DIFFERENTIATION AND INTEGRATION BY USING MATRIX INVERSION

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Abstract. In the paper certain examples of applications of the matrix inverses for generating and calculating the integrals are presented.

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Introduction

The first part of our discussion concerns the linear mappings defined on the finite-dimensional space of solutions of the following system of differential equations

$$\begin{cases} f_1' = a_{1,1}f_1 + \cdots + a_{1,n}f_n \\ f_2' = a_{2,1}f_1 + \cdots + a_{2,n}f_n \\ \vdots \\ f_n' = a_{n,1}f_1 + \cdots + a_{n,n}f_n. \end{cases} \tag{1}$$

Suppose that functions $g_1, g_2, \dots, g_n$ form the solution of the above system of equations and matrix $A = [a_{i,j}]_{n\times n}$ of system (1) is nonsingular. Let us consider the linear mapping T of the linear space of solutions $(f_1, f_2, \dots, f_n)^T$ of system (1) onto itself defined in the following way:

$$T\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} = \begin{pmatrix} a_{1,1}f_1 + \cdots + a_{1,n}f_n \\ \vdots \\ a_{n,1}f_1 + \cdots + a_{n,n}f_n \end{pmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{2}$$

Matrix A is nonsingular, so there exists its inverse A^{-1}. In particular, the following equality occurs:

$$T\left(A^{-1}\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}\right) = AA^{-1}\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}. \tag{3}$$

Therefore, if $A^{-1}\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix}$, then functions G_i are the primitive functions of $g_i (i = 1,2, \ldots, n)$.

The current article is inspired by Swartz's paper [1] where the author gives some simple examples of using this procedure, among others, for generating the integrals of functions $h_1 = \mathrm{e}^{ax} \sin bx$, $h_2 = e^{ax} \cos bx$, for which he obtained the formula (the integration constants are omitted and this rule will oblige henceforward):

$$\begin{pmatrix} \int h_1 dx \\ \int h_2 dx \end{pmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{e^{ax}}{a^2+b^2} \begin{pmatrix} a \sin bx - b \cos bx \\ b \sin bx + a \cos bx \end{pmatrix}. \tag{4}$$

1. Generalization of Swartz's example

Let us start from the generalization of the example mentioned above. Let us take the functions

$$\begin{aligned} g_1(x) &= \cosh ax \sin bx\,, \quad g_2(x) = \cosh ax \cos bx\,, \\ g_3(x) &= \sinh ax \cos bx\,, \quad g_4(x) = \sinh ax \sin bx\,. \end{aligned} \tag{5}$$

Note that the differentiation operator for these functions is of the form

$$T \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} g_1' \\ g_2' \\ g_3' \\ g_4' \end{pmatrix} = \begin{bmatrix} 0 & b & 0 & a \\ -b & 0 & a & 0 \\ 0 & a & 0 & -b \\ a & 0 & b & 0 \end{bmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}. \tag{6}$$

If $ab \neq 0$, then the inverse of matrix of operator T has the form

$$A^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} 0 & -b & 0 & a \\ b & 0 & a & 0 \\ 0 & a & 0 & b \\ a & 0 & -b & 0 \end{bmatrix}. \tag{7}$$

Now we can easily integrate functions g_i, e.g.

$$\int g_1(x) dx = \frac{-b g_2(x) + a g_4(x)}{a^2 + b^2}.$$

2. Integrals of $\sin^n x$ for odd n

Consider the second derivatives of functions $g(x) = \sin^n x, n \geq 2$. We have

$$(\sin^n x)'' = (n \sin^{n-1} x \cos x)' = n(n-1)\sin^{n-2} x - n^2 \sin^n x. \qquad (8)$$

