Journal of Applied Mathematics and Computational Mechanics 2015, 14(4), 167-174
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2015.4.17 e-ISSN 2353-0588

EFFECT OF CRACK PARAMETERS ON FREE VIBRATIONS OF THE BERNOULLI-EULER BEAM

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Abstract. The paper presents the problem of free vibration of the cantilever Bernoulli-Euler beam with a crack. The beam along the length has a variable cross-sectional area, and the crack parameters (thickness, depth and location) undergo changes. The work includes an analytical solution of the vibration problem and the results obtained with the use of FEM and CATIA program.

Keywords: *Bernoulli-Euler beam, crack, free vibration, mathematical modelling, CATIA*

1. Introduction

The beams, whose geometry and/or material properties change along their length, are important for instance in the design of aircraft, robot arms and tall buildings, where they are used both to reduce weight or volume, and to increase strength and stability. The result is that the vibration problem of beams were and are the subject of the work of many authors [1-11]. In order to solve the linear boundary problems, the authors applied various methods, like the Green's function method [1, 2], the Lagrange multiplier formalism [3, 4], FEM [5, 6] or others [7, 8].

Structural elements can fail by crack formation. In the case when the crack extends in the element, it will produce a number of changing parameters, such as a decrease in the rigidity of the system and the vibration frequency or increasing damping. These changes allow for the determination of the location and size of the crack using data collected on one or at most a few vibration frequencies and can be used for diagnostic devices to detect a failure in the earliest stage.

Damage analysis of mechanical systems, the cracked beam, is an interesting issue for many researchers [1, 4-7, 9, 10]. To determine the location and size of crack in [6, 9, 10] the experimental modal analysis was proposed. The analysis of the influence of crack parameters on the beam structural behavior can be considered more generally as the problem of optimal boundary conditions [11].

The subject of the paper is a beam of linearly cross-sectional area with a symmetric crack localized at one point of the beam's length. A theoretical analysis of the free vibration problem of this system has been presented. Using the Green's function method, the frequency equation and the mode shapes have been determined. The paper presents numerical results obtained on the basis of the computational model (FEM) prepared in CATIA program.

2. Formulation and theoretical solution of the problem

The subject of interest is the cantilever beam of length L, varying cross sectional area along the length and crack at one point of the beam. The model discussed beam is shown in Figure 1.

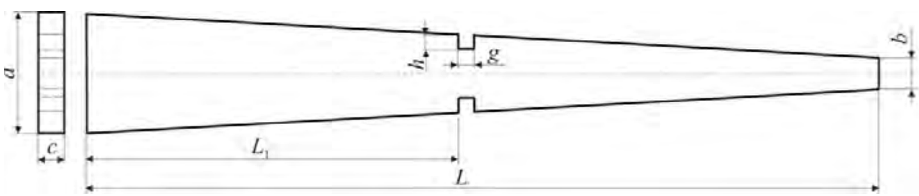

Fig. 1. A scheme of the system under study

Let us consider the model of the beam approximated by a stepped beam with constant geometrical and physical parameters of each $n = n_1 + n_2$ segments.

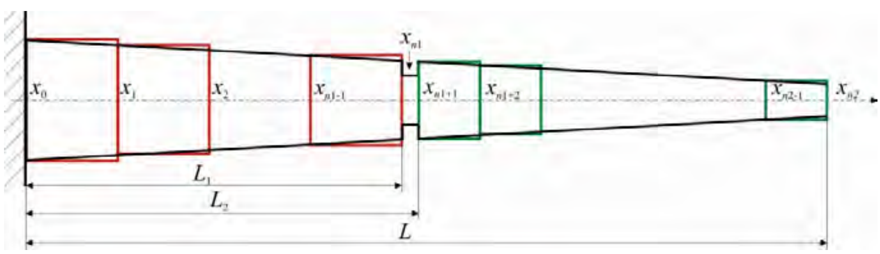

Fig. 2. A sketch of stepped beam

According to the Bernoulli-Euler theory, differential equation of motion of the considered system is:

$$EI_i \frac{\partial^4 y_i(x,t)}{\partial x^4} + \rho A_i \frac{\partial^2 y_i(x,t)}{\partial t^2} = \\ = -s_{i-1}(t)\delta(x - x_{i-1}) + s_i(t)\delta(x - x_i) + m_{i-1}(t)\delta'(x - x_{i-1}) - m_i(t)\delta'(x - x_i) \tag{1}$$

