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FINDING THE PROBABILISTICALLY-TEMPORAL CHARACTERISTICS OF MARKOV G-NETWORK WITH BATCH REMOVAL OF POSITIVE CUSTOMERS

***Mikhail Matalytski*[1], *Victor Naumenko*[2], *Dmitry Kopats*[2]**

[1] *Institute of Mathematics, Czestochowa University of Technology Częstochowa, Poland*
[2] *Faculty of Mathematics and Computer Science, Grodno State University Grodno, Belarus*
m.matalytski@gmail.com, victornn86@gmail.com

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Abstract. The investigation of a Markov queueing network with positive and negative customers and positive customers batch removal has been carried out in the article. The purpose of the research is analysis of such a network at the non-stationary regime, finding the time-dependent state probabilities and mean number of customers. In the first part of this article, a description of the G-network operation is provided with one-line queueing systems. When a negative customer arrives to the system, the count of positive customers is reduced by a random value, which is set by some probability distribution. Then for the non-stationary state probabilities a Kolmogorov system was derived of difference-differential equations. A technique for finding the state probabilities and the mean number of customers of the investigated network, based on the use of an apparatus of multi-dimensional generating functions has been proposed. The theorem about the expression for the generating function has been given. A model example has been calculated.

Keywords: *G-network, positive and negative customers, customers batch removal, non-stationary regime, generation function, state probabilities*

1. Introduction

G-networks or queueing networks with positive and negative customers was introduced by Gelenbe [1]. Such networks with batch removal of positive customers are used in the development of computer systems and networks models, in which defective problems (negative customers) destroy tasks or user jobs (positive customers). Task handlers are the processes running on servers; customers - user jobs for processing [2].

Let's consider an open G-queueing network with n single-line queueing systems (QS). To the S_i system external environment (QS S_0) a simple flow of customers arrives with the rate λ_{0i}^{+} and a simple flow of negative customers with the rate λ_{0i}^{-}, $i = \overline{1,n}$. All customer flows that arrive to the network are independent.

The service durations of positive customers in i-th QS distributed exponentially with the rate μ_i, $i=\overline{1,n}$.

In the articles [3-5], a G-network at a transient (non-stationary) regime has been investigated. At that negative customer, arrivining to the system, where there were positive customers, immediately destroy one of them, and then immediately left the network, not having received service in the QS. In the article [6] was conducted analysis at non-stationary regime of G-network with signals, and in the article [