Of course for $n = 1$ there is $(\sin x)'' = -\sin x$. Thus we can write the second derivative operator for the odd powers of function $\sin x$, from 1 to odd k, in the following matrix form:

$$A_k \begin{pmatrix} \sin x \\ \sin^3 x \\ \vdots \\ \sin^k x \end{pmatrix} = \begin{pmatrix} (\sin x)'' \\ (\sin^3 x)'' \\ \vdots \\ (\sin^k x)'' \end{pmatrix} = \begin{pmatrix} -\sin x \\ 6\sin x - 9\sin^3 x \\ \vdots \\ k(k-1)\sin^{k-2} x - k^2 \sin^k x \end{pmatrix} =$$
$$= \begin{bmatrix} -1^2 & 0 & \cdots & 0 & 0 \\ 3 \cdot 2 & -3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & k(k-1) & -k^2 \end{bmatrix} \begin{pmatrix} \sin x \\ \sin^3 x \\ \vdots \\ \sin^k x \end{pmatrix}. \qquad (9)$$

The determinant of the obtained matrix is equal to: $\det A_k = (-1)^{\frac{k+1}{2}}(k!!)^2 \neq 0$, $k \in N$. The inverse matrix is of the form

$$A_k^{-1} = [a_{i,j}]_{\frac{k+1}{2} \times \frac{k+1}{2}}, \qquad (10)$$

where

$$a_{i,j} = \begin{cases} 0, & i < j, \\ -\frac{1}{(2i-1)^2}, & i = j, \\ -\frac{(n-1)!!}{n!!}\frac{1}{(2i-1)}\frac{(2i-3)!!}{(2i-2)!!}, & i > j. \end{cases} \qquad (11)$$

We can deduce that for odd n there occurs (with respect to the linear element):

$$\int^2 \sin^n x\, dx = -\frac{(n-1)!!}{n!!}\sum_{i=0}^{(n-1)/2} \frac{1}{2i+1}\frac{(2i-1)!!}{(2i)!!}\sin^{2i+1} x, \qquad (12)$$

where $\int^2 f(x)\,dx := \int(\int f(x)\,dx)dx$. For example, we get

$$\int^2 \sin^5 x\, dx = -\frac{8}{15}\sin x - \frac{4}{45}\sin^3 x - \frac{1}{25}\sin^5 x. \qquad (13)$$

We note that from (12) by differentiating we obtain (see [2, 3]):

$$\int \sin^n x\, dx = -\frac{(n-1)!!}{n!!}\cos x \sum_{i=0}^{(n-1)/2} \frac{(2i-1)!!}{(2i)!!}\sin^{2i} x. \qquad (14)$$

For example, we have

$$\int \sin^5 x\,dx = -\frac{8}{15}\cos x\ \left(1+\frac{1}{2}\sin x+\frac{3}{8}\sin^4 x\right).$$

3. The case of the even powers of sin x

Consider functions of the form

$$g_n(x) = \sin^n x - \frac{n-1}{n}\sin^{n-2} x, \tag{15}$$

for $n = 2,4,6,\ldots$. Acting on the vector $\begin{pmatrix} g_2(x)\\ g_4(x)\\ \vdots\\ g_k(x)\end{pmatrix}$, where k is even, with the second derivative operator, like it was done in equation (9), we get the following transformation matrix:

$$B_k\begin{pmatrix} g_2(x)\\ g_4(x)\\ \vdots\\ g_k(x)\end{pmatrix} = \begin{pmatrix} g_2(x)''\\ g_4(x)''\\ \vdots\\ g_k(x)''\end{pmatrix} = \tag{16}$$
$$= \begin{bmatrix} -2^2 & 0 & \ldots & 0 & 0\\ 2^2\cdot 3/4 & -4^2 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & (k-2)^2(k-1)/k & -k^2\end{bmatrix}\begin{pmatrix} g_2(x)\\ g_4(x)\\ \vdots\\ g_k(x)\end{pmatrix}.$$

The above matrix is invertible and its inversion is of the form

$$B_k^{-1} = \left[b_{i,j}\right]_{\frac{k}{2}\times\frac{k}{2}}, \tag{17}$$

where

$$b_{i,j} = \begin{cases} 0, & i<j,\\ -\frac{1}{(2i)^2}, & i=j,\\ -\frac{1}{(2i)^2}\frac{(2i-1)!!}{(2j-1)!!}\frac{(2j)!!}{(2i)!!}, & i>j. \end{cases} \tag{18}$$