where: $y_i(x,t)$ is the transverse displacement, EI_i, ρA_i are the flexural rigidity and the mass per unit length of the i-th beam segment, respectively. Moreover, $\delta(\cdot)$ is the Dirac's delta function, $\delta'(\cdot)$ is the doublet function, the functions $s_i(t)$ and $m_i(t)$

are the shear force and bending moment acting on the right end of the i-th segment. The transverse displacements functions y_1, y_n satisfy homogeneous boundary conditions given in the form:

$$\mathbf{B}_0\left[y_1(x,t)\right]\Big|_{x=0} = \mathbf{0}, \quad \mathbf{B}_1\left[y_n(x,t)\right]\Big|_{x=L} = \mathbf{0} \tag{2}$$

and continuity conditions at points x_i, $i = 1,...,n-1$:

$$y_i(x_i,t) = y_{i+1}(x_i,t), \quad \left.\frac{\partial y_i(x,t)}{\partial x}\right|_{x=x_i} = \left.\frac{\partial y_{i+1}(x,t)}{\partial x}\right|_{x=x_i} \tag{3}$$

The dividing points x_i are: $\frac{L_1}{n_1}i$ for $i = 0,\ldots,n_1$, L_2 for $i = n_1+1$, $L_2 + \frac{L-L_2}{n_2-1}(i-n_1-1)$ for $i = n_1+2,\ldots,n$. EI_i, ρA_i for $\alpha = \frac{b}{a} > 0$ are as follows:

$$EI_i = \begin{cases} EI_0\left(\frac{\alpha-1}{L}x_{i-1}+1\right)^3 & i = 1,\ldots,n_1-1, n_1+1,\ldots,n \\ \frac{c}{12}(a-2h)^3 & i = n_1 \end{cases} \tag{4}$$

$$\rho A_i = \begin{cases} \rho A_0\left(\frac{\alpha-1}{L}x_{i-1}+1\right) & i = 1,\ldots,n_1-1, n_1+1,\ldots,n \\ c(a-2h) & i = n_1 \end{cases} \tag{5}$$

For the free vibration of the beam, we assume that $y_i(x,t) = Y_i(x)e^{j\omega t}$, $s_i(t) = \overline{S}_i\, e^{j\omega t}$, $m_i(t) = \overline{M}_i\, e^{j\omega t}$ $(j^2 = -1)$. Further, taking that $S_i = \overline{S}_i / EI_i$, $M_i = \overline{M}_i / EI_i$, $S_0 = S_n = M_0 = M_n = 0$ and $\Omega_i = \sqrt[4]{\frac{\rho A_i}{EI_i}\omega^2}$, where ω is the natural frequency, we finally get:

$$Y_i^{IV}(x) - \Omega_i^4 Y_i(x) = -S_{i-1}\delta(x-x_{i-1}) + S_i\delta(x-x_i) + M_{i-1}\delta'(x-x_{i-1}) - M_i\delta'(x-x_i) \tag{6}$$

$$\mathbf{B}_0\left[Y_1(x)\right]\Big|_{x=0} = \mathbf{0}, \quad \mathbf{B}_1\left[Y_n(x)\right]\Big|_{x=L} = \mathbf{0} \tag{7}$$

$$Y_i(x_i) = Y_{i+1}(x_i), \quad Y_i'(x)\Big|_{x=x_i} = Y_{i+1}'(x)\Big|_{x=x_i} \tag{8}$$

The solution of (6) determined with the use of the Green's function method [1, 2] has a form ($\mu_{i-1} = EI_{i-1}/EI_i$):

$$Y_i(x) = -\mu_{i-1} S_{i-1} G_i(x, x_{i-1}) + S_i G_i(x, x_i) + \mu_{i-1} M_{i-1} G'_{i,\zeta}(x, x_{i-1}) - M_i G'_{i,\zeta}(x, x_i) \quad (9)$$

