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ON K-SUPERQUADRATIC SET-VALUED FUNCTIONS

Katarzyna Troczka-Pawelec

Institute of Mathematics, Częstochowa Jan Długosz University
Częstochowa, Poland
k.troczka@ajd.czest.pl

Abstract. In this paper we consider K-superquadratic set-valued functions. We will present here some connections between K-boundedness of K-superquadratic set-valued functions and K-semicontinuity of multifunctions of this kind.

Keywords: *set-valued functions, quadratic set-valued functions, subquadratic real-valued functions*

Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function f is called superquadratic if it fulfils inequality

$$2f(x) + 2f(y) \le f(x + y) + f(x - y), \qquad x, y \in X. \tag{1}$$

If the sign $\le$ in (1) is replaced by $\ge$, then f is called subquadratic. The continuity problem of functions of this kind was considered in [1]. This problem was also considered in the class of set-valued functions. By the set-valued functions we understand functions of the type $F : X \to 2^Y$, where X and Y are given sets. Throughout this paper set-valued functions will be always denoted by capital letters.

A set-valued function F is called subquadratic if it satisfies inclusion

$$F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \qquad x, y \in X \tag{2}$$

and superquadratic, if it satisfies inclusion defined in this form

$$2F(x) + 2F(y) \subset F(x + y) + F(x - y), \qquad x, y \in X. \tag{3}$$

For single-valued real functions properties of subquadratic and superquadratic functions are quite analogous and, in view of the fact that if a function f is subquadratic, then the function $-f$ is superquadratic and conversely, it is not necessary to investigate functions of these two kinds individually. In the case of set-valued functions the situation is different. Even if properties of subquadratic and superquadratic set-valued functions are similar, we have to prove them separately.

If the sign $\subset$ in the inclusions above is replaced by =, then F is called quadratic set-valued function. The class of quadratic set-valued functions is an important subclass of the class of subquadratic and superquadratic set-valued functions. Quadratic set-valued functions have already extensive bibliography (see Smajdor [2], Henney [3] and Nikodem [4]).

The continuity problem of subquadratic and superquadratic set-valued functions was considered in [5] and [6].

Adding a cone K in the space of values of a set-valued function F lets us consider a K-superquadratic set-valued function, that is, solution of the inclusion

$$F(x+y)+F(x-y)\subset 2F(x)+2F(y)+K, \qquad x,y\in X \tag{4}$$

where F is defined on 2-divisible topological group X with non-empty, compact and convex values in a locally convex topological vector space Y.

The concept of K-superquadraticity is related to real-valued superquadratic functions. Note, in the case when F is a single-valued real function and $K=[0;\infty)$ we obtain the standard definition of superquadratic functionals (1).

If a set-valued function F satisfies the following inclusion

$$2F(x)+2F(y)\subset F(x+y)+F(x-y)+K, \qquad x,y\in X \tag{5}$$

then is called K-subquadratic. The K-continuity problem of multifunction of this kind was considered in [7] and [8]. It was proved that a K-subquadratic set-valued function, which is K-continuous at zero, $F(0)=\{0\}$ and locally K-bounded in X is K-continuous every where in X.

In this paper we will consider similar problem for K-superquadratic set-valued functions. Likewise as in functional analysis we can look for connections between K-boundedness and K-semicontinuity of set-valued functions of this kind. Assuming $K=\{0\}$ in (4) and (5) we obtain the inclusions (2) and (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. We consider the family $n(Y)$ of all non-empty subsets of Y as a topological space with the Hausdorff topology. In this topology the set

$$N_W(A):=\{B\in n(Y): A\subset B+W, B\subset A+W\}$$

where W runs the base of neighbourhoods of zero in Y, form a base of neighbourhoods of a set $A\in n(Y)$. By $cc(Y)$ we denote the family of all compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

Now we present here some definitions for the sake of completeness.

Recall that a set $K \subset Y$ is called a cone if $K + K \subset K$ and $sK \subset K$ for all $s \in (0;\infty)$.