Therefore for even *n* we get the formula (exact to the linear element):

$$\begin{aligned} \int^2 g_n(x)dx &= -\frac{(n-1)!!}{n!!n^2}\sum_{i=1}^{n/2}\frac{(2i)!!}{(2i-1)!!}\left(\sin^{2i} x - \frac{2i-1}{2i}\sin^{2i-2} x\right)\\ &= -\frac{(n-1)!!}{n!!n^2}\left(\frac{n!!}{(n-1)!!}\sin^n x - 1 + \right.\\ &\quad \left. + \sum_{i=1}^{n/2-1}\sin^{2i} x\left[\frac{(2i)!!}{(2i-1)!!} - \frac{(2i+2)!!}{(2i+1)!!}\frac{(2i+1)}{(2i+2)}\right]\right)\\ &= \frac{(n-1)!!}{n!!n^2} - \frac{1}{n^2}\sin^n x = -\frac{1}{n^2}\sin^n x\,, \end{aligned} \tag{19}$$

which implies the following integral identity

$$\begin{aligned}\int^2 \sin^n x\,dx &= \frac{(n-1)!!}{n!!}\left(\int^2 dx + \sum_{k=1}^{n/2} \frac{(2k)!!}{(2k-1)!!}\int^2 g_{2k}(x)dx\right)\\ &= \frac{(n-1)!!}{n!!}\left(\frac{x^2}{2} - \sum_{k=1}^{n/2} \frac{(2k)!!}{(2k-1)!!(2k)^2}\sin^{2k} x\right).\end{aligned} \tag{20}$$

Hence, by differentiating we get (see [2, 3]):

$$\int \sin^n x\,dx = \frac{(n-1)!!}{n!!}\left(x - (\cos x)\sum_{k=1}^{n/2} \frac{(2k-2)!!}{(2k-1)!!}\sin^{2k-1} x\right). \tag{21}$$

For example, we obtain

$$\int \sin^n x\,dx = \frac{15}{48}\left(x - \cos x\left(\sin x + \frac{2}{3}\sin^3 x + \frac{6}{15}\sin^5 x\right)\right). \tag{22}$$

4. Integral of $\tan^n x$

Let V be the linear space of sequences $\{f_n(x)\}_{n=0}^{\infty}$ of differentiable functions $f_n\colon (a,b) \to R$. Let $\mathbb{A}\colon V \to V$ be a linear operator satisfying equation

$$\begin{pmatrix} (\tan x)' \\ (\tan^2 x)' \\ (\tan^3 x)' \\ \vdots \end{pmatrix} = \mathbb{A}\begin{pmatrix} 1 \\ \tan x \\ \tan^2 x \\ \vdots \end{pmatrix}. \tag{23}$$

If $\mathbb{A}$ is represented by infinite matrix A, then from (23) matrix A has the form

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & \vdots \\ 0 & 2 & 0 & 2 & 0 & \vdots \\ 0 & 0 & 3 & 0 & 3 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{24}$$

It is easy to show that

$$A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 & \vdots \\ 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 & \vdots \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{5} & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{25}$$

However, matrix A^{-1} does not represent the inverse operator $\mathbb{A}^{-1}$, since the following relations hold

$$\begin{pmatrix} (\tan x)' \\ (\tan^2 x)' \\ (\tan^3 x)' \\ \vdots \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{pmatrix} 1 \\ \tan x \\ \tan^2 x \\ \vdots \end{pmatrix}, \quad (26)$$

resulting from the formula

$$\frac{d}{dx}\left(\frac{1}{n+1}\tan^{n+1} x - \frac{1}{n+3}\tan^{n+3} x\right) = \tan^n x - \tan^{n+4} x. \quad (27)$$

Moreover, we get from this, after summing over powers in the interval $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ and by uniform convergence (see [4]), that

$$\tan^n x = \frac{d}{dx}\sum_{k=0}^{\infty}\left(\frac{1}{n+4k+1}\tan^{n+4k+1} x - \frac{1}{n+4k+3}\tan^{n+4k+3} x\right), \quad (28)$$

or equivalently

$$\begin{aligned} \int_0^x \tan^n y dy &= \sum_{k=0}^{\infty}\left(\frac{1}{n+4k+1}\tan^{n+4k+1} x - \frac{1}{n+4k+3}\tan^{n+4k+3} x\right) \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{n+2k+1}\tan^{n+2k+1} x, \end{aligned} \quad (29)$$

for every $x \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ and $n = 0,1,2,\dots$.