Each function of G_i satisfies the nonhomogeneous equation

$$\frac{\partial^4 G_i(x,\zeta)}{\partial x^4} - \Omega_i^4 G_i(x,\zeta) = \delta(x-\zeta) \quad (10)$$

and can be written as a sum $G_i(x,\zeta) = G_{i\,0}(x,\zeta) + G_{i\,1}(x,\zeta) H(x-\zeta)$ (solution of the homogeneous equation $G_{i,xxxx}(x,\zeta) - \Omega_i^4 G_i(x,\zeta) = 0$ and solution of equation (10)), where

$$\begin{gathered} G_{i\,0}(x,\zeta) = c_{i1}\cos\Omega_i x + c_{i2}\sin\Omega_i x + c_{i3}\cosh\Omega_i x + c_{i4}\sinh\Omega_i x \\ G_{i\,1}(x,\zeta) = \left(2\Omega_i^3\right)^{-1}\left[\sinh\Omega_i(x-\zeta) - \sin\Omega_i(x-\zeta)\right] \end{gathered} \quad (11)$$

Functions $G_i(x,\zeta)$ satisfy the same boundary conditions as functions $Y_i(x)$ at $x = x_i$ for $i = 0,\ldots,n$ and in the considered problem they are:

for $i = 2,\ldots,n$ (C-C beam) $G'_{i,xx}(x,\zeta)\Big|_{x=x_{i-1}}^{x=x_i} = G'_{i,xxx}(x,\zeta)\Big|_{x=x_{i-1}}^{x=x_i} = 0$

for $i = 1$ (C-F beam): $G_1(x_0,\zeta) = G'_{1,x}(x_0,\zeta) = 0$ $G'_{1,xx}(x_1,\zeta) = G'_{1,xxx}(x_1,\zeta) = 0$

Under these conditions the constant values c_{ij}, $j = 1,\ldots,4$ are determined.

Taking into account continuity conditions (8) in (9), for $i = 1,\ldots,n-1$ we obtain a system of equations:

$$\begin{aligned} &-\mu_{i-1}S_{i-1}G_i(x_i,x_{i-1}) + S_i\left[G_i(x_i,x_i) + \mu_i G_{i+1}(x_i,x_i)\right] - S_{i+1}G_{i+1}(x_i,x_{i+1}) + \\ &+\mu_{i-1}M_{i-1}G_{i,\zeta}(x_i,x_{i-1}) + M_i\left[-G_{i,\zeta}(x_i,x_i) - \mu_i G_{i+1,\zeta}(x_i,x_i)\right] + M_{i+1}G_{i+1,\zeta}(x_i,x_{i+1}) = 0 \\ &-\mu_{i-1}S_{i-1}G_{i,x}(x_i,x_{i-1}) + S_i\left[G_{i,x}(x_i,x_i) + \mu_i G_{i+1,x}(x_i,x_i)\right] - S_{i+1}G_{i+1,x}(x_i,x_{i+1}) + \\ &+\mu_{i-1}M_{i-1}G_{i,\zeta x}(x_i,x_{i-1}) + M_i\left[-G_{i,\zeta x}(x_i,x_i) - \mu_i G_{i+1,\zeta x}(x_i,x_i)\right] + M_{i+1}G_{i+1,\zeta x}(x_i,x_{i+1}) = 0 \end{aligned} \quad (12)$$

or in the matrix form:

$$\mathbf{AX} = \mathbf{0} \quad (13)$$

where $\mathbf{X} = \left[S_1, S_2,\ldots,S_{n-1}, M_1, M_2,\ldots,M_{n-1}\right]^T$ and $\mathbf{A} = \left[A_{ij}\right]_{2(n-1)\times 2(n-1)}$. The elements of the main matrix are as follows:

for $i = 1$

$a_{11} = G_1(x_1,x_1) + \mu_1 G_2(x_1,x_1)$, $a_{12} = -G_2(x_1,x_2)$,

$a_{1n} = -G_{1,\zeta}(x_1,x_1) - \mu_1 G_{2,\zeta}(x_1,x_1)$, $a_{1,n+1} = G_{2,\zeta}(x_1,x_2)$,

$a_{21} = G_{1,x}(x_1,x_1) + \mu_1 G_{2,x}(x_1,x_1)$, $a_{22} = -G_{2,x}(x_1,x_2)$,

