Journal of Applied Mathematics and Computational Mechanics 2013, 12(1), 121-130

M/M/n/(m,V) QUEUEING SYSTEMS WITH A REJECTION MECHANISM BASED ON AQM

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Abstract. $M/M/n/(m,V)$ queueing systems with service time independent of customer volume are well known models used in computer science. In real computer systems (computer networks etc.) we often deal with the overload problem. In computer networks we solve the problem using AQM techniques, which are connected with introducing some accepting function that lets us reject in random way some part of the arriving customers. It causes reduction of each customer's mean waiting time and let us avoid jams in consequence. Unfortunately, in this way the loss probability increases. In this paper we investigate the analogous model based on some generalization of $M/M/n/(m,V)$ queueing system. We obtain formulas for a stationary number of customers distribution function and loss probability and we do some computations in special cases.

Keywords: *Markovian process, queueing systems with non-homogeneous customers, active queue management*

Introduction

We investigate some modification of $M/M/n/(m,V)$ queueing system [1]. Denote as ζ the random size of a customer (packet) arriving to the system and as $L(x) = P\{\zeta < x\}$ - the distribution function of this random variable. Assume that a customer's service time and its volume are independent and the summary volume of all customers present in the system at time instant t $\sigma(t)$ is limited by constant value $V > 0$ i.e. $\sigma(t) \leq V$ for the arbitrary t. In addition, we assume that every customer arriving to the system can belong to one of the following classes: 1) customers from the first class are always accepted if there are free servers or free waiting places and there is enough free memory space; 2) customers from the second class can be rejected even if there are free servers or free waiting places and enough free memory space to accept them on the base of accepting function $H(x)$ which is non-increasing, left-sided continuous and satisfies conditions $H(0) \leq 1$, $H(V) \geq 0$ and $H(x) = 0$ for $x > V$.

If we compare the performance of the above system to the classical $M/M/n/m$ one [2, 3] we notice that there is no difference at the moments of the end of service, but at the moments when the customers arrive to the system we have to take into account the possibility of a customer's loss that is connected with limitation of total volume and (for customers of the second class) rejection mechanism. More precisely, the arriving customer from the first class is lost if there are no free devices and no waiting places or in the case of $x + y > V$, where x denotes the size of the arriving customer and y denotes the summary volume of all customers present in the system before the arriving moment, whereas the customer from the second class is accepted to the system with probability $H(x + y)$, if there are free devices or free waiting places and in the case of $x + y \le V$.

1. Random process and functions describing the performance of the system

Let us denote by $\eta(t)$ the number of customers present in the system at time instant t. Let $\sigma_1(t), \sigma_2(t), ..., \sigma_{\eta(t)}(t)$ be the volumes of customers numbered $1,2,...,\eta(t)$ in accordance of their arrival order; a be the entrance rate of the Poisson arriving process; μ be the parameter of exponential distributing service time; q $(q \in [0,1])$ be the probability that the arriving customer belongs to the first class.

Then the system under consideration can be described by the following Markovian process:

$$\left(\eta(t), \sigma_1(t), \sigma_2(t), ..., \sigma_{\eta(t)}(t)\right). \tag{1}$$

Process (1) can be characterized by the following functions:

$$P_k(t) = P\{\eta(t) = k\}, k = \overline{0, n+m}; \tag{2}$$

$$G_k(y,t)dy = P\{\eta(t) = k, \sigma(t) \in [y, y+dy)\}, k = \overline{1, n+m}, \tag{3}$$

where $\sigma(t) = \sum_{i=1}^{\eta(t)} \sigma_i(t)$.