From formula (28) we also get the matrix form $I(\mathbb{A})$ of operator $\mathbb{A}^{-1}$, i.e. the inverse operator of operator $\mathbb{A}$ (under assumption of its existence):

$$I(\mathbb{A}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots \\ \frac{1}{5} & 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (30)$$

Formula (29) is our main analytic result in this section. Why do we think so? Because, as we show now, this formula is a generalization of the classical MacLaurin's formulae for $\ln(x+1)$, i.e.

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad (31)$$

for $-1 < x \le 1$, found independently by Nicolaus Mercator and Saint-Vincent (see sections 10-9 and 10-10 in [5] and page 387 in [6]), and for $\arctan x$, i.e.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad (32)$$

for $-1 < x \le 1$, which is known as the Gregory series.

This connection should not be surprising because of the known complex relation (see [7]):

$$\arctan z = \frac{1}{2}\ln\left(\frac{i+z}{i-z}\right), \tag{33}$$

where $z/i \in \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$ and where the principal branch of the logarithm is under consideration. On the cuts we have

$$\arctan(iy) = \pm\frac{\pi}{2} + \frac{i}{2}\ln\left(\frac{y+1}{y-1}\right), \tag{34}$$

for $y \in (-\infty, -1) \cup (1, \infty)$ and where the upper/lower sign corresponds to the right/left side of the set determining y. More precisely, the connection between the arctan function and log function is obvious and the section concerns the real and imaginary parts of $\arctan z$, since we have

$$\arctan z = \frac{1}{2}\arctan\left(\frac{2x}{1-x^2-y^2}\right) + \frac{1}{4} i \ln\left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right), \tag{35}$$

where $z = x + iy$, $|z| < 1$. First, from equation (29) for $n = 1$ we get

$$-\ln\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2(k+1)} \tan^{2(k+1)} x \tag{36}$$

or

$$\ln\cos\left(\arctan\sqrt{x}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-x)^k}{k}, \tag{37}$$

which by (31) implies the well known identity (since $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$ for $\alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$):

$$\ln\cos\left(\arctan\sqrt{x}\right) = -\frac{1}{2}\ln(x+1), \tag{38}$$

i.e.

$$\sqrt{x+1}\cos(\arctan x) \equiv 1, \tag{39}$$

for every $x \in [0,1]$.

But this formula holds for every $x \geq 0$ since $\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$ for every $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In other words, formula (37) is equivalent to (31). For $n = 0$ from (29) we get

$$\frac{\arctan x}{x} = \sum_{k=0}^{\infty}\frac{(-x^2)^k}{2k+1}, \tag{40}$$

which implies (32). Hence, for $x = 1$ we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{41}$$

That is the classical Gregory-Leibniz-Nilakantha's formula (see [8]). Generalizations of the Gregory power series (32) are discussed in papers [9] and [10].

As seen from equation (29), the values of integrals $\int_0^{\frac{\pi}{4}} \tan^n x\, dx$ for $n \geq 2$ are the translations of numbers $\int_0^{\frac{\pi}{4}} \tan x\, dx$ or $\int_0^{\frac{\pi}{4}} dx$, depending on parity of n. For even n we have

$$\int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}} dx - \sum_{k=0}^{(n-2)/2} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} - \sum_{k=0}^{(n-2)/2} \frac{(-1)^k}{2k+1}, \tag{42}$$

whereas for odd n we get

$$\int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}} \tan x\, dx - \sum_{k=1}^{(n-1)/2} \frac{(-1)^k}{2k} = \frac{1}{2}\left(\ln 2 - \sum_{k=1}^{(n-1)/2} \frac{(-1)^k}{2k}\right). \tag{43}$$

5. Final remark

Some other applications of the matrix obtained by n-times differentiation of product functions and composition functions are discussed in paper [11]. In turn, in paper [12] the technique of the inverse matrix was used for calculating the integral $\int \sec^{2n+1} x\, dx$, similarly as in the present study. The obtained formulae were used there for generating the trigonometric identities.

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