Definition 1.1

A cone K in a topological vector space Y is said to be a normal cone if there exists a base Λ of zero in Y such that

$$W = (W + K) \cap (W - K)$$

for all $W \in \Lambda$.

Definition 1.2

An s.v.f. $F : X \to n(Y)$ is said to be K-upper semi-continuous (abbreviated K-u.s.c.) at $x_0 \in X$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 1.3

An s.v.f. $F : X \to n(Y)$ is said to be K-lower semi-continuous (abbreviated K-l.s.c.) at $x_0 \in X$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 1.4

An s.v.f. $F : X \to n(Y)$ is said to be K-continuous at $x_0 \in X$ if it is both K-u.s.c. and K-l.s.c. at $x_0 \in X$. It is said to be K-continuous if it is K-continuous at each point of X.

Note that K-continuity of F in the case where $K = \{0\}$ means its continuity with respect to the Hausdorff topology on $n(Y)$.

We will frequently use the following lemma.

Lemma 1.1 ([7])

Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A + C \subset B + C + K$. If B is convex and C is bounded then $A \subset \overline{B + K}$.

The main result

In the proof of the main theorem, which will be presented here, we will often use four known lemmas (see Lemma 1.1, Lemma 1.3, Lemma 1.6 and Lemma 1.9 in [9]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s+t)A = sA + tA$ holds for every $s,t \geq 0$ (or $s,t \leq 0$). The second lemma says that in a real vector space Y for two convex subsets A, B the set $A+B$ is also convex. The next lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A+B$ is closed. For any sets $A, B \subset Y$ where Y denotes the same space as above, the inclusion $\overline{A}+\overline{B} \subset \overline{A+B}$ holds and equality holds if and only if the set $\overline{A}+\overline{B}$ is closed.

Note that for the cone K the following remark holds.

Remark 2.1
Let Y be a real topological vector space. If K is a closed cone, then it is a cone with zero.

Let us adopt the following three definitions which are natural extension of the concept of the lower and upper boundedness for real-valued functions.

Definition 2.1
An s.v. function $F: X \to n(Y)$ is said to be K-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$.

Definition 2.2
An s.v. function $F: X \to n(Y)$ is said to be K-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$.

Definition 2.3
An s.v. function $F: X \to n(Y)$ is said to be locally K-lower (upper) bounded in X if for every $x \in X$ there exists a neighbourhood U_x of zero in X such that F is K-lower (upper) bounded on a set $x + U_x$. It is said to be locally K-bounded in X if it is both locally K-lower and locally K-upper bounded in X.

Theorem 2.1
Let X be a 2-divisible topological group, Y locally convex topological real vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F: X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- bounded in X then it is K-l.s.c. in X.

Proof:

Suppose that F is not K-l.s.c. at a point $z \in X$ i.e. there exists a neighbourhood V of zero in Y such that for every neighbourhood U of zero in X, we can find $x_u \in X$ for which

$$F(z) \not\subset F(z + x_u) + V + K.$$

Take a balanced convex neighbourhood W of zero in Y such that

$$W \subset V$$

and

$$\overline{F(z + x_u) + W + K} \subset F(z + x_u) + V + K.$$

Then also

$$F(z) \not\subset \overline{F(z + x_u) + W + K}. \tag{6}$$

We shall show by induction that

$$F(z) + 2^s(2^s - 1)F(x_u) \not\subseteq \overline{F(z + 2^s x_u) + 2^s W + K} \tag{7}$$

for every neighbourhood U of zero in X and $s \in N_0 := \{0,1,2,3\ldots\}$. For $k = 0$ condition (7) holds with respect to (6). We assume that (7) holds for $s = k$ and for every neighbourhood U of zero in X. Let $s = k+1$. Putting $y = x$ in (4) and using condition $F(0) = \{0\}$ we have

$$F(2\text{x}) \subset 4F(x) + K.$$

An easy induction shows

$$F(2^n x) \subset 4^n F(x) + K, \tag{8}$$

for $x \in X$ and for all positive integers n. By K-superquadraticity of F and (8), we have

$$\begin{gathered} F(z + 2^{k+1} x_u) + F(z) = F(z + 2^k x_u + 2^k x_u) + F(z + 2^k x_u - 2^k x_u) \subset \\ 2\text{F}(z + 2^k x_u) + 2\text{F}(2^k x_u) + K \subset \\ 2\text{F}(z + 2^k x_u) + 2^{2\text{k}+1} F(x_u) + K. \end{gathered} \tag{9}$$

