FREE VIBRATION ANALYSIS
OF FUNCTIONALLY GRADED BEAMS

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Abstract. In this paper free vibration of axially functionally graded (FG) beams consisting of two segments is studied. The FG beams under consideration are characterized by axially varying cross-section areas and/or functionally grading material properties. Numerical example for a beam that is clamped at both ends is presented.

Keywords: free vibration problem, functionally graded beams

Introduction

FG beams are characterized by uniaxially or spatially variable material properties. A list of papers on FG beams is very extensive. For example in [1] an analytical solution of a static cantilever functionally graded beam is formulated under the assumption that all the elastic moduli of the material have the same variations along the beam-thickness direction. Many papers deal with free and forced vibration analysis of FG beams, e.g. [2, 3]. For axially graded beams similar problems have become more complicated because of the governing equations with functional coefficients. We can mention here [4] where free vibration of exponentially graded beams is analyzed. Such problems are investigated also in [5] by using the Fredholm integral equations method. By expanding the mode shapes as power series, the resulting Fredholm equations were solved. In paper [6] free vibration and stability analysis of axially functionally graded tapered Timoshenko beams are carried out through a finite element approach.

In this contribution the formulation and solution of the vibration problem concerns the beams in the framework of Bernoulli-Euler theory. It is assumed that the changes of the cross-sectional area and material properties in the beam segments have an exponential form. The analytical solution of the problem is used for numerical analysis. The effect of selected parameters characterized the system on the free vibration frequencies of the clamped-clamped beam is investigated.

1. Formulation of the problem

Consider a functionally graded beam of length L (along the x direction) consisting of two segments, having a non-uniform cross-section area $A(x)$, moment of inertia $I(x)$, modulus of elasticity $E(x)$ and material density $\rho(x)$. The equation of motion of the beam is given by [7]

$$\frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2 u}{\partial x^2}\right] + \rho(x)A(x)\frac{\partial^2 u_1}{\partial t^2} = 0 \tag{1}$$

where $u(x,t)$ is the transverse deflection of the beam at the position x and in time t.

It is assumed that

$$E(x)I(x) = \begin{cases} D_1 e^{2\beta_1\frac{x}{L}} & \text{for} \quad 0 \le x \le \eta L \\ D_2 e^{2\beta_2\frac{x}{L}} & \text{for} \quad \eta L < x \le L \end{cases}$$
$$\rho(x)A(x) = \begin{cases} m_1 e^{2\beta_1\frac{x}{L}} & \text{for} \quad 0 \le x \le \eta L \\ m_2 e^{2\beta_2\frac{x}{L}} & \text{for} \quad \eta L < x \le L \end{cases} \tag{2}$$

where $0 < \eta < 1$ and β_1, β_2 are the dimensionless gradient parameters, D_1, D_2, m_1, m_2 are real constants. Substituting (2) into equation (1) and assuming that $u_1(x,t) = u(x,t)$ for $0 \le x \le \eta L$ and $u_2(x,t) = u(x,t)$ for $\eta L < x \le L$, one obtains the governing equations of the beam vibrations in the form

$$\frac{\partial^2}{\partial x^2}\left[D_1 e^{2\beta_1\frac{x}{L}}\frac{\partial^2 u_1}{\partial x^2}\right] + m_1 e^{2\beta_1\frac{x}{L}}\frac{\partial^2 u_1}{\partial t^2} = 0, \qquad 0 \le x \le \eta L$$
$$\frac{\partial^2}{\partial x^2}\left[D_2 e^{2\beta_2\frac{x}{L}}\frac{\partial^2 u_2}{\partial x^2}\right] + m_2 e^{2\beta_2\frac{x}{L}}\frac{\partial^2 u_2}{\partial t^2} = 0, \qquad \eta L < x \le L \tag{3}$$