It is obvious that for $k = \overline{1, n+m}$ we have

$$P_k(t) = \int_0^V G_k(y,t)dy. \tag{4}$$

In the steady state (if $t \to \infty$) that exists if only $\rho = a/(n\mu) < \infty$ we can introduce the limits of the functions (2)-(3)

$$p_k = P\{\eta = k\}, k = \overline{0, n+m}; \tag{5}$$

$$g_k(y)dy = P\{\eta = k, \sigma \in [y, y+dy)\}, k = \overline{1, n+m}\,, \tag{6}$$

where η, σ are the stationary equivalents of processes $\eta(t)$ and $\sigma(t)$ consequently ($\eta(t) \Rightarrow \eta$ and $\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergence).
It is obvious that in the steady state we obtain the formula

$$p_k = \int_0^V g_k(y)dy, k = \overline{1, n+m}\,. \tag{7}$$

2. Equations and solution

If we analyse the behaviour of process (1) we can obtain the following equations:

$$P_0^{'}(t) = -aP_0(t)\left[qL(V) + (1-q)\int_0^V H(x)dL(x)\right] + \mu P_1(t)\,; \tag{8}$$

$$\begin{aligned} P_1^{'}(t) &= aP_0(t)\left[qL(V) + (1-q)\int_0^V H(x)dL(x)\right] - \\ &- a\left\{q\int_0^V G_1(y,t)L(V-y)dy + (1-q)\int_{y=0}^{V} G_1(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} - \\ &- \mu P_1(t) + 2\mu P_2(t)\,; \end{aligned} \tag{9}$$

$$\begin{aligned} P_k^{'}(t) &= a\left\{q\int_0^V G_{k-1}(y,t)L(V-y)dy + \right. \\ &\left. + (1-q)\int_{y=0}^{V} G_{k-1}(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} - \\ &- a\left\{q\int_0^V G_k(y,t)L(V-y)dy + (1-q)\int_{y=0}^{V} G_k(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} - \\ &- k\mu P_k(t) + (k+1)\mu P_{k+1}(t), k = \overline{2, n-1}\,; \end{aligned} \tag{10}$$

$$P_k'(t) = a\left\{q\int_0^V G_{k-1}(y,t)L(V-y)dy + \right.$$
$$\left. + (1-q)\int_{y=0}^{V} G_{k-1}(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} -$$
$$- a\left\{q\int_0^V G_k(y,t)L(V-y)dy + (1-q)\int_{y=0}^{V} G_k(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} -$$
$$- n\mu P_k(t) + n\mu P_{k+1}(t), k = \overline{n, n+m-1}; \tag{11}$$

$$P_{n+m}'(t) = a\left\{q\int_0^V G_{n+m-1}(y,t)L(V-y)dy + \right.$$
$$\left. + (1-q)\int_{y=0}^{V} G_{n+m-1}(y,t)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} - n\mu P_{n+m}(t). \tag{12}$$

In the steady state from equations (8)-(12) we obtain the following equations:

$$0 = -ap_0\left[qL(V) + (1-q)\int_0^V H(x)dL(x)\right] + \mu p_1; \tag{13}$$

$$0 = ap_0\left[qL(V) + (1-q)\int_0^V H(x)dL(x)\right] -$$
$$- a\left\{q\int_0^V g_1(y)L(V-y)dy + (1-q)\int_{y=0}^{V} g_1(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} -$$
$$- \mu p_1 + 2\mu p_2; \tag{14}$$

$$0 = a\left\{q\int_0^V g_{k-1}(y)L(V-y)dy + (1-q)\int_{y=0}^{V} g_{k-1}(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} -$$
$$- a\left\{q\int_0^V g_k(y)L(V-y)dy + (1-q)\int_{y=0}^{V} g_k(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy\right\} -$$
$$- k\mu p_k + (k+1)\mu p_{k+1}, k = \overline{2, n-1}; \tag{15}$$

$$0 = a\left\{ q\int_0^V g_{k-1}(y)L(V-y)dy + (1-q)\int_{y=0}^V g_{k-1}(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy \right\} -$$

$$- a\left\{ q\int_0^V g_k(y)L(V-y)dy + (1-q)\int_{y=0}^V g_k(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy \right\} -$$

$$- n\mu p_k + n\mu p_{k+1},\ k = \overline{n, n+m-1}; \tag{16}$$

$$0 = a\left\{ q\int_0^V g_{n+m-1}(y)L(V-y)dy + (1-q)\int_{y=0}^V g_{n+m-1}(y)\left[\int_{x=0}^{V-y} H(x+y)dL(x)\right]dy \right\} -$$