$a_{2,n} = -G_{1,\zeta x}(x_1,x_1) - \mu_1 G_{2,\zeta x}(x_1,x_1)$, $a_{2,n+1} = G_{2,\zeta x}(x_1,x_2)$

for $i = 2,\ldots,n-2$

$$a_{2i-1,i-1} = -\mu_{i-1}G_i(x_i, x_{i-1}),\ a_{2i-1,i} = G_i(x_i, x_i) + \mu_i G_{i+1}(x_i, x_i),$$
$$a_{2i-1,i+1} = -G_{i+1}(x_i, x_{i+1}),\ a_{2i-1,n+i-2} = \mu_{i-1}G_{i,\zeta}(x_i, x_{i-1}),$$
$$a_{2i-1,n+i-1} = -G_{i,\zeta}(x_i, x_i) - \mu_i G_{i+1,\zeta}(x_i, x_i),\ a_{2i-1,n+i} = G_{i+1,\zeta}(x_i, x_{i+1}),$$
$$a_{2i,i-1} = -\mu_{i-1}G_{i,x}(x_i, x_{i-1}),\ a_{2i-1,i} = G_{i,x}(x_i, x_i) + \mu_i G_{i+1,x}(x_i, x_i),$$
$$a_{2i,i+1} = -G_{i+1,x}(x_i, x_{i+1}),\ a_{2i,n+i-2} = \mu_{i-1}G_{i,\zeta x}(x_i, x_{i-1}),$$
$$a_{2i,n+i-1} = -G_{i,\zeta x}(x_i, x_i) - \mu_i G_{i+1,\zeta x}(x_i, x_i),\ a_{2i,n+i} = G_{i+1,\zeta x}(x_i, x_{i+1})$$

for $i = n-1$

$$a_{2n-3,n-2} = -\mu_{n-2}G_{n-1}(x_{n-1}, x_{n-2}),\ a_{2n-3,n-1} = G_{n-1}(x_{n-1}, x_{n-1}) + \mu_{n-1}G_n(x_{n-1}, x_{n-1}),$$
$$a_{2n-3,2n-3} = \mu_{n-2}G_{n-1,\zeta}(x_{n-1}, x_{n-2}),$$
$$a_{2n-3,2n-2} = -G_{n-1,\zeta}(x_{n-1}, x_{n-1}) - \mu_{n-1}G_{n,\zeta}(x_{n-1}, x_{n-1}),$$
$$a_{2n-2,n-2} = -\mu_{n-2}G_{n-1,x}(x_{n-1}, x_{n-2}),\ a_{2n-2,n-1} = G_{n-1,x}(x_{n-1}, x_{n-1}) + \mu_{n-1}G_{n,x}(x_{n-1}, x_{n-1})$$
$$a_{2n-2,2n-3} = \mu_{n-2}G_{n-1,\zeta x}(x_{n-1}, x_{n-2}),$$
$$a_{2n-2,2n-2} = -G_{n-1,\zeta x}(x_{n-1}, x_{n-1}) - \mu_{n-1}G_{n,\zeta x}(x_{n-1}, x_{n-1}).$$

The nontrivial solution of the equation (13) exists for the nonsingular matrix **A**, its yielding to the frequency equation:

$$\det \mathbf{A} = 0 \tag{14}$$

(14) is then solved numerically with respect to the frequencies ω_m. The mode shapes corresponding to ω_m are in the form of (9) for the coefficients $S_2,\ldots,S_{n-1},M_1,\ldots,M_{n-1}$ dependent on S_1.

3. Sample numerical results

Using the above formulae, an algorithm and a computer program that will allow for the determination of the influence of crack parameters on the free vibrations of the cantilever beam with variable cross-sectional area can be developed.

In the paper, the results of numerical calculations from CATIA program have been presented. The obtained natural frequencies of the cantilever beam with parameters $L = 500$ mm, $a = 75$ mm, $b = 20$ mm, $c = 5$ mm for the specified positions and sizes of crack have been illustrated on 3D plots (Figs. 3-8). It allowed one to determine how the change of the crack parameters (g - width, h - depth, L_1 - distance between the crack and mounting points of the beam) affect the first two natural frequencies of the beam. One of the three parameters is constant while the other two undergo changes on each graph.