Using a separation of variables according to

$$u_1(x,t) = U_1(x)\cos\omega t, \qquad u_2(x,t) = U_2(x)\cos\omega t \tag{4}$$

where $U_i(x)$, $i = 1, 2$, are the corresponding amplitude functions and ω is the eigenfrequency of the beam. Substitution of equations (4) into equations (3) yields

$$\frac{d^2}{dx^2}\left[D_1 e^{2\beta_1 \frac{x}{L}} \frac{d^2 U_1}{dx^2}\right] - m_1 \omega^2 L^4 e^{2\beta_1 \frac{x}{L}} U_1 = 0, \qquad 0 \le x \le \eta L$$

$$\frac{d^2}{dx^2}\left[D_2 e^{2\beta_2 \frac{x}{L}} \frac{d^2 U_2}{dx^2}\right] - m_2 \omega^2 L^4 e^{2\beta_2 \frac{x}{L}} U_2 = 0, \qquad \eta L < x \le L$$

(5)

Introducing the non-dimensional coordinate $\xi = \dfrac{x}{L}$ and dimensionless quantities

$\Omega^4 = \dfrac{m_1}{D_1} L^4 \omega^2$, $\mu = \sqrt{\dfrac{m_2 D_1}{m_1 D_2}}$, equations (5) can be written in the form

$$\frac{d^2}{d\xi^2}\left[e^{2\beta_1 \xi} \frac{d^2 U_1}{d\xi^2}\right] - \Omega^4 e^{2\beta_1 \xi} U_1 = 0, \qquad 0 \le \xi \le \eta$$

$$\frac{d^2}{d\xi^2}\left[e^{2\beta_2 \xi} \frac{d^2 U_2}{d\xi^2}\right] - \mu^2 \Omega^4 e^{2\beta_2 \xi} U_2 = 0, \qquad \eta < \xi \le 1$$

(6)

After some transformations we can rewrite equations (6) as follows

$$\frac{d^4 U_1}{d\xi^4} + 4\beta_1 \frac{d^3 U_1}{d\xi^3} + 4\beta_1^2 \frac{d^2 U_1}{d\xi^2} - \Omega^4 U_1 = 0, \qquad 0 \le \xi \le \eta$$

$$\frac{d^4 U_2}{d\xi^4} + 4\beta_2 \frac{d^3 U_2}{d\xi^3} + 4\beta_2^2 \frac{d^2 U_2}{d\xi^2} - \mu^2 \Omega^4 U_2 = 0, \qquad \eta < \xi \le 1$$

(7)

Equations (7) are completed by boundary and continuity conditions. The continuity conditions are

$$U_1\left(\eta^+\right) = U_2\left(\eta^-\right) \qquad \frac{dU_1}{d\xi}\left(\eta^+\right) = \frac{dU_2}{d\xi}\left(\eta^-\right)$$

$$\frac{d^2 U_1}{d\xi^2}\left(\eta^+\right) = \frac{d^2 U_2}{d\xi^2}\left(\eta^-\right) \qquad \frac{d^3 U_1}{d\xi^3}\left(\eta^+\right) = \frac{d^3 U_2}{d\xi^3}\left(\eta^-\right)$$

(8)

and the boundary conditions for clamped-clamped beam are

$$U_1(0) = 0 \qquad \frac{\partial U_1}{\partial \xi}(0) = 0$$

$$U_2(1) = 0 \qquad \frac{\partial U_2}{\partial \xi}(1) = 0$$

(9)

2. Solution of the problem

In this section a solution of the vibration problem (7)-(9) is presented. The general solution of equations (7) has the form

$$U_1(\xi) = e^{-\beta_1\xi}\left(C_1\cos\delta_1\xi + C_2\sin\delta_1\xi + C_3\cosh\delta_2\xi + C_4\sinh\delta_2\xi\right), \quad 0 \le \xi \le \eta$$
$$U_2(\xi) = e^{-\beta_2\xi}\left(C_5\cos\delta_3\xi + C_6\sin\delta_3\xi + C_7\cosh\delta_4\xi + C_8\sinh\delta_4\xi\right), \quad \eta < \xi \le 1 \tag{10}$$

where

$$\delta_1 = \sqrt{\Omega^2 - \beta_1^2}, \quad \delta_2 = \sqrt{\Omega^2 + \beta_1^2}, \quad \delta_3 = \sqrt{\mu\Omega^2 - \beta_2^2}, \quad \delta_4 = \sqrt{\mu\Omega^2 + \beta_2^2} \tag{11}$$

