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THE QUEUEING SYSTEM $M_2^X/M/n$ WITH HYSTERETIC CONTROL OF THE INPUT FLOW INTENSITY

***Yuriy Zhernovyi*[1], *Bohdan Kopytko*[2]**

[1] *Ivan Franko National University of Lviv, Lviv, Ukraine*
[2] *Institute of Mathematics, Czestochowa University of Technology Częstochowa, Poland*
[1] *yu_zhernovyi@yahoo.com*, [2] *bohdan.kopytko@im.pcz.pl*

Abstract. We consider a multi-channel queueing system with unlimited queue and with exponentially distributed service time and the intervals between the arrival of customers batches, which uses a hysteretic control mechanism of the input flow intensity. The system receives two independent flows of customers, one of which is blocked in an overload mode. An algorithm for finding the stationary distribution of the number of customers and stationary characteristics (the mean queue length, the mean waiting time in the queue, the probability of customers loss) is proposed. The obtained results are verified with the help of a simulation model constructed with the assistance of GPSS World tools.

Keywords: *multi-channel queueing system, batch arrival of customers, hysteretic control of the input flow intensity, stationary characteristics*

Introduction

For the purpose of preventing overloads in the information and telecommunication systems simulated by means of queueing systems, hysteretic load control with several types of thresholds is used [1]. We can control both an input flow and its parameters, and service process.

According to [1], the queueing systems with hysteretic control may be adequate models for evaluating the quality of functioning of SIP servers under overloads. Detailed description of the use of such systems to the modeling of SIP servers operating under overload conditions is given in [1-3].

Because of the practical importance, a large number of publications is devoted to the study of queueing systems with hysteretic control. In particular, a sufficiently detailed overview of the results obtained in this direction can be found in [4-7]. Most studies examined a single-channel system with an arbitrary distribution of service time.

In this paper we consider a multi-channel queueing system $M_2^X/M/n$ with the independent flows of two types of customers. In this system the control is applied only to the parameters of input flows. Switchings on the overload mode are carried out not at the moments of the end of customers service, and at the moments of arrival of customers in the system. It is more natural to practical applications.

When constructing a random process describing the number of customers in the system, we share its states corresponding to the normal mode and the overload mode. It allows us to write the system of equations for the stationary probabilities and construct an algorithm to solve it. To find the stationary characteristics of the queue, we apply the apparatus of generating functions. Using the same approach in [8] the stationary characteristics of the system $\mathrm{M}^{\mathrm{X}}/\mathrm{M}/1$ with hysteretic control of service intensity are found.

1. Description of the system

Let us consider an n-channel $\mathrm{M}_2^{\mathrm{X}}/\mathrm{M}/\mathrm{n}$ queueing system that receives independent flows of two types of customer batches. With probability a_k (respectively b_k) the number of customers in a batch of the first flow (respectively of the second flow) is equal to k $(k \geq 1)$, and

$$\sum_{k=1}^{\infty} a_k = 1, \quad \sum_{k=1}^{\infty} b_k = 1; \qquad a_{(s)} = \sum_{k=1}^{\infty} k^s a_k < \infty, \quad b_{(s)} = \sum_{k=1}^{\infty} k^s b_k < \infty, \quad s = 1, 2.$$

The time intervals between the time of arrival of customers batches of the flow number k are independent random variables distributed exponentially with parameter λ_k $(k = 1, 2)$. In the total flow being a superposition of the first and second type of flows, the time intervals between the time of arrival of customers batches have exponential distribution with parameter $\lambda = \lambda_1 + \lambda_2$ [9, p. 83]. The service time of each customer is distributed exponentially with parameter μ. The first-in first-out (FIFO) service discipline is used.

A two-threshold hysteretic control mechanism of arriving load is realized in the system. This mechanism operates as follows. We fix two numbers h_1 and h_2, for which the following inequalities $n < h_1 < h_2$ hold. From the moment of arrival in the system of the first customer and until the moment of presence of h_2 customers for the first time, the system is functioning in a normal mode and customers of both types are accepted for service. As soon as in the system there will be h_2 customers, the system passes into an overload mode, accept of customers of the second type stops and customers of the first type are accepted only. This mode continues until the moment when the number of customers in the system is reduced to $(h_1 - 1)$. At this moment the system passes into a normal mode of functioning and starts over again to accept customers of both types. The normal mode operates up to the moment where in the system there will be h_2 customers. Then the system goes into overload mode, again accept of customers of the second type stops and so on.

