounded representation type for any directions of arrows in the diagram S_1.

Proof. Let M be a finitely generated right A-module that is given by the set: $\{t_1,\ldots,t_m;\ l_1,\ldots,l_{n-1};\ \mathbf{T}\}$. Renumbering the vertices of the poset S in such a way that the point $1 \in S$ has the weight $H_m(O)$ and is connected with the point i by an arrow, i.e. the diagram has the following form:



Then the matrix $\mathbf{T} = (\mathbf{T}_{ij})$ is an upper block-rectangular matrix with elements in D partitioned into n horizontal and n vertical strips in which $\mathbf{T}_{ii}$ are identical matrices ($i = 1,\ldots,n$) and in the first horizontal strip all blocks are zero except the first and the i-th blocks.

The matrix of transformations $\mathbf{U} = (\mathbf{U}_{ii})$ is a block diagonal matrix, where $\mathbf{U}_{11}$ is an invertible matrix with entries in $H_m(O)$, and $\mathbf{U}_{ii}$ ($i = 2,\ldots,n$) are invertible matrices with entries in D.

Consider the matrices $\mathbf{T}'$ and $\mathbf{U}'$ that are obtained from the matrices $\mathbf{T}$ and $\mathbf{U}$ correspondingly by erasing the first horizontal and the first vertical strip.

Reducing the matrix $\mathbf{T}$ by the matrix $\mathbf{U}$ is equivalent to the representation of the quiver Q_1 corresponding to the diagram

$$\bullet \!-\!\!-\!\!-\!\!-\!\!-\!\!-\! \bullet \quad \cdots \quad \bullet \!-\!\!-\!\!-\!\!-\!\!-\!\!-\! \bullet$$

By the Gabriel theorem, this quiver has $\frac{n(n-1)}{2}$ indecomposable representations. In accordance with these representations the matrix $\mathbf{T}_{1i}$ is partitioned into $2n$-2 vertical strips, and the partial relations between these strips are linear. Therefore reduction of the matrix $\mathbf{T}$ by the matrix $\mathbf{U}$ leads to reduction of matrix $\mathbf{T}_{1i}$ partitioned by $2n$-2 vertical linear ordering strips by matrix $\mathbf{U}_{11}$ with entries in $H_m(O)$. This matrix problem is equivalent to the following.

Given a rectangular matrix $\mathbf{B} = (\mathbf{B}_{ij})$ *with entries in a skew field D that is partitioned into* $2n$-2 *horizontal strips and m vertical strips.*

Over this matrix one performs admissible transformations of the following types:

1) *right O-elementary transformations of rows inside any horizontal strip;*
2) *left D-elementary transformations of columns inside any vertical strip;*
3) *addition of columns in the i-th vertical strip multiplied on the right by elements of D to columns in the j-th horizontal strip, if* $i \le j$*;*
4) *addition of rows in the i-th horizontal strip multiplied on the left by elements of O to rows in the j-th horizontal strip,* $i \le j$*;*
5) *addition of an arbitrary row of the j-th horizontal strip multiplied on the left by elements of R = rad(O) to any row of the i-th horizontal strip, if* $i \le j$.

By means of these transformations the matrix $\mathbf{B}$ can be reduced to the form in which any block $\mathbf{B}_{ij}$ has the following form:

E	**O**
O	**O**

and up and down of the matrix $\mathbf{E}$ in the matrix $\mathbf{B}$ we have the zero matrices. This means that the matrix $\mathbf{T}_{1i}$ is decomposed into a direct sum of matrices of the form

E	**O**
O	**O**

Thus, for any indecomposable finitely generated A-module M the corresponding matrix $\mathbf{T}$ has a finite fixed number of elements distinct of zero that is only depend on n. Therefore A is a ring of bounded representation type.

Lemma 2. Let O be a discrete valuation ring with a skew field of fractions D, and the radical $R = \pi O = O\pi$. Then the ring

$$A = \begin{pmatrix} H_{n_1}(O) & 0 & M_1 \\ 0 & H_{n_2}(O) & M_2 \\ 0 & 0 & T(S_1) \end{pmatrix} \qquad (2)$$

corresponding to the diagram:

and $T(S_1)$ is a ring corresponding to the diagram

where M_i is a $(H_{n_i}(O), T(S_1))$-bimodule (i = 1,2), is a ring of bounded representation type for any directions of arrows in the diagram S_1.

