FREE LONGITUDINAL VIBRATION OF A DOUBLE-NANOROD SYSTEM

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Abstract. In this paper a solution to the problem of the free longitudinal vibration of a double-nanorod-system (DNRS) is presented. The nanorods of the system are coupled by many translational springs. The clamped-clamped and clamped-free boundary conditions are employed. The problem of vibration is solved by using the Green's function method. The natural frequencies were numerically calculated.

Keywords: *double-nanorod system, longitudinal vibration, Green's function method*

Introduction

The theory of nonlocal elasticity is often used for the analysis of vibration and instability of nanostructures like: nanorods, nanotubes, nanobeams, etc. This theory was introduced to nanotechnology by Peddieson et al. [1]. The vibration analysis of nanostructures has been of great interest because of their promising mechanical, chemical, electrical, optical properties and their applications, for example in nanoelectromechanical, nanodevices and nanooptomechanical systems.

The forced axial vibrations of nanorods are induced by the axial external forces. The frequencies of the longitudinal free vibration of nanorods system are important parameters which characterize the behavior of this nanorod during the enforced vibration. The free vibrations of the complex nanorods system were studied in papers [2] by Marmu and Adhikari. The authors present an investigation on the longitudinal vibration of the two nanorods which are coupled by longitudinally directed distributed springs. The nonlocal frequencies of vibration by using an analytical method have been derived. The study was an inspiration for the authors of the present paper to investigate the free vibration of a double-nanorod-system. The consideration deals with the vibration of nanorods coupled by longitudinal directed discrete springs. In order to solve the vibration problem, the Green's function method is applied [3]. The problem of free vibration to the similar system as a classical model of a double-rod-system has been presented in reference [4].

1. Formulation of the problem

The system of two nanorods which are coupled by longitudinally directed n-discrete springs is considered. The equations of motion for the longitudinal vibration of the nanorods can be written in the form [1, 2]:

$$
\begin{aligned}
&E_1A_1u_1''(x_1,t)-\rho_1A_1\ddot{u}_1(x_1,t)=\sum_{j=1}^{n}k_j\left[u_1(x_{1j},t)-u_2(x_{2j},t)\right]\delta(x_1-x_{1j})\\
&-(e_0a)^2\sum_{j=1}^{n}k_j\left[u_1''(x_{1j},t)-u_2''(x_{2j},t)\right]\delta(x_1-x_{1j})-(e_0a)^2\rho_1A_1\ddot{u}_1''(x_1,t)\\
&E_2A_2u_2''(x_2,t)-\rho_2A_2\ddot{u}_2(x_2,t)=-\sum_{j=1}^{n}k_j\left[u_1(x_{1j},t)-u_2(x_{2j},t)\right]\delta(x_2-x_{2j})\\
&+(e_0a)^2\sum_{i=1}^{n}k_j\left[u_1''(x_{1j},t)-u_2''(x_{2j},t)\right]\delta(x_2-x_{2j})-(e_0a)^2\rho_2A_2\ddot{u}_2''(x_2,t)
\end{aligned}
\tag{1}
$$

where: $u_i(x,t)$ is the axial displacement, $\rho_i(x)$ is the mass density, $E_i(x)$ is the modulus of elasticity, $A_i(x)$ is the area of cross-section of the i-th nanorod, x_1, x_2 are axial positions along the nanorods, x_{1j}, x_{2j}, $j=1,2...n$ are points of the nanorods which are joined by a j-th spring, e_0 is a constant appropriate to nanorods material and a is an internal characteristic size. When $e_0a=0$, the equations (1) are reduced to equations of classical model of the rods system [3]. The functions $u_i(x,t)$ satisfies the boundary conditions

$$u_i(0,t)=\frac{\partial u_i}{\partial x}(L_i,t)=0; \qquad i=1,2 \tag{2}$$

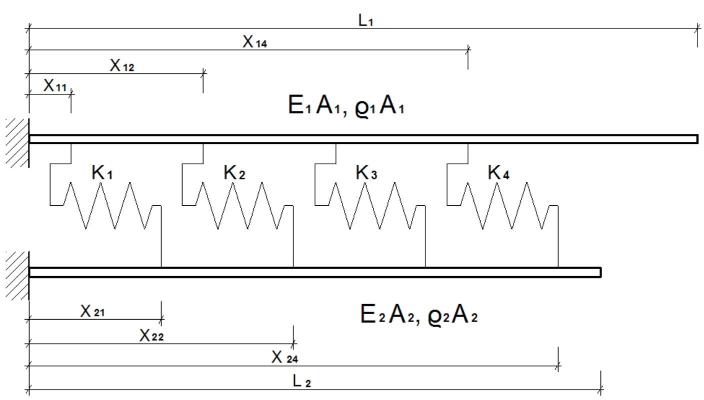

Fig. 1. Double nanorod configuration: clamped-free boundary condition

2. Solution of the problem

In order to find the natural frequencies of the double-nanorods system, one assumes a solution of the problem in the form:

$$u_i\left(x,t\right)=U_i\left(x\right)\cdot\cos\omega t \qquad i=1,2 \tag{3}$$

where ω is the circular frequency. Introducing new variable $\xi_i=\dfrac{x_i}{L}$ into equations (1), the following equations are obtained:

$$\begin{aligned}
&\left(1-\mu^2\Omega^2\right)U_1''\left(\xi_1\right)+\Omega^2 U_1\left(\xi_1\right)=\sum_{j=1}^{n}K_j\left[U_1\left(\xi_{1j}\right)-U_2\left(\xi_{2j}\right)\right]\delta\left(\xi_1-\xi_{1j}\right)\\
&\qquad-\mu^2\sum_{j=1}^{n}K_j\left[U_1''\left(\xi_{1j}\right)-U_2''\left(\xi_{2j}\right)\right]\delta\left(\xi_1-\xi_{1j}\right)\\
&\left(1-r^2\mu^2\Omega^2\right)U_2''\left(\xi_2\right)+r^2\,\Omega^2 U_2\left(\xi_2\right)=-a^2\sum_{j=1}^{n}K_j\left[U_1\left(\xi_{1j}\right)-U_2\left(\xi_{2j}\right)\right]\delta\left(\xi_2-\xi_{2j}\right)\\
&\qquad+a^2\mu^2\sum_{j=1}^{n}K_j\left[U_1''\left(\xi_{1j}\right)-U_2''\left(\xi_{2j}\right)\right]\delta\left(\xi_2-\xi_{2j}\right)
\end{aligned} \tag{4}$$

where:

$$\Omega^2=\frac{\rho_1 A_1 L_1^2}{E_1 A_1}\omega^2,\qquad K_j=\frac{k_j L_1}{E_1 A_1},\qquad a^2=\frac{A_1 E_1 L_2}{A_2 E_2 L_1},\qquad r^2=\frac{\rho_2 E_1 L_2^2}{\rho_1 E_2 L_1^2},\qquad \mu=\frac{e_0 a}{L_1}$$

The functions U_1 and U_2 satisfy the boundary conditions which are obtained from equations (2)-(3)

$$U_i(0)=U_i'(1)=0; \qquad i=1,2 \tag{5}$$

The solution of the boundary problem (4)-(5) can be expressed with the aid of Green's function and has the form:

$$\begin{aligned}
&U_1\left(\xi_1\right)=\sum_{j=1}^{n}K_j\left[U_1\left(\xi_{1j}\right)-U_2\left(\xi_{2j}\right)\right]G_1\left(\xi_1,\xi_{1j}\right)+\\
&\qquad-\mu^2\sum_{j=1}^{n}K_j\left[U_1''\left(\xi_{1j}\right)-U_2''\left(\xi_{2j}\right)\right]G_1\left(\xi_1,\xi_{1j}\right)\\
&U_2\left(\xi_2\right)=-a^2\sum_{j=1}^{n}K_j\left[U_1\left(\xi_{1j}\right)-U_2\left(\xi_{2j}\right)\right]G_2\left(\xi_2,\xi_{2j}\right)\\
&\qquad+a^2\mu^2\sum_{j=1}^{n}K_j\left[U_1''\left(\xi_{1j}\right)-U_2''\left(\xi_{2j}\right)\right]G_2\left(\xi_2,\xi_{2j}\right)
\end{aligned} \tag{6}$$