We assume that the following inequalities $h_1 - n \ge 2$ and $h_2 - h_1 \ge 2$ hold. These assumptions are introduced only in order not to consider cases for which the formulas are different from those shown here, and in no way detract from the generality of the obtained results.

2. Stationary distribution of the number of customers

Let $\alpha = \lambda / \mu$, $\alpha_i = \lambda_i / \mu$ $(i = 1,2)$, $\gamma = \alpha / n$, $\gamma_i = \alpha_i / n$, $(i = 1,2)$, then $\rho = \gamma_1 a_{(1)} + \gamma_2 b_{(1)}$ and $\rho_1 = \gamma_1 a_{(1)}$ are coefficients of system loading in a normal mode and in an overload mode respectively.

Let us enumerate the system's states as follows (see Fig. 1): s_0 corresponds to the empty system; s_i $(1 \le i \le h_2 - 1)$, to the system with i customers that operates in the normal mode; x_i $(i \ge h_1)$, to the system with i customers that operates in the overload mode.

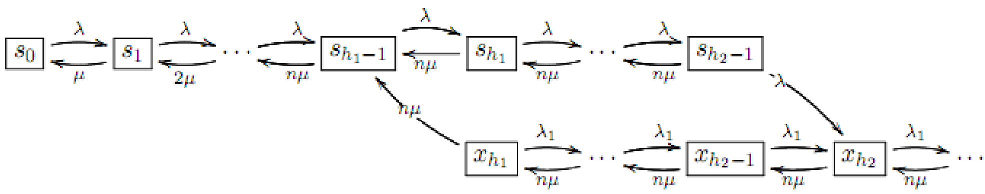

Fig. 1. Diagram of the system's states in the case $a_1 = b_1 = 1$

Let $p_i(t)$ and $q_i(t)$ be the probability of the fact that at instant t the system is in state s_i and x_i respectively. Assuming that the limits $p_i = \lim_{t\to\infty} p_i(t)$ $(0 \le i \le h_2 - 1)$ and $q_i = \lim_{t\to\infty} q_i(t)$ $(i \ge h_1)$ exist (the ergodicity conditions to be obtained below), we can represent the system of equations for stationary probabilities p_i and q_i in the form

$$-\lambda p_0 + \mu p_1 = 0;$$

$$-(\lambda + i\mu) p_i + \sum_{k=0}^{i-1} p_k (\lambda_1 a_{i-k} + \lambda_2 b_{i-k}) + (i+1)\mu p_{i+1} = 0 \quad (1 \le i \le n-1);$$

$$-(\lambda + n\mu) p_i + \sum_{k=0}^{i-1} p_k (\lambda_1 a_{i-k} + \lambda_2 b_{i-k}) + n\mu p_{i+1} = 0 \quad (n \le i \le h_2 - 2;\ \ i \ne h_1 - 1); \quad (1)$$

$$-(\lambda + n\mu) p_{h_1-1} + \sum_{k=0}^{h_1-2} p_k (\lambda_1 a_{h_1-1-k} + \lambda_2 b_{h_1-1-k}) + n\mu (p_{h_1} + q_{h_1}) = 0;$$

$$-(\lambda + n\mu) p_{h_2-1} + \sum_{k=0}^{h_2-2} p_k (\lambda_1 a_{h_2-1-k} + \lambda_2 b_{h_2-1-k}) = 0;$$

$$-(\lambda_1+n\mu)q_{h_1}+n\mu q_{h_1+1}=0;$$

$$-(\lambda_1+n\mu)q_i+\lambda_1\sum_{k=h_1}^{i-1}q_k a_{i-k}+n\mu q_{i+1}=0 \quad (h_1+1\le i\le h_2-1);$$

$$-(\lambda_1+n\mu)q_{h_2}+\lambda_1\sum_{k=h_1}^{h_2-1}q_k a_{h_2-k}+\sum_{k=0}^{h_2-1}p_k(\lambda_1 a_{h_2-k}+\lambda_2 B_{h_2-1-k})+n\mu q_{h_2+1}=0; \tag{2}$$