$$- n\mu p_{n+m}. \tag{17}$$

Introduce the following notation: $R(z) = \int_0^z H(V-z+x)dL(x)$. Then equations (13)-(17) can be rewritten as it follows:

$$0 = -ap_0\big[qL(V) + (1-q)R(V)\big] + \mu p_1; \tag{18}$$

$$0 = ap_0\big[qL(V) + (1-q)R(V)\big] -$$

$$- a\left[q\int_0^V g_1(y)L(V-y)dy + (1-q)\int_0^V g_1(y)R(V-y)dy) \right] - \mu p_1 + 2\mu p_2; \tag{19}$$

$$0 = a\left[q\int_0^V g_{k-1}(y)L(V-y)dy + (1-q)\int_0^V g_{k-1}(y)R(V-y)dy \right] -$$

$$- a\left[q\int_0^V g_k(y)L(V-y)dy + (1-q)\int_0^V g_k(y)R(V-y)dy \right] -$$

$$- k\mu p_k + (k+1)\mu p_{k+1},\ k = \overline{2, n-1}; \tag{20}$$

$$0 = a\left[q\int_0^V g_{k-1}(y)L(V-y)dy + (1-q)\int_0^V g_{k-1}(y)R(V-y)dy \right] -$$

$$- a\left[q\int_0^V g_k(y)L(V-y)dy + (1-q)\int_0^V g_k(y)R(V-y)dy \right] -$$

$$- n\mu p_k + n\mu p_{k+1},\ k = \overline{n, n+m-1}; \tag{21}$$

$$0 = a\left[q\int_0^V g_{n+m-1}(y)L(V-y)dy + (1-q)\int_0^V g_{n+m-1}(y)R(V-y)dy \right] - n\mu p_{n+m}\,. \quad (22)$$

If we denote: $\Phi(x) = qL(x) + (1-q)R(x)$ then equations (18)-(22) take the form:

$$0 = -ap_0\Phi(V) + \mu p_1\,; \quad (23)$$

$$0 = ap_0\Phi(V) - a\int_0^V g_1(y)\Phi(V-y)dy - \mu p_1 + 2\mu p_2\,; \quad (24)$$

$$0 = a\int_0^V g_{k-1}(y)\Phi(V-y)dy -$$
$$- a\int_0^V g_k(y)\Phi(V-y)dy - k\mu p_k + (k+1)\mu p_{k+1}\,, k = \overline{2, n-1}\,; \quad (25)$$

$$0 = a\int_0^V g_{k-1}(y)\Phi(V-y)dy -$$
$$- a\int_0^V g_k(y)\Phi(V-y)dy - n\mu p_k + n\mu p_{k+1}\,, k = \overline{n, n+m-1}\,; \quad (26)$$

$$0 = a\int_0^V g_{n+m-1}(y)\Phi(V-y)dy - n\mu p_{n+m}\,. \quad (27)$$

Denote as $F_*^{(k)}(x)$ the k-th order Stieltjes convolution of distribution functions $F(x)$ and in addition we introduce the following notation:

$$N(k) = \begin{cases} \dfrac{(n\rho)^k}{k!}, & k = \overline{1, n-1} \\ \dfrac{n^n \rho^k}{n!}, & k = \overline{n, n+m} \end{cases}.$$

By direct substitution we can check that the solution of the system (23)-(27) has the form

$$g_k(y)dy = p_0 N(k) d\Phi_*^{(k)}(y), k = \overline{1, n+m}\,. \quad (28)$$

Using (7), we finally obtain the formula

$$p_k = p_0 N(k)\Phi_*^{(k)}(V), k = \overline{1, n+m}\,. \quad (29)$$

On the base of normalisation condition we additionally obtain

$$p_0 = \left[1 + \sum_{k=1}^{n+m} N(k)\Phi_*^{(k)}(V)\right]^{-1}. \tag{30}$$

3. Loss probability

If we consider systems with losses of customers, we usually obtain formula for loss probability p_L using equilibrium condition [4] which has the following probabilistic sense: in the steady state mean number of customers that arrive to the system and are not lost in the fixed time unit is equal to mean number of customers serviced during this time unit. In this case the equilibrium condition for analysed system has the form

$$a(1 - p_L) = \mu \sum_{k=1}^{n-1} kp_k + n\mu\left(1 - \sum_{k=0}^{n-1} p_k\right). \tag{31}$$

The solution of (31) leads to the following formula:

$$p_L = 1 - (n\rho)^{-1} \sum_{k=1}^{n-1} kp_k - \rho^{-1}\left(1 - \sum_{k=0}^{n-1} p_k\right), \tag{32}$$

where p_k are definite by (29).