Proof. Let M be a finitely generated right A-module that is given by the set:

$$\{t_{11},\dots,t_{1n_1};t_{21},\dots,t_{2n_2};l_1,\dots,l_{n-1};\mathbf{T}\}$$

Renumber the vertices of the poset S in such a way that the point $1 \in S$ has weight $H_{n_1}(O)$ and is connected with the point i by an arrow, and point $2 \in S$ has the weight $H_{n_2}(O)$ and is connected with the point j by an arrow, i.e. the diagram has the following form:

Then the matrix $\mathbf{T} = (\mathbf{T}_{ij})$ is an upper block-rectangular matrix with elements in D partitioned into n+1 horizontal and vertical strips in which $\mathbf{T}_{ii}$ are identical matrices (i = 1,…,n+1) and in the first (second) horizontal strip all blocks are zero except the first (second) and the i-th (j-th) blocks.

The matrix of transformations $\mathbf{U} = (\mathbf{U}_{ii})$ is a block diagonal matrix, where $\mathbf{U}_{11}$ is an invertible matrix with entries in $H_{n_i}(O)$ (i = 1,2), and $\mathbf{U}_{ii}$ (i = 3,…,n+1) are invertible matrices with entries in D.

Consider the matrices $\mathbf{T}'$ and $\mathbf{U}'$ that are obtained from the matrices $\mathbf{T}$ and $\mathbf{U}$ correspondingly by erasing the first horizontal and the first vertical strip. The reduction of the matrix $\mathbf{T}'$ by the matrix $\mathbf{U}'$ leads to the matrix problem described in the previous lemma. In according with this problem the matrix $\mathbf{T}_{1i}$ is partitioned into t horizontal strips. Thus we obtain the following matrix problem.

Given a block-rectangular matrix $\mathbf{B} = (\mathbf{B}_{ij})$ *with entries in a skew field D that is partitioned into t horizontal strips and n_2 vertical strips* (*where* $t = n+n_1-1$).

Over this matrix one performs admissible transformations of the following types:

1) *right O-elementary transformations of columns inside any vertical strip;*
2) *left D-elementary transformations of rows inside any horizontal strip $\mathbf{B}_{ij}$, if $i \notin \{k+1,\dots,k+n_1\}$;*
3) *left O-elementary transformations of rows inside any horizontal strip $\mathbf{B}_{ij}$, if $i \in \{k+1,\dots,k+n_1\}$;*
4) *addition of columns in the i-th vertical strip multiplied on the left by elements of D to rows in the j-th vertical strip, if $i \le j$;*
5) *addition of columns in the j-th vertical strip multiplied on the right by elements of $R = \mathrm{rad}\, O$ to columns in the i-th vertical strip, if $i \le j$;*
6) *addition of rows in the $k+i$-th horizontal strip multiplied on the right by elements of O to rows in the $k+j$-th horizontal strip, if $i \le j$, and $i, j \in \{1,2,\dots,n_1\}$;*
7) *addition of rows in the $k+j$-th horizontal strip multiplied on the right by elements of $R = \mathrm{rad}\, O$ to rows in the $k+i$-th horizontal strip, if $i \le j$, and $i, j \in \{1,2,\dots n_1\}$;*
8) *addition of rows in the i-th horizontal strip multiplied on the right by elements of D to rows in the j-th horizontal strip, if $i \le j$, and $i, j \in \{k+1, k+2,\dots,k+n_1\}$.*

Using these transformations, the matrix $\mathbf{B}$ can be reduced to the form in which any block $\mathbf{B}_{ij}$ has one of the following forms:

E	**O**
O	**O**

or

$\pi^{m}\mathbf{E}$	**O**
O	**O**

and up and down, on the left and on the right of the matrix $\mathbf{E}$ (or $\pi^{m}\mathbf{E}$) in the matrix $\mathbf{B}$ we have the zero matrices. This means that the matrix $\mathbf{T}_{1i}$ is decomposed into a direct sum of matrices of the these forms.

Thus, A is a ring of bounded representation type.