$$-(\lambda_1+n\mu)q_i+\lambda_1\left(\sum_{k=h_1}^{i-1}q_k+\sum_{k=0}^{h_2-1}p_k\right)a_{i-k}+n\mu q_{i+1}=0 \quad (i\ge h_2);$$

$$\sum_{k=0}^{h_2-1}p_k+\sum_{k=h_1}^{\infty}q_k=1. \tag{3}$$

Here

$$B_i=1-\sum_{k=1}^{i}b_k \quad (i\ge 0), \quad B_0=1.$$

Let us introduce the notations

$$A_i=1-\sum_{k=1}^{i}a_k \quad (i\ge 0), \quad A_0=1; \quad \tilde{p}_k=p_k/p_0 \quad (0\le k\le h_2-1);$$
$$\tilde{q}_k=q_k/p_0 \quad (k\ge h_1); \quad P_s=\sum_{k=0}^{s}\tilde{p}_k; \quad L_n=\sum_{k=1}^{n}k\tilde{p}_k; \quad Q_{h_2-1}=\sum_{k=h_1}^{h_2-1}\tilde{q}_k. \tag{4}$$

Theorem 1. When $a_{(1)}<\infty$, $b_{(1)}<\infty$ and $\rho_1<1$, the stationary probabilities $p_i=p_0\tilde{p}_i$ $(0\le i\le h_2-1)$ and $q_i=p_0\tilde{q}_i$ $(i\ge h_1)$ exist and are determined from the recurrence relations

$$p_0=\frac{n-\alpha_1 a_{(1)}}{nP_n-L_n+\alpha_2\sum_{i=0}^{h_2-1}\sum_{k=0}^{i}\tilde{p}_k B_{i-k}}; \tag{5}$$

$$\tilde{p}_{i+1}=\frac{1}{i+1}\sum_{k=0}^{i}\tilde{p}_k(\alpha_1 A_{i-k}+\alpha_2 B_{i-k}) \qquad (0\le i\le n-1); \tag{6}$$

$$\tilde{p}_{i+1}=\frac{1}{n}\sum_{k=0}^{i}\tilde{p}_k(\alpha_1 A_{i-k}+\alpha_2 B_{i-k}) \qquad (n\le i\le h_1-2); \tag{7}$$

$$\tilde{p}_{i+1}=(1+\gamma)\tilde{p}_i-\sum_{k=0}^{i-1}\tilde{p}_k(\gamma_1 a_{i-k}+\gamma_2 b_{i-k}) \qquad (h_1\le i\le h_2-2); \tag{8}$$

$$\tilde{p}_{h_2-1}=\frac{1}{1+\gamma}\sum_{k=0}^{h_2-2}\tilde{p}_k(\gamma_1 a_{h_2-1-k}+\gamma_2 b_{h_2-1-k}); \tag{9}$$

$$\tilde{q}_{h_1} = \sum_{k=0}^{h_2-1} \tilde{p}_k (\gamma_1 A_{h_2-1-k} + \gamma_2 B_{h_2-1-k}); \tag{10}$$

$$\tilde{q}_{i+1} = \tilde{q}_{h_1} + \gamma_1 \sum_{k=h_1}^{i} \tilde{q}_k A_{i-k} \qquad (h_1 \le i \le h_2 - 1); \tag{11}$$

$$\tilde{q}_{i+1} = \gamma_1 \left(\sum_{k=h_1}^{i} \tilde{q}_k + \sum_{k=0}^{h_2-1} \tilde{p}_k \right) A_{i-k} \qquad (i \ge h_2); \tag{12}$$

and $\tilde{p}_{h_1}$ is determined from the equation

$$\frac{1}{1+\gamma} \sum_{k=0}^{h_2-2} \tilde{p}_k (\gamma_1 a_{h_2-1-k} + \gamma_2 b_{h_2-1-k}) = (1+\gamma)\tilde{p}_{h_2-2} - \sum_{k=0}^{h_2-3} \tilde{p}_k (\gamma_1 a_{h_2-2-k} + \gamma_2 b_{h_2-2-k}). \tag{13}$$

Proof. The successive summation of the first i $(0 \le i \le h_1 - 2)$ equations of system (1) yields relations (6) and (7). We continue this process for $h_1 - 1 \le i \le h_2 - 2,$ to obtain the equalities

$$n\mu(\tilde{p}_{i+1} + \tilde{q}_{h_1}) = \sum_{k=0}^{i} \tilde{p}_k (\lambda_1 A_{i-k} + \lambda_2 B_{i-k}) \qquad (h_1 - 1 \le i \le h_2 - 2), \tag{14}$$

which yield relations (8). Equality (9) follows from the last equation of system (1). We equate the expressions for $\tilde{p}_{h_2-1}$ that follow from (8) and (9) to obtain equation (13). In order to find $\tilde{p}_{h_1}$ it is necessary to express all $\tilde{p}_k$ for $h_1 + 1 \le k \le h_2 - 2$ in this equation by $\tilde{p}_{h_1}$.