4. Generalization

The analysed model may be generalized. Consider an analogous system in which every customer may belong to one of n classes that vary because of different accepting function. Denote as $H_i(x)$ accepting function for the i-th class ($i = \overline{1,n}$) and as q_i - the probability that the arriving customer belongs to the i-th class. If we want the customers of chosen class to be never rejected (if there is enough memory to accept it) we may assume that $H_i(x) \equiv 1$ for $x \in [0,V]$ (as it was in the system analysed above). If we do an analogous computation we also obtain formulas (29) and (32) but the formula for $\Phi(x)$ has the form

$$\Phi(x) = \sum_{i=1}^{n} q_i R_i(x),$$

where $R_i(z) = \int_0^z H_i(V - z + x)dL(x)$.

5. Analysis of some special cases

5.1. System M/M/n/(0,V)

Assume that $m = 0$ and sizes of arriving customers have the exponential distribution function with the parameter f i.e. $L(x) = 1 - e^{-fx}$ ($f > 0$). In addition we suppose that accepting function is constant i.e. $H(x) = \text{const} = h_0$ for $x \in [0, V]$. Then we obtain the formula

$$p_k = p_0 \frac{(n\rho)^k}{k!} \left(1 - e^{-fV}\right)^k \left(-1 + e^{fV} - fV\right)\left(q + (1-q)h_0\right)^k, k = \overline{1, n+m}. \quad (33)$$

If we, for example, assume that $n = 2$, $\rho = 1$, $q = 0.5$, $f = 1$ and $h_0 = 0.3$, we obtain, using (32), the results presented in Table 1.

Table 1

Loss probability in M/M/n/(2,V) system with rejection mechanism based on AQM

V	p_L
0	1.000000
1	0.857118
2	0.694752
3	0.600905
4	0.557482
5	0.538539
6	0.530445
7	0.527037
8	0.525620
9	0.525038
10	0.524801

5.2. System M/M/1/(∞ ,V)

Assume that $n = 1$ and $m = \infty$ and sizes of arriving customers have also the exponential distribution function with the parameter f and the accepting function is also constant ($H(x) = \text{const} = h_0$). Then we obtain the following formula for loss probability p_L:

$$p_L = \frac{1 - \rho\left(q + (1-q)h_0\right)}{e^{fV\left(1 - \rho\left(q + (1-q)h_0\right)\right)} - \rho\left(q + (1-q)h_0\right)}. \quad (34)$$

If we, for example, assume that $\rho = 2$, $q = 0.5$, $f = 1$ and $h_0 = 0.6$, we obtain, using (34), the results presented in Table 2.

Table 2

Loss probability in M/M/1/(∞ ,V) system with rejection mechanism based on AQM

V	p_L
0	1.000000
1	0.570783
2	0.461963
3	0.418206
4	0.397540
5	0.387044
6	0.381515
7	0.378548
8	0.376939
9	0.376062
10	0.375582

Conclusions

In the presented paper we investigate *M*/*M*/*n*/(*m*,*V*) queueing system with a rejection mechanism based on AQM technique. The formulas for a stationary number of customers present in the system and loss probability were obtained. Unfortunately, we assumed that distribution function of every arriving customer's size is the same. In real situations the distribution function should be different for every class but then from the mathematical point of view the analysis seems to be impossible. In addition, we only obtained general formulas for loss probability, not formulas for loss probabilities for each class which may be more interesting but may also be complicated.

The future task is to analyse some more special cases in which we are able to obtain function $\Phi_*^{(k)}(V)$ in its exact form or we may use numerical techniques to approximate loss probabilities in the other cases. It is also worth analysing the influence rejection of mechanism on mean waiting time of the arriving (and not lost) customers.

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