Substituting functions (10) into (8) and (9) we obtain a set of equations with unknowns $C_1,...,C_8$. Let us introduce the following denotations

$$
\begin{aligned}
m_{ki} &= [\beta_k, \delta_i] \cdot \mathbf{v}_i & n_{ki} &= [-\delta_i, \beta_k] \cdot \mathbf{v}_i \\
p_{kj} &= [\beta_k, -\delta_j] \cdot \mathbf{v}_j & r_{kj} &= [-\delta_j, \beta_k] \cdot \mathbf{v}_j \\
\overline{m}_{ki} &= \left[(\beta_k^2 - \delta_i^2), 2\beta_k\delta_i\right] \cdot \mathbf{v}_i & \overline{n}_{ki} &= \left[-2\beta_k\delta_i, (\beta_k^2 - \delta_i^2)\right] \cdot \mathbf{v}_i \\
\overline{p}_{kj} &= \left[(\beta_k^2 + \delta_j^2), -2\beta_k\delta_j\right] \cdot \mathbf{v}_j & \overline{r}_{kj} &= \left[-2\beta_k\delta_j, (\beta_k^2 + \delta_j^2)\right] \cdot \mathbf{v}_j \\
\widetilde{m}_{ki} &= \left[(3\beta_k\delta_i^2 - \beta_k^3), (-3\beta_k^2\delta_i + \delta_i^3)\right] \cdot \mathbf{v}_i & \widetilde{n}_{ki} &= \left[(3\beta_k^2\delta_i - \delta_i^3), (3\beta_k\delta_i^2 - \beta_k^3)\right] \cdot \mathbf{v}_i \\
\widetilde{p}_{kj} &= \left[(-3\beta_k\delta_j^2 - \beta_k^3), (3\beta_k^2\delta_j + \delta_j^3)\right] \cdot \mathbf{v}_j & \widetilde{r}_{kj} &= \left[(3\beta_k^2\delta_j + \delta_j^3), (-3\beta_k\delta_j^2 - \beta_k^3)\right] \cdot \mathbf{v}_j
\end{aligned}
\tag{12}
$$

where $\mathbf{v}_i = [\cos\delta_i\eta, \sin\delta_i\eta]$, $\mathbf{v}_j = [\cosh\delta_j\eta, \sinh\delta_j\eta]$, $k = 1,2$, $i = 1,3$, $j = 2,4$.
Under denotations (12) and $w = e^{(\beta_1-\beta_2)\eta}$ this set of equations can be written in the matrix form

$$\mathbf{A}(\omega) \cdot \mathbf{C} = 0 \tag{13}$$

where $\mathbf{C} = [C_1...C_8]^T$ and

$$
\mathbf{A} = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_1 & 0 & \delta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\delta_3 & \sin\delta_3 & \cosh\delta_4 & \sinh\delta_4 \\
0 & 0 & 0 & 0 & -\delta_3\sin\delta_3 & \delta_3\cos\delta_3 & \delta_4\sinh\delta_4 & \delta_4\cosh\delta_4 \\
\cos\delta_1\eta & \sin\delta_1\eta & \cosh\delta_2\eta & \sinh\delta_2\eta & -w\cos\delta_3\eta & -w\sin\delta_3\eta & -w\cosh\delta_4\eta & -w\sinh\delta_4\eta \\
m_{11} & n_{11} & p_{12} & r_{12} & -wm_{23} & -wn_{23} & -wp_{24} & -wr_{24} \\
\overline{m}_{11} & \overline{n}_{11} & \overline{p}_{12} & \overline{r}_{12} & -w\overline{m}_{23} & -w\overline{n}_{23} & -w\overline{p}_{24} & -w\overline{r}_{24} \\
\widetilde{m}_{11} & \widetilde{n}_{11} & \widetilde{p}_{12} & \widetilde{r}_{12} & -w\widetilde{m}_{23} & -w\widetilde{n}_{23} & -w\widetilde{p}_{24} & -w\widetilde{r}_{24}
\end{bmatrix}
$$

A non-trivial solution to equation (13) exists when the determinant of matrix **A** is equal to zero. Then the frequency equation of the beam vibration is

$$\det \mathbf{A}(\omega) = 0 \tag{14}$$

Equation (14) is then solved numerically.