The summation of the last equality from (14) and last equation of system (1) yields relation (10). Equalities (11) are obtained as a result of the successive summation of equations (2) for $h_1 \le i \le h_2 - 1$.

Multiplying the last equality from (11) by $n\mu$ and successive summing of the result with equation (2) for $i \ge h_2$, taking into account (10), we obtain relations (12).

Divide both sides of (14) by $n\mu$ and add the result with (11) for $h_1 \le i \le h_2 - 1$. Taking into account the first of the relations (14), we obtain the equalities

$$\tilde{p}_{i+1} + \tilde{q}_{i+1} = \gamma_1 \left(\sum_{k=0}^{i} \tilde{p}_k A_{i-k} + \sum_{k=h_1}^{i} \tilde{q}_k A_{i-k} \right) + \gamma_2 \sum_{k=0}^{i} \tilde{p}_k B_{i-k} \quad (h_1 - 1 \le i \le h_2 - 2). \tag{15}$$

In order to find p_0, first with the help of (6), (7), (15), (11) (taking into account (10)) and (12) we obtain the equality

$$\sum_{k=1}^{n} kp_k + n\left(\sum_{k=n+1}^{h_2-1} p_k + \sum_{k=h_1}^{\infty} q_k\right) =$$
$$= \alpha_1\left(\sum_{i=0}^{h_2-1}\sum_{k=0}^{i} + \sum_{i=h_2}^{\infty}\sum_{k=0}^{h_2-1}\right) p_k A_{i-k} + \alpha_2 \sum_{i=0}^{h_2-1}\sum_{k=0}^{i} p_k B_{i-k} + \alpha_1 \sum_{i=h_1}^{\infty}\sum_{k=h_1}^{i} q_k A_{i-k}. \quad (16)$$

Then taking into account the definitions of A_i, B_i, $a_{(1)}$, $b_{(1)}$ and normalization condition (3) we calculate the sums

$$\left(\sum_{i=0}^{h_2-1}\sum_{k=0}^{i} + \sum_{i=h_2}^{\infty}\sum_{k=0}^{h_2-1}\right) p_k A_{i-k} = \sum_{i=0}^{h_2-1} p_i \sum_{k=0}^{\infty} A_k = a_{(1)} \sum_{k=0}^{h_2-1} p_k;$$
$$\sum_{i=h_1}^{\infty}\sum_{k=h_1}^{i} q_k A_{i-k} = \sum_{i=h_1}^{\infty} q_i \sum_{k=0}^{\infty} A_k = a_{(1)}\left(1 - \sum_{k=0}^{h_2-1} p_k\right). \quad (17)$$

In view of (17) we rewrite (16) as

$$\alpha_1 a_{(1)} + \alpha_2 \sum_{i=0}^{h_2-1}\sum_{k=0}^{i} p_k B_{i-k} = \sum_{k=1}^{n} kp_k + n\left(\sum_{k=n+1}^{h_2-1} p_k + 1 - \sum_{k=0}^{h_2-1} p_k\right) = \sum_{k=1}^{n} kp_k + n\left(1 - \sum_{k=0}^{n} p_k\right).$$

Taking into account the definitions of P_s and L_n (see (4)), we transform the obtained equation to the form

$$\alpha_1 a_{(1)} + \alpha_2 p_0 \sum_{i=0}^{h_2-1}\sum_{k=0}^{i} \tilde{p}_k B_{i-k} = n + p_0(L_n - nP_n). \quad (18)$$

Relation (18) implies formula (5), which yields positive values of p_0 only under the condition $\alpha_1 a_{(1)} < n$, i.e., when $\rho_1 < 1$.