Lemma 3. Let O be a discrete valuation ring with a skew field of fractions D, and the radical $R = \pi O = O\pi$. Then a ring

$$A = \begin{pmatrix} H_m(O) & M_1 & M_2 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \qquad (3)$$

corresponding to the diagram

$$\bullet \longleftarrow \odot \longrightarrow \bullet$$

where M_i is a ($H_m(O)$, D)-bimodule ($i = 1,2$), is a ring of bounded representation type.

Proof. Let M be a finitely generated right A-module that is given by the set:

$$\{t_1,\dots,t_m;\ l_1,l_2;\ \mathbf{T}\},$$

in which a block-rectangular matrix **T** has the following form:

E	$\mathbf{T}_{12}$	$\mathbf{T}_{13}$
O	**E**	**O**
O	**O**	**E**

and $\mathbf{T}_{12}$,$\mathbf{T}_{13}$ are matrices with entries in D.

The matrix of transformations **U** has the following form:

$\mathbf{U}_{11}$	**O**	**O**
O	$\mathbf{U}_{22}$	**O**
O	**O**	$\mathbf{U}_{33}$

where $\mathbf{U}_{11}$ is an invertible matrix with entries in $H_m(O)$, and $\mathbf{U}_{ii}$ ($i = 2,3$) are invertible matrices with entries in D.

The reduction of the matrix **T** by the matrix **U** leads to the following matrix problem:

Given a matrix $\mathbf{T}'$ *with entries in D partitioned into* 2 *vertical strips and m horizontal strips.*

Over these matrices one performs admissible transformations of the following types:

1) *right D-elementary transformations of columns inside any vertical strip;*
2) *left O-elementary transformations of rows inside any horizontal strip;*
3) *addition of rows of the i-th horizontal strip multiplied by elements of O to rows of the j-th horizontal strip;*
4) *addition of rows of the j-th horizontal strip multiplied by elements of* $R = \mathrm{rad}\, O$ *to rows of the i-th horizontal strip if* $i \le j$.

Using these transformations the matrix $\mathbf{T}'$ can be reduced to the direct sum of the following matrices:

$$[\mathbf{E}\,|\,\mathbf{O}]; \qquad [\mathbf{O}\,|\,\mathbf{E}]; \qquad [\mathbf{E}\,|\,\mathbf{E}]; \qquad \left[\begin{array}{c|c} \mathbf{E} & \pi^k\mathbf{E} \\ \mathbf{O} & \mathbf{E} \end{array}\right]$$

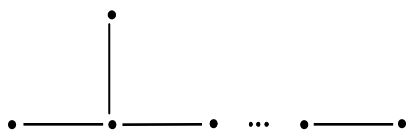
Thus, A is a ring of bounded representation type.

Lemma 4. Let O be a discrete valuation ring with a skew field of fractions D and the radical $R = \pi O = O\pi$. Then the ring

$$A = \begin{pmatrix} H_m(O) & M \\ 0 & T(S_1) \end{pmatrix} \tag{4}$$

corresponding to the diagram:

and $T(S_1)$ is a ring corresponding to the diagram of the poset S_1



where M is a $(H_m(O), T(S_1))$-bimodule, is a ring of bounded representation type for any directions of arrows in the diagram S_1.

Proof. Let M be a finitely generated right A-module that is given by the set:

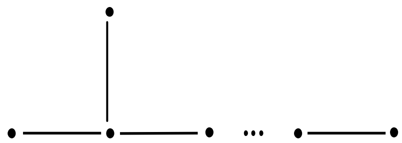
$$\{t_1,\dots,t_m;\ l_1,\dots,l_{n-1};\ \mathrm{T}\}.$$

Renumber the vertices of the poset S in such a way that the point $1 \in S$ has the weight $H_m(O)$.

Then the matrix $\mathbf{T} = (\mathbf{T}_{ij})$ is an upper block-rectangular matrix with entries in D partitioned into n horizontal and n vertical strips in which $\mathbf{T}_{ii}$ are identical matrices ($i = 1,\dots,n$) and in the first horizontal strip all blocks are zero except the first and the i-th blocks.