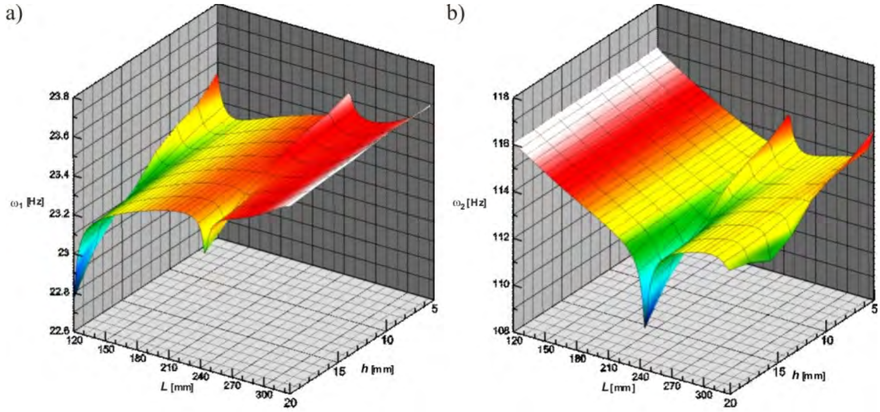

Fig. 3. Free vibrations frequencies for g = 1 mm

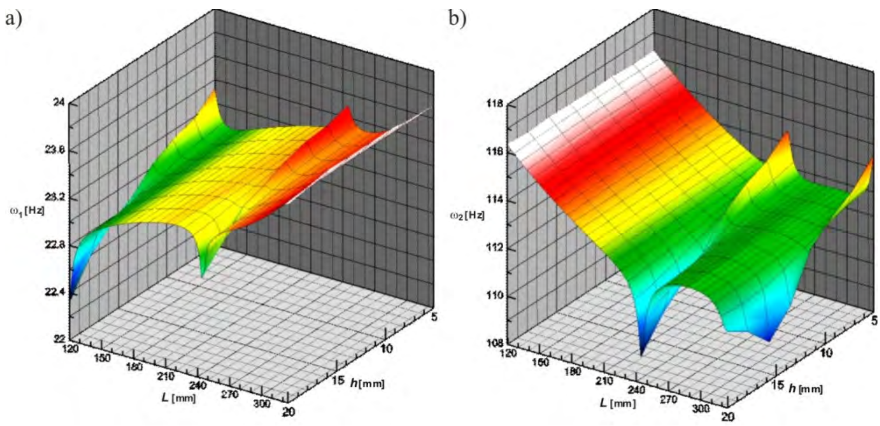

Fig. 4. Free vibrations frequencies for g = 10 mm

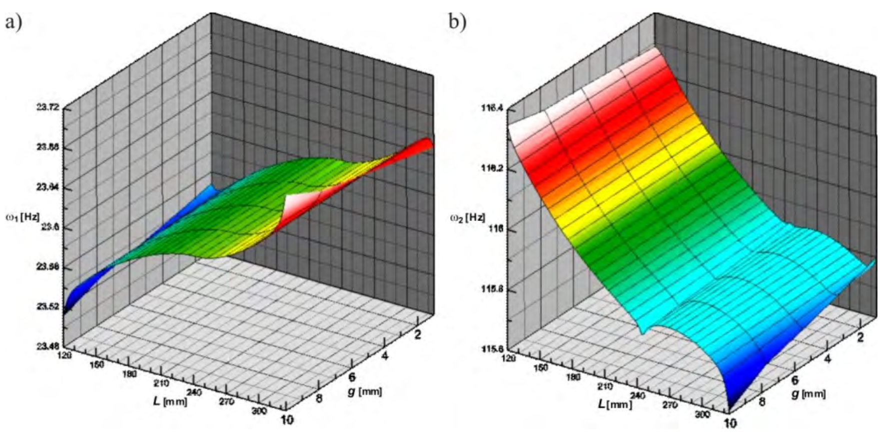

Fig. 5. Free vibrations frequencies for h = 5 mm

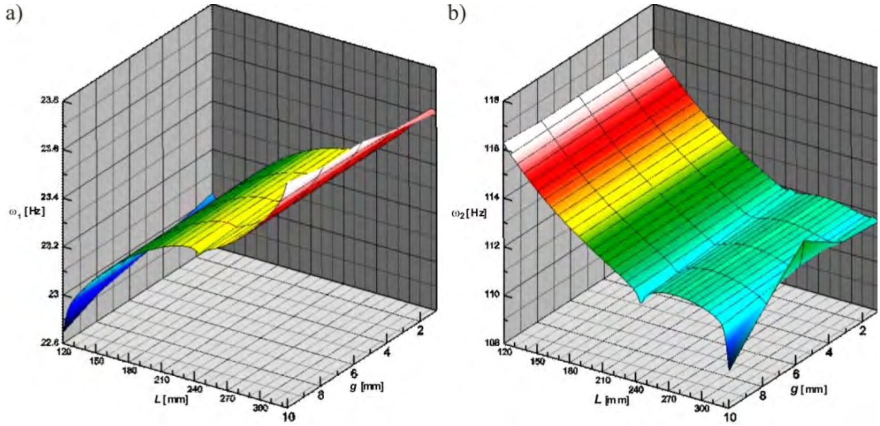

Fig. 6. Free vibrations frequencies for h = 15 mm

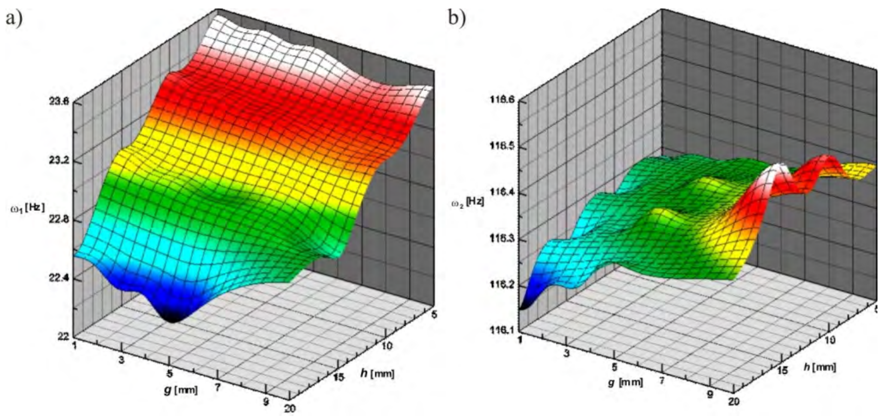

Fig. 7. Free vibrations frequencies for L_1 = 120 mm

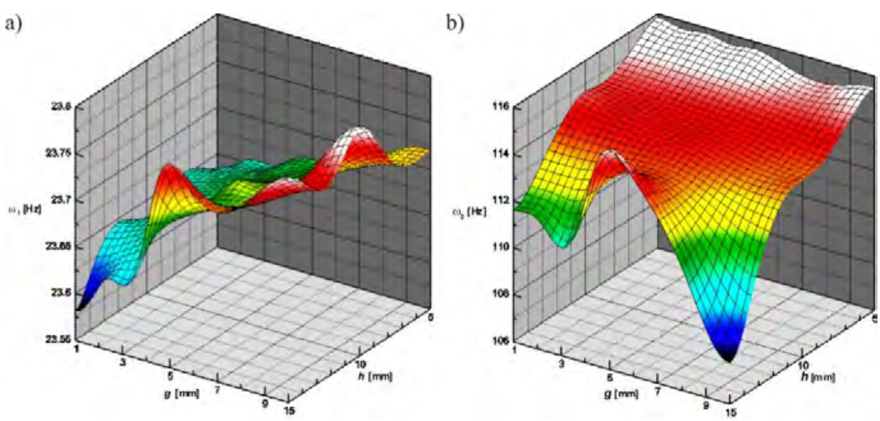

Fig. 8. Free vibrations frequencies for L_1 = 325 mm

By analyzing the influence of changes of the crack parameters, it can be seen that the increase of thickness, depth and offset value of the crack from the base causes an increase of the first and decrease of the second free vibration frequencies. The thickness and depth of the crack have a negligible effect on the natural frequencies of the system except when the crack is located 240 mm away from the clamp. Conversely, changing the distance of the damage from the base of the beam has a significant influence on the first two frequencies of the system.

4. Conclusions

The presented theoretical solution allows one to perform the numerical research of not only the influence of dimensional changes of the crack (or cracks) but also its (their) location on the beam on the free vibration frequencies of the system.

On the basis of the conducted numerical studies in the CATIA software, it was found that the thickness and depth of the crack have little effect on the system, but the location of damage causes significant changes in the natural frequencies of the beam. However, the smallest change in the vibration frequencies of the system may indicate damage. Therefore, use of the above considerations can develop a non-destructive method of the identification of the damage beam element of any cross section and any number of cracks.

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