Thus, when the conditions $a_{(1)} < \infty$, $b_{(1)} < \infty$ and $\rho_1 < 1$ are fulfilled, the system of equilibrium equations (1) and (2) has a nontrivial solution such that equality (3) is valid. Let $\{\xi(t),\ t \ge 0\}$ be a random process with discrete states s_i $(0 \le i \le h_2 - 1)$ and x_i $(i \ge h_1)$. Since all the states of this irreducible Markovian process communicate, the regularity criterion [9, p. 40] implies the regularity of this process, and it follows from the Foster ergodic theorem [9, p. 48] that the conditions of Theorem 1 are sufficient for the existence of stationary probabilities p_i $(0 \le i \le h_2 - 1)$ and q_i $(i \ge h_1)$. The theorem is proved.

3. Generating function and its derivative

Let $N(F)(z)$ and $D(F)(z)$ denote the numerator and denominator, respectively, in the expression for certain function $F(z)$. Consider the generating functions

$$P(z)=\sum_{i=0}^{h_2-1} p_i z^i; \qquad Q(z)=\sum_{i=h_1}^{\infty} q_i z^i; \qquad A(z)=\sum_{i=1}^{\infty} a_i z^i.$$

Theorem 2. Generating function $Q(z)$ is determined in the form

$$Q(z)=\frac{N(Q)(z)}{D(Q)(z)}; \qquad D(Q)(z)=n\mu(1-z)-\lambda_1 z(1-A(z));$$

$$N(Q)(z)=n\mu q_{h_1} z^{h_1}-\lambda_1 zA(z)P(z)+\lambda_1\sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} a_i z^{k+i+1}-\lambda_2 z^{h_2+1}\sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}. \tag{19}$$

Proof. The multiplication of the ith equation of system (2) by z^i $(i \ge h_1)$ and summation yield

$$-(\lambda_1+n\mu)Q(z)+n\mu\frac{Q(z)}{z}-n\mu q_{h_1} z^{h_1-1}+\lambda_1\sum_{i=h_1+1}^{\infty} z^i \sum_{k=h_1}^{i-1} q_k a_{i-k}+$$
$$+\lambda_1\sum_{i=h_2}^{\infty} z^i \sum_{k=0}^{h_2-1} p_k a_{i-k}+\lambda_2 z^{h_2}\sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}=$$
$$=-(\lambda_1+n\mu)Q(z)+n\mu\frac{Q(z)}{z}-n\mu q_{h_1} z^{h_1-1}+\lambda_1 A(z)(P(z)+Q(z))-$$
$$-\lambda_1\sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} a_i z^{k+i}+\lambda_2 z^{h_2}\sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}=0.$$

Solving the last equation, we find $Q(z)$ in form (19). The theorem is proved.

Let us simplify the calculation of the derivative of function $Q(z)$.

Lemma 1. For the derivative of function $Q(z)$ for $z=1$ the following formula

$$Q'(1)=\frac{N''(Q')(1)}{D''(Q')(1)}=-\frac{(n\mu-\lambda_1 a_{(1)})N''(Q)(1)+\lambda_1(a_{(1)}+a_{(2)})N'(Q)(1)}{2(n\mu-\lambda_1 a_{(1)})^2} \tag{20}$$

is valid where

$$N'(Q)(1) = n\mu h_1 q_{h_1} - \lambda_1 \left(a_{(1)} \sum_{k=0}^{h_2-1} p_k - \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} (k+i+1) a_i \right) -$$
$$- \lambda_1 \sum_{k=0}^{h_2-1} (k+1) p_k - \lambda_2 (h_2+1) \sum_{k=0}^{h_2-1} p_k B_{h_2-1-k};$$
$$N''(Q)(1) = n\mu h_1 (h_1-1) q_{h_1} - \lambda_1 \left(2a_{(1)} \sum_{k=1}^{h_2-1} k p_k + \sum_{k=1}^{h_2-1} k(k+1) p_k + (a_{(1)} + a_{(2)}) \sum_{k=0}^{h_2-1} p_k \right) +$$
$$+ \lambda_1 \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} (k+i)(k+i+1) a_i - \lambda_2 h_2 (h_2+1) \sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}.$$

Proof. We calculate the derivative $Q'(1)$ taking into account the equalities

$$A'(1) = \sum_{k=1}^{\infty} k a_k = a_{(1)}; \qquad A''(1) = \sum_{k=2}^{\infty} k(k-1) a_k = a_{(2)} - a_{(1)}.$$

Since $A(1) = \sum_{k=1}^{\infty} a_k = 1$, as a result of addition of the equations (1), we see that $N(Q)(1) = 0$.