In view of the fact that for any sets $A, B \subset Y$, $\overline{A} + \overline{B} \subset \overline{A + B}$ we get

$$\overline{F(z + 2^k x_u) + 2^k W + K} + K \subset \overline{F(z + 2^k x_u) + 2^k W + K}$$

and consequently

$$\overline{\overline{F(z+2^k x_u)+2^k W+K}+K} \subset \overline{F(z+2^k x_u)+2^k W+K}. \tag{10}$$

By (7) and (10), we obtain

$$F(z)+2^k(2^k-1)F(x_u) \not\subseteq \overline{\overline{F(z+2^k x_u)+2^k W+K}+K}. \tag{11}$$

Notice that for a cone K the equality $aK=K$ holds, for every $a\in(0;\infty)$. Hence

$$2F(z)+2^{k+1}(2^k-1)F(x_u) \not\subseteq \overline{\overline{2F(z+2^k x_u)+2^{k+1}W+K}+K}. \tag{12}$$

By (12) and Lemma 1.1

$$\begin{gathered}2F(z)+2^{k+1}(2^k-1)F(x_u)+2^{2k+1}F(x_u) \not\subseteq \\ \overline{2F(z+2^k x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)+K.\end{gathered}$$

In view of Remark 2.1 K is a cone with zero. Therefore, by above

$$\begin{gathered}2F(z)+2^{k+1}(2^k-1)F(x_u)+2^{2k+1}F(x_u)+K \not\subseteq \\ \overline{2F(z+2^k x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)+K.\end{gathered} \tag{13}$$

In view of the fact that the sum of closed and compact sets is closed and for any sets $A,B\subset Y$, $\overline{A}+\overline{B}=\overline{A+B}$ in the case where $\overline{A}+\overline{B}$ is a closed set, we get

$$\begin{gathered}\overline{2F(z+2^k x_u)+2^{k+1}W+K}+2^{2k+1}F(x_u)= \\ \overline{2F(z+2^k x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)}.\end{gathered} \tag{14}$$

Since K is a cone, by (9) we obtain

$$\begin{gathered}\overline{2F(z+2^k x_u)+2^{k+1}W+K+2^{2k+1}F(x_u)} \supset \\ \overline{F(z+2^{k+1}x_u)+F(z)+2^{k+1}W+K}.\end{gathered} \tag{15}$$

Since F has closed values, we get

$$\begin{gathered}F(z)+\overline{F(z+2^{k+1}x_u)+2^{k+1}W+K}+K\subset \\ \overline{F(z+2^{k+1}x_u)+F(z)+2^{k+1}W+K}+K.\end{gathered} \tag{16}$$

Consequently, using (13)-(16), we conclude

$$\begin{gathered}2F(z)+2^{k+1}(2^k-1)F(x_u)+2^{2k+1}F(x_u)+K \not\subseteq \\ F(z)+\overline{F(z+2^{k+1}x_u)+2^{k+1}W+K}+K.\end{gathered}$$

By convexity of the sets $F(x_u), F(z)$ we obtain

$$F(z)+\overline{F(z)+2^{k+1}(2^{k+1}-1)F(x_u)}+K \not\subseteq F(z)+\overline{F(z+2^{k+1}x_u)+2^{k+1}W+K}+K.$$

Therefore

$$F(z)+2^s(2^s-1)F(x_u)\not\subseteq\overline{F(z+2^s x_u)+2^s W+K}$$

for $s=k+1$, so that (7) is generally valid for all integers $s\geq 0$.