The matrix of transformations $\mathbf{U} = (\mathbf{U}_{ii})$ is a block diagonal matrix, where $\mathbf{U}_{ii}$ is an invertible matrix with entries in $H_m(O)$, and $\mathbf{U}_{ii}$ ($i = 2,\dots,n$) are invertible matrices with entries in D.

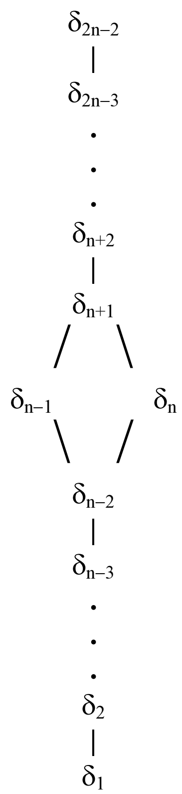
Consider the matrices $\mathbf{T}'$ and $\mathbf{U}'$ that are obtained from the matrices $\mathbf{T}$ and $\mathbf{U}$ correspondingly by erasing the first horizontal and the first vertical strip. Note that the reduction of the matrix $\mathbf{T}'$ by the matrix $\mathbf{U}'$ leads to the representation of the quiver Q of the form:



By the Gabriel theorem, this quiver has $(n-1)n$ indecomposable representations. In accordance with these representations the block $\mathbf{T}_{1i}$ is partitioned into $2(n-1)$ vertical strips:

$\mathbf{B}_1$	$\mathbf{B}_2$	…	$\mathbf{B}_{2n-2}$

Moreover, the admissible transformations on the columns of blocks $\mathbf{B}_i$ are in one-to-one correspondence with the poset $S_2 = \{\delta_1,\delta_2,\ldots,\delta_{2n-2}\}$ whose diagram has the following form:



That is, if $\delta_i \leq \delta_j$ in S_2 then any column in the block $\mathbf{B}_i$ can be added to any row of the block $\mathbf{B}_j$. Taking into account the form of the matrix $\mathbf{T}$, we obtain that our problem leads to the following matrix problem.

Given a rectangular matrix $\mathbf{B}$ *with entries in a skew field D which is partitioned into* $2(n-1)$ *vertical strips and m horizontal strips.*

Over this matrix one performs admissible transformations of the following types:

1) *left O-elementary transformations of rows inside any horizontal strip;*
2) *right D-elementary transformations of columns inside any vertical strip;*
3) *addition of rows of the i-th horizontal strip multiplied on the left by elements of O to rows of the j-th horizontal strip, if* $\delta_i \leq \delta_j$ *in* S_2;
4) *addition of columns of the i-th vertical strip multiplied on the right by elements of D to columns of the j-th vertical strip, if* $\delta_i \leq \delta_j$ *in* S_2;
5) *addition of rows of the j-th horizontal strip multiplied on the left by elements of* $R = \mathrm{rad}\, O$ *to rows of the i-th horizontal strip, if* $i \leq j$.

Using these transformations the matrix $\mathbf{B}$ can be reduced to the form in which the corresponding matrix $\mathbf{T}_{1i}$ is decomposed into a direct sum of the matrices of the following form:

$$[\mathbf{E}\,|\,\mathbf{O}]; \qquad [\mathbf{E}\,|\,\mathbf{E}]; \qquad \left[\begin{array}{c|c} \mathbf{E} & \pi^k\mathbf{E} \\ \mathbf{O} & \mathbf{E} \end{array}\right]$$

Thus, A is a ring of bounded representation type.

Now the sufficiency of the main theorem [10, Theorem 1] follows from theorem 5 and lemmas 1-4.

Conclusions

In this paper we prove the sufficiency in the main theorem [10, Theorem 1] for the right hereditary semiperfect semidistributive ring $A = A(\mathrm{S}, O)$. Taking into account that the necessity in this theorem was proved in the previous paper [10], the full proof of this theorem is obtained in the case when all discrete valuation rings corresponding to minimal elements of the poset S are the same. This theorem gives the structure of these rings in terms of Dynkin diagrams, and also of diagrams with weights that were introduced in [14]. The proof of this theorem was obtained using the reduction to mixed matrix problems over a discrete valuation ring and its skew field of fractions and using the results about representations of quivers and species in [1, 6-8].

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