So, function $Q(z)$ specified by formulas (19) satisfies the equalities

$$N(Q)(1) = D(Q)(1) = N(Q')(1) = D(Q')(1) = N'(Q')(1) = D'(Q')(1) = 0; \quad D''(Q')(1) \neq 0. \tag{21}$$

The first part of formulas (20) follows from (21). Since

$$N(Q')(z) = N'(Q)(z) \cdot D(Q)(z) - N(Q)(z) \cdot D'(Q)(z);$$
$$N'(Q')(z) = N''(Q)(z) \cdot D(Q)(z) - N(Q)(z) \cdot D''(Q)(z);$$
$$N''(Q')(z) = N'''(Q)(z) \cdot D(Q)(z) + N''(Q)(z) \cdot D'(Q)(z) -$$
$$- N'(Q)(z) \cdot D''(Q)(z) - N(Q)(z) \cdot D'''(Q)(z),$$

taking into account (21) we obtain the equality

$$N''(Q')(1) = N''(Q)(1) \cdot D'(Q)(1) - N'(Q)(1) \cdot D''(Q)(1) =$$
$$= -(n\mu - \lambda_1 a_{(1)}) N''(Q)(1) - \lambda_1 (a_{(1)} + a_{(2)}) N'(Q)(1).$$

Thus, the lemma is proved.

4. Stationary characteristics

Using the stationary distribution of the number of customers and the derivative $Q'(1)$ we can obtain simple formulas for the main stationary characteristics of the system.

Given that

$$Q'(1) = \sum_{k=h_1}^{\infty} kq_k,$$

the formulas for the stationary mean queue length in the system and the stationary mean number of customers in the system

$$\mathbf{E}(\mathrm{Q}) = \sum_{k=n+1}^{h_2-1} (k-n)p_k + \sum_{k=h_1}^{\infty} (k-n)q_k, \qquad \mathbf{E}(\mathrm{N}) = \sum_{k=1}^{h_2-1} kp_k + \sum_{k=h_1}^{\infty} kq_k,$$

we reduce to the form

$$\mathbf{E}(\mathrm{Q}) = Q'(1) + \sum_{k=n+1}^{h_2-1} kp_k - n\left(1 - \sum_{k=0}^{n} p_k\right), \qquad \mathbf{E}(\mathrm{N}) = \sum_{k=1}^{h_2-1} kp_k + Q'(1).$$

The formula for the stationary mean number of served customers per unit of time

$$\overline{N}_{\text{sv}} = \mu\left(\sum_{k=1}^{n-1} kp_k + n\left(\sum_{k=n}^{h_2-1} p_k + \sum_{k=h_1}^{\infty} q_k\right)\right).$$

after using of the normalization condition (3) takes the form

$$\overline{N}_{\text{sv}} = \mu\left(n(1-p_0) - \sum_{k=1}^{n-1}(n-k)p_k\right). \tag{22}$$

On the other hand, the served load is determined by the formula

$$\overline{N}_{\text{sv}} = \lambda_1 a_{(1)} + \lambda_2 \sum_{i=0}^{h_2-1} \sum_{k=0}^{i} p_k B_{i-k}. \tag{23}$$

Equating the expressions (22) and (23), we again arrive at the balance equation (18), by which we determine the stationary probability p_0 of free system.

The stationary mean waiting time in the queue equals

$$\mathbf{E}(\mathrm{W}) = \frac{\mathbf{E}(\mathrm{Q})}{\overline{N}_{\text{sv}}},$$

and the stationary loss probability of customers π and the stationary loss probability of customers of the second type π_2 have the form

$$\pi = 1 - \frac{\overline{N}_{\text{sv}}}{\lambda_1 a_{(1)} + \lambda_2 b_{(1)}}, \qquad \pi_2 = 1 - \frac{1}{b_{(1)}} \sum_{i=0}^{h_2-1} \sum_{k=0}^{i} p_k B_{i-k}.$$

Recall that customers of first type are not lost. To find π_2 we have used relation (23) and the equality $\overline{N}_{\text{sv}} = \lambda_1 a_{(1)} + \lambda_2 b_{(1)}(1-\pi_2)$.