3. Numerical example

In this section we present some numerical results. Table 1 shows the first four non-dimensional frequency parameters Ω_i (i = 1, 2, 3, 4) of the FG clamped--clamped beam for various values of parameters β_1, β_2 with fixed $\eta = 0.5$ and $\mu = 5$. It can be observed that an increase of the value of β_2 with fixed β_1 causes an increase of the frequency parameters Ω_i (i = 1, 2, 3, 4). Moreover, with an increase in β_1 (β_2 is fixed) the frequency parameters Ω_i (i = 1, 2, 3) are decreasing. The frequency parameter Ω_4 changes non-monotonic. Diagrams of frequency parameter values Ω_i (i = 1, 2, 3) as a functions of η for $\beta_1 = 0.5$, $\beta_2 = 1$ and $\mu = 1.75,\ 2.75,\ 4$, respectively, are presented in Figure 1.

Table 1

The first four non-dimensional free vibration frequencies for $\eta = 0.5$ and $\mu = 5$

| β_2 \ β_1	−1	0	1	2
−1	2.47146	2.23620	1.90799	2.00000
	4.71453	4.54801	4.38694	4.23994
	7.04476	6.94461	6.86846	6.81638
	8.73202	8.67374	8.64946	8.66247
0	2.62109	2.44218	2.22418	2.00000
	4.83775	4.68940	4.54715	4.41785
	7.11824	7.02212	6.95020	6.90233
	8.76384	8.70849	8.68687	8.70233
1	2.77642	2.63385	2.47146	2.28476
	4.97371	4.84086	4.71453	4.60025
	7.20404	7.11210	7.04476	7.00159
	8.80375	8.75122	8.73202	8.74946
2	2.94629	2.82991	2.70269	2.56454
	5.12346	5.00398	4.89131	4.79002
	7.30102	7.21334	7.15087	7.11293
	8.85185	8.80211	8.78498	8.80387
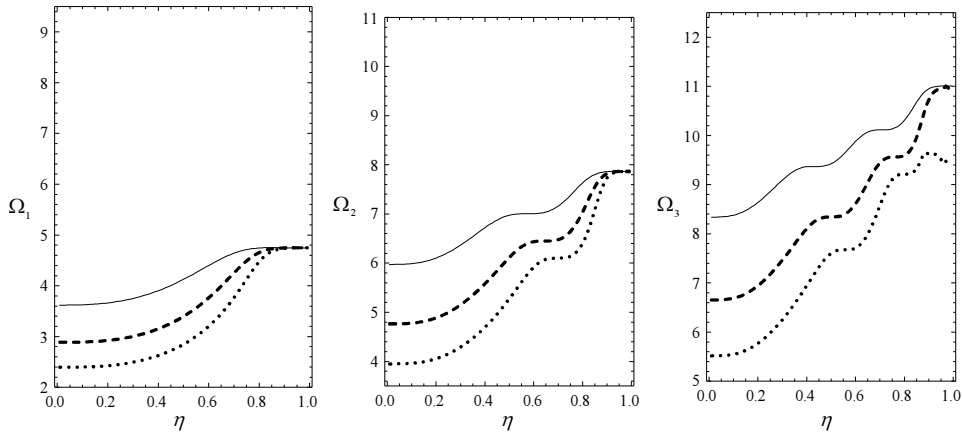
Fig 1. Frequency parameter values Ω_i for the first three modes of vibration as a function
of η for $\mu = 1.75$ (solid line), $\mu = 2.75$ (dashed line), $\mu = 4$ (dotted line)

Conclusions

The free vibration problem of the FG beam consisting of two segments each of exponentially varying cross section areas and material properties is the subject of this paper. For a clamped-clamped beam the characteristic equation is derived. The numerical examples have shown the influence of gradient variation on the frequencies of the FG beam. The proposed approach can be applied to the vibration problem of beam consisting of an arbitrary number of segments.

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