Since K is a normal cone, there exists a base Λ of neighbourhoods of zero in Y such that $M=(M+K)\cap(M-K)$ for all $M\in\Lambda$. We can choose $W_1\in\Lambda$ and balanced neighbourhood W_2 of zero in Y such that

$$W_2\subset W_1\subset W.$$

Because $M\in\Lambda$. is K-lower bounded on a neighbourhood of z, then there exists a neighbourhood U_0 of zero in X and bounded set $B_1\subset Y$ such that

$$F(z+t)\subset B_1+K,\quad t\in U_0.$$

Since the set B_1 is bounded, there exists $\lambda_1>0$ such that

$$B_1\subset\frac{1}{\lambda_1}W_2.$$

Therefore, from the above

$$F(z+t)\subset\frac{1}{\lambda_1}W_2+K,\quad t\in U_0.$$

Because F is K-upper bounded on a neighbourhood of z, then there exists a neighbourhood U_1 of zero in X and bounded set $B_2\subset Y$ such that

$$F(z+t)\subset B_2-K,\quad t\in U_1.$$

Since the set B_2 is bounded, there exists $\lambda_2>0$ such that

$$B_2\subset\frac{1}{\lambda_2}W_2.$$

Therefore, from the above

$$F(z+t)\subset\frac{1}{\lambda_2}W_2-K,\quad t\in U_1.$$

Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Since W_2 is balanced, we get

$$F(z+t) \subset \frac{1}{\lambda} W_2 + K \subset \frac{1}{\lambda} W_1 + K, \quad t \in U_0 \tag{17}$$

and

$$F(z+t) \subset \frac{1}{\lambda} W_2 - K \subset \frac{1}{\lambda} W_1 - K, \quad t \in U_1. \tag{18}$$

By (17) and (18), we obtain

$$F(z+t) \subset \left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right), \quad t \in U_0 \cap U_1. \tag{19}$$

Because the set $M \in \Lambda$, we have

$$\left(\frac{1}{\lambda} W_1 + K\right) \cap \left(\frac{1}{\lambda} W_1 - K\right) = \frac{1}{\lambda} W_1$$

and consequently the following inclusion holds

$$F(z+t) \subset \frac{1}{\lambda} W_1 \subset \frac{1}{\lambda} W, \tag{20}$$

for every $t \in U_0 \cap U_1$.

Let $k \in N$ be so large that

$$2^k > \frac{3}{\lambda}. \tag{21}$$

Since F is K-u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood U_2 of zero in X such that

$$U_2 \subset \frac{1}{2^k}(U_0 \cap U_1) \tag{22}$$

and

$$F(t) \subset \frac{1}{\lambda 2^k (2^k - 1)} W + K, \quad t \in U_2. \tag{23}$$

There exists $x_u \in U_2$ such that (7) holds.
By (22)

$$2^k x_u \in (U_0 \cap U_1) \tag{24}$$

and by (23)

$$F(x_u) \subset \frac{1}{\lambda 2^k (2^k - 1)} W + K. \tag{25}$$

Let $a \in F(z + 2^k x_u), b \in F(z), c \in F(x_u)$. By (20), (21), (24), (25), we obtain

$$b + 2^k (2^k - 1) c - a \in \frac{1}{\lambda} W + \frac{1}{\lambda} W + K + \frac{1}{\lambda} W \subset 2^k W + K.$$

Therefore

$$b + 2^k (2^k - 1) c \in F(z + 2^k x_u) + 2^k W + K.$$

We have proved that

$$F(z) + 2^k (2^k - 1) F(x_u) \subset F(z + 2^k x_u) + 2^k W + K,$$

which contradicts (7).

Conclusions

This article is the introduction to the discussion on the K-continuity problem for K-superquadratic set-valued functions. In the theory of K-subquadratic and K-superquadratic set-valued functions an important role is played by theorems giving possibly weak conditions under which such multifunctions are K-continuous. In this paper we have presented some connections between K-boundedness of K-superquadratic set-valued functions and K-semicontinuity of multifunctions of this kind.

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