5. The case of ordinary input flow

Since the ordinary flow of customers following equalities hold

$$a_1 = b_1 = 1; \qquad a_k = b_k = 0 \quad (k \ge 2); \qquad A_k = B_k = 0 \quad (k \ge 1),$$

the recurrence relations (6)-(12) are simplified and allow us to obtain expressions for $\tilde{p}_k$ $(1 \le k \le h_2 - 1)$ and $\tilde{q}_k$ $(k \ge h_1)$ in explicit form

$$\tilde{p}_k = \frac{\alpha^k}{k!} \quad (1 \le k \le n); \qquad \tilde{p}_k = \frac{\alpha^k}{n! n^{k-n}} \quad (n+1 \le k \le h_1 - 1);$$

$$\tilde{p}_k = \frac{\sum\limits_{i=k+1-h_1}^{h_2-h_1} \gamma^i}{\sum\limits_{i=0}^{h_2-h_1} \gamma^i} \tilde{p}_{h_1-1} \quad (h_1 \le k \le h_2 - 1); \qquad \tilde{q}_k = \frac{\gamma(1-\gamma_1^{k-h_1+1})}{1-\gamma_1} \tilde{p}_{h_2-1} \quad (h_1 \le k \le h_2);$$

$$\tilde{q}_k = \frac{\gamma \gamma_1^{k-h_2}(1-\gamma_1^{h_2-h_1+1})}{1-\gamma_1} \tilde{p}_{h_2-1} \quad (k \ge h_2 + 1).$$

Taking into account that

$$\sum_{k=h_2}^{\infty} \tilde{q}_k = \frac{\tilde{q}_{h_2}}{1-\gamma_1},$$

and using the normalization condition (3) we obtain the formula for p_0

$$p_0 = \frac{1}{P_{h_2-1} + Q_{h_2-1} + \dfrac{\tilde{q}_{h_2}}{1-\gamma_1}}.$$

Condition for the existence of stationary probabilities in the case of the ordinary input flow takes the form $\gamma_1 < 1$.

Since

$$\frac{1}{q_{h_2}} \sum_{k=h_2}^{\infty} k q_k = \sum_{k=h_2}^{\infty} (k-h_2)\gamma_1^{k-h_2} + h_2 \sum_{k=h_2}^{\infty} \gamma_1^{k-h_2} = \sum_{k=0}^{\infty} k\gamma_1^k + \frac{h_2}{1-\gamma_1} = \frac{\gamma_1 + h_2(1-\gamma_1)}{(1-\gamma_1)^2},$$

for the stationary mean queue length in the system we obtain the expression

$$\mathbf{E}(\mathrm{Q}) = \sum_{k=n+1}^{h_2-1} (k-n)p_k + \sum_{k=h_1}^{h_2-1} (k-n)q_k + \frac{\gamma_1 + (1-\gamma_1)(h_2-n)}{(1-\gamma_1)^2} q_{h_2}.$$

6. Examples of calculation of the stationary characteristics

Let us present the results of calculations of stationary characteristics of the system, performed using the obtained analytical relations.

Calculations are performed for the three-channel queueing system ($n = 3$) for the values of thresholds $h_1 = 7$, $h_2 = 12$ and following values of parameters characterizing the intensity of service and the intensity of customers flow: $\mu = 1$; $\lambda = 4,5$; $a_1 = 0,5$, $a_2 = 0,3$, $a_3 = 0,2$; $b_1 = 0,2$, $b_2 = 0,3$, $b_3 = 0,5$; $a_k = b_k = 0$ ($k \geq 4$). So, customers arrive in batches numbering from one to three and with different probabilities distributions of the number of customers in the batch for each flow. The mean number of customers in the batch for the first type and second type of flows equals $a_{(1)} = 1,7$ and $b_{(1)} = 2,3$ respectively.

For a fixed value $\lambda = 4,5$ of the parameter of the exponential distribution of time intervals between the moments of arrival of customers batches of total flow we consider the following pairs of values of the parameters λ_1 and λ_2 for flows of the first and the second type:

1) $\lambda_1 = 0,5$; $\lambda_2 = 4$; $\overline{N\lambda} = 10,05$;

2) $\lambda_1 = 1$; $\lambda_2 = 3,5$; $\overline{N\lambda} = 9,75$;

3) $\lambda_1 = 1,5$; $\lambda_2 = 3$; $\overline{N\lambda} = 9,45$.