Assuming $\xi_1 = \xi_{1i}$, $\xi_2 = \xi_{2i}$, $(i = 1, 2, ...n)$ in the equations (6) and in the second order derivative of the functions $U_1(\xi_1)$ and $U_2(\xi_2)$ we obtain a system of equations

$$
\begin{aligned}
U_1(\xi_{1i}) &= \sum_{j=1}^{n} K_j \left[U_1(\xi_{1j}) - U_2(\xi_{2j})\right] G_1(\xi_{1i}, \xi_{1j}) + \\
&\quad - \mu^2 \sum_{j=1}^{n} K_j \left[U_1''(\xi_{1j}) - U_2''(\xi_{2j})\right] G_1(\xi_{1i}, \xi_{1j}) \\
U_2(\xi_{2i}) &= -a^2 \sum_{j=1}^{n} K_j \left[U_1(\xi_{1j}) - U_2(\xi_{2j})\right] G_2(\xi_{2i}, \xi_{2j}) \\
&\quad + a^2 \mu^2 \sum_{j=1}^{n} K_j \left[U_1''(\xi_{1j}) - U_2''(\xi_{2j})\right] G_2(\xi_{2i}, \xi_{2j})
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
U_1''(\xi_{1i}) &= \sum_{j=1}^{n} K_j \left[U_1(\xi_{1j}) - U_2(\xi_{2j})\right] \frac{\partial^2 G_1}{\partial \xi_1^2}(\xi_{1i}, \xi_{1j}) + \\
&\quad - \mu^2 \sum_{j=1}^{n} K_j \left[U_1''(\xi_{1j}) - U_2''(\xi_{2j})\right] \frac{\partial^2 G_1}{\partial \xi_1^2}(\xi_{1i}, \xi_{1j}) \\
U_2''(\xi_{2i}) &= -a^2 \sum_{j=1}^{n} K_j \left[U_1(\xi_{1j}) - U_2(\xi_{2j})\right] \frac{\partial^2 G_2}{\partial \xi_2^2}(\xi_{2i}, \xi_{2j}) \\
&\quad + a^2 \mu^2 \sum_{j=1}^{n} K_j \left[U_1''(\xi_{1j}) - U_2''(\xi_{2j})\right] \frac{\partial^2 G_2}{\partial \xi_2^2}(\xi_{2i}, \xi_{2j})
\end{aligned}
\tag{8}
$$

After substracting of the equations (7) and the equations (8) we have a system

$$
\begin{aligned}
V_i &= \sum_{j=1}^{n} K_j (V_j - \mu^2 W_j) A_{ij} \\
W_i &= \sum_{j=1}^{n} K_j (V_j - \mu^2 W_j) B_{ij}; \qquad i = 1, 2, ...n
\end{aligned}
\tag{9}
$$

where: $V_i = U_1(\xi_{1i}) - U_2(\xi_{2i})$, $W_i = U_1''(\xi_{1i}) - U_2''(\xi_{2i})$

$$
A_{ij} = G_1(\xi_{1i}, \xi_{1j}) + a^2\, G_2(\xi_{2i}, \xi_{2j}); \quad B_{ij} = \frac{\partial^2 G_1}{\partial \xi_{1i}^2}(\xi_{1i}, \xi_{1j}) + a^2 \frac{\partial^2 G_2}{\partial \xi_{2i}^2}(\xi_{2i}, \xi_{2j})
$$

The equation system (9) can be written in the matrix form

$$
\mathbf{D} \cdot \mathbf{Z} = \mathbf{0} \tag{10}
$$

where: $\mathbf{Z} = [V_1 \;\; V_2 \;\; V_3 \;\; .... \;\; V_n \;\; W_1 \;\; W_2 \;\; W_3 \;\; .... \;\; W_n]^T$; $\mathbf{D} = \begin{bmatrix} \mathbf{A}\text{-}\mathbf{E} & -\mu^2 \mathbf{A} \\ \mathbf{B} & \mu^2 \mathbf{B}\text{-}\mathbf{E} \end{bmatrix}$

$$\mathbf{A}=\begin{bmatrix} K_1A_{11} & \cdots & K_nA_{1n} \\ K_1A_{21} & \cdots & K_nA_{2n} \\ \vdots & \vdots & \vdots \\ K_1A_{n1} & \ldots & K_nA_{nn} \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} K_1B_{11} & \cdots & K_nB_{1n} \\ K_1B_{21} & \cdots & K_nB_{2n} \\ \vdots & \vdots & \vdots \\ K_1B_{n1} & \ldots & K_nB_{nn} \end{bmatrix}$$

The non-trivial solutions of equation (10) exist if and only if the determinant of matrix **D** is zero

$$\det \boldsymbol{D} = 0 \tag{11}$$

The roots Ω_k of equation (11) are called natural frequencies and can be determined numerically.