Here $\overline{N\lambda} = \lambda_1 a_{(1)} + \lambda_2 b_{(1)}$ is an intensity of the total flow of customers during the normal operating mode of the system.

The stationary probabilities $\mathbf{p}_k$ $(k \geq 0)$ of availability in the system of k customers computed for each pair $1) - 3)$ of values λ_1 and λ_2 are shown in Table 1. Bold are maximum values of $\mathbf{p}_k$ which are reached for $k = 9$, $k = 11$ and $k = 12$ respectively.

Table 2 shows the values of the stationary characteristics computed for the same pairs of values λ_1 and λ_2. For comparison, the values of these parameters obtained by simulation system GPSS World [10, 11] for the value of the simulation time $t = 3 \cdot 10^5$ also are presented in Table 2.

Analysis of the obtained results shows that, despite the close values of the total flow intensity $\overline{N\lambda}$ for cases $1) - 3)$, the characteristics of the queue of the system for the considered input data deteriorate significantly with increasing flow intensity $\lambda_1 a_{(1)}$ of the first type customers.

Table 3 shows the values of the stationary mean queue length **E**(Q) calculated using the GPSS World for various laws of distribution of service time: exponential distribution with parameter $\mu = 1$, uniform distribution on the interval [0, 2], uniform distribution on the interval [0.5, 1.5] and deterministic value of 1.

Table 1

Stationary distributions of the number of customers in the system for different pairs of values λ_1 and λ_2

λ_1	λ_2	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
0.5	4	0.0002	0.0010	0.0026	0.0052	0.0115	0.0251	0.0545	0.1186	0.1394	**0.1504**
1	3.5	0.0002	0.0007	0.0019	0.0037	0.0080	0.0173	0.0371	0.0799	0.1032	0.1202
1.5	3	0.0001	0.0003	0.0008	0.0015	0.0033	0.0070	0.0148	0.0315	0.0441	0.0557
λ_1	λ_2	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	…
0.5	4	0.1476	0.1423	0.1230	0.0373	0.0212	0.0107	0.0048	0.0024	0.0012	…
1	3.5	0.1272	**0.1300**	0.1219	0.0708	0.0526	0.0374	0.0260	0.0184	0.0130	…
1.5	3	0.0643	0.0714	**0.0744**	0.0615	0.0565	0.0510	0.0458	0.0413	0.0372	…

Table 2

Stationary system characteristics for different pairs of values λ_1 and λ_2 in comparison with the results of simulation

λ_1	λ_2	**E**(Q)	**E**(Q) (GPSS)	**E**(W)	**E**(W) (GPSS)	π	π (GPSS)	π_2	π_2 (GPSS)
0.5	4	6.324	6.454	2.112	2.156	0.702	0.702	0.767	0.767
1	3.5	8.103	8.184	2.704	2.739	0.693	0.694	0.839	0.841
1.5	3	14.482	14.428	4.830	4.814	0.683	0.683	0.935	0.935

Table 3

Values of the stationary mean queue length E(Q) for different pairs of values λ_1 and λ_2 calculated using the GPSS World for various laws of distribution of service time: 1 - exponential distribution with parameter μ = 1; 2 - uniform distribution on the interval [0, 2]; 3 - uniform distribution on the interval [0.5, 1.5]; 4 - deterministic value of 1

λ_1	λ_2	1	2	3	4
0.5	4	6.454	6.307	6.234	4.429
1	3.5	8.184	7.438	7.725	6.990
1.5	3	14.428	12.754	11.958	11.633

Conclusions

Thus, in this paper we have presented a simple algorithm for finding of the stationary distribution of the number of customers and stationary characteristics of a queueing system $M_2^X/M/n$. The system receives two independent nonordinary flows of customers and uses a hysteresis strategy of control of the input flow intensity. The results of test calculations are given. The obtained relations can be used to solve optimization problems associated with a hysteretic control mechanism.

Analyzing the data from Table 3, we conclude that the formulas for the case of exponential service time distribution obtained in this paper can be used for evaluative analysis of systems of type $M_2^X/G/n$ with hysteretic control of the input flow intensity.

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