3. The Green's function determination

The Green's function $G_1(\xi_1,\xi_{1j})$ is a solution of the following boundary problem [3]:

$$(1-\mu^2\Omega^2)G_{1,\xi\xi}+\Omega^2 G_1=\delta(\xi_1-\eta_1) \tag{12a}$$

$$G_1\Big|_{\xi_1=0}=G_1\Big|_{\xi_1=1}=0 \tag{12b}$$

We search the function G_1 in the form

$$G_1(\xi_1,\eta_1)=G_{10}(\xi_1,\eta_1)+G_{11}(\xi_1,\eta_1)\cdot H(\xi_1-\eta_1) \tag{13}$$

where $H(\xi_1-\eta_1)$ is the Heaviside function.

It can be shown that both functions G_{10} and G_{11} satisfy the homogeneous differential equation:

$$(1-\mu^2\Omega^2)G_{1l,\xi_1\xi_1}+\Omega^2 G_{1l}=0; \qquad l=1,2 \tag{14}$$

Moreover, the function G_{11} satisfies the conditions

$$\begin{aligned} &G_{11}(\eta_1,\eta_1)=0 \\ &G_{11,\xi_1}\Big|_{\xi_1=\eta_1}=\frac{1}{1-\mu^2\Omega^2} \end{aligned} \tag{15}$$

The solution of the boundary problem for G_{11} is

$$G_{11}(\xi_1,\eta_1)=\frac{-\sin\nu_1\eta_1}{\nu_1(1-\mu^2\Omega^2)}\cos\nu_1\xi_1+\frac{\cos\nu_1\eta_1}{\nu_1(1-\mu^2\Omega^2)}\sin\nu_1\xi_1 \tag{16}$$

where $\nu_1^2=\dfrac{\Omega^2}{1-\mu^2\Omega^2}$. It results that the general solution of differential equation (12) can be written in the form:

$$G_1(\xi_1,\eta_1)=C_1\cos\nu_1\xi_1+C_2\sin\nu_1\xi_1+G_{11}(\xi_1,\eta_1)\cdot H(\xi_1-\eta_1) \tag{17}$$

The constants C_1 and C_2 are determined from boundary conditions (12b). Finally we have

$$\begin{aligned}G_1(\xi_1,\eta_1)=\frac{1}{\nu_1(1-\mu^2\Omega^2)}\Big[-\sin\nu_1(L_1-\eta_1)\cdot\frac{\sin\nu_1\xi_1}{\sin\nu_1L_1}+\\+\sin(\nu_1(\xi_1-\eta_1))H(\xi_1-\eta_1)\Big]\end{aligned} \tag{18}$$

The Green's function G_2 we find by replacing Ω^2 by $r^2\Omega^2$ in Eq. (18):

$$\begin{aligned}G_2(\xi_2,\eta_2)=\frac{1}{\nu_2(1-\mu^2r^2\Omega^2)}\Big[-\sin\nu_2(L_2-\eta_2)\cdot\frac{\sin\nu_2\xi_2}{\sin\nu_2L_2}+\\+\sin(\nu_2(\xi_2-\eta_2))H(\xi_2-\eta_2)\Big]\end{aligned}$$

where $\nu_2^2=\dfrac{r^2\Omega^2}{1-r^2\mu^2\Omega^2}$.

The Green's functions G_1 and G_2 corresponding to a nanorod clamped at the left end and free at the right end ($G_1\big|_{\xi_1=0}=\dfrac{\partial G_1}{\partial\xi}\big|_{\xi_1=L_1}=0$) are:

$$\begin{aligned}G_1(\xi_1,\eta_1)=\frac{-1}{\nu_1(1-\mu^2\Omega^2)}\Big[\cos\nu_1(L_1-\eta_1)\cdot\frac{\sin\nu_1\xi_1}{\cos\nu_1L_1}+\\+\sin(\nu_1(\xi_1-\eta_1))H(\xi_1-\eta_1)\Big]\\G_2(\xi_2,\eta_2)=\frac{-1}{\nu_2(1-\mu^2r^2\Omega^2)}\Big[\cos\nu_2(L_2-\eta_2)\cdot\frac{\sin\nu_2\xi_2}{\cos\nu_2L_2}+\\+\sin(\nu_2(\xi_2-\eta_2))H(\xi_2-\eta_2)\Big]\end{aligned} \tag{19}$$

3. Numerical example

Numerical results have been obtained for a system of two nanorods of identical length and the same physical properties. The system consists of clamped-free nanorods whose free ends are connected by one longitudinally directed spring. Four different values of a spring stiffness coefficient in computation were assumed: $K_1 = 0.1;\ 1;\ 10;\ 100$. For such a system four dimensionless natural vibration frequencies as functions of parameter μ were calculated and these are plotted in Figure 2. The computations have been performed by using the package Maple [5].

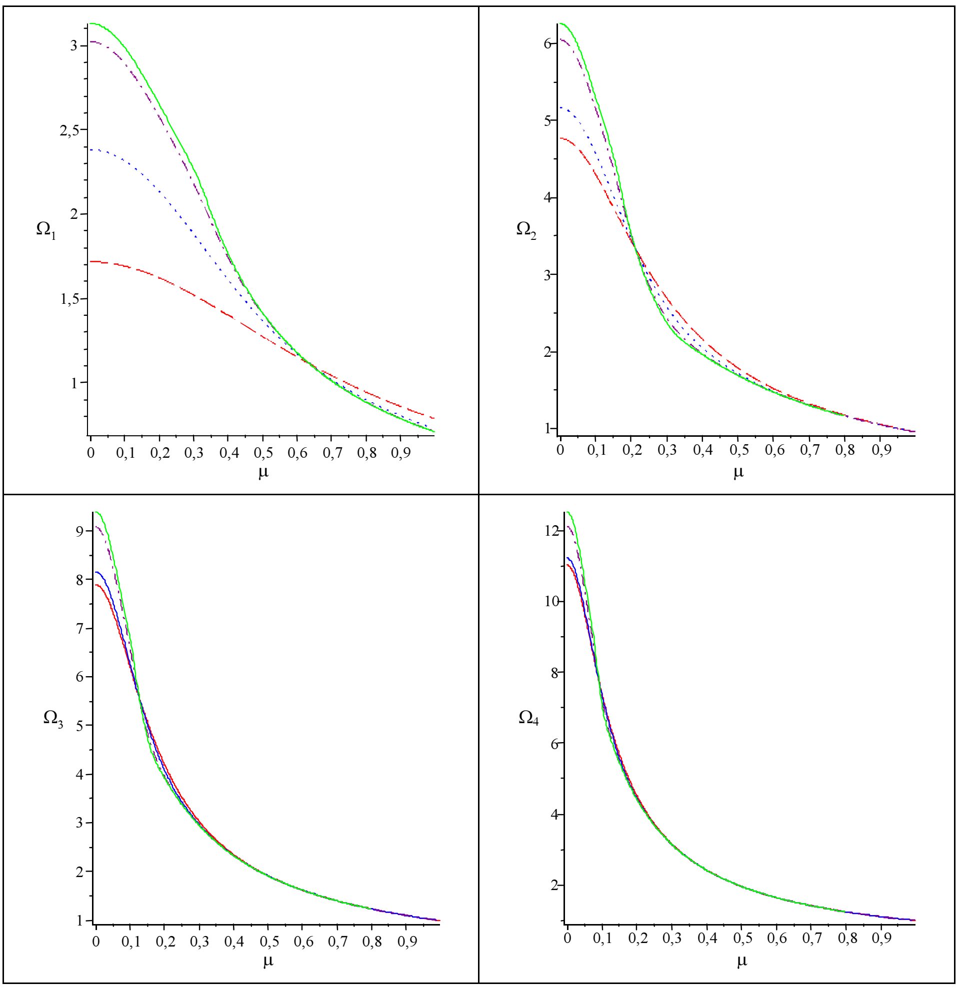

Fig. 2. The first four dimensionless natural vibration frequencies as functions of μ

The figure shows that as the parameter μ increases, the frequencies decrease for all spring stiffnesses considered. The frequencies Ω_n obtained for $\mu = 0$ correspond to the classical model of the rod system.

Conclusions

The Green function method was applied to solve the problem of longitudinal vibration of a double-nanorod coupled by translational springs. Clamped-clamped and clamped-free boundary conditions were employed. Although the number of coupling springs considered in the presented examples was limited to one, the approach can be used to solve the problems of vibration of systems consisting of many nanorods and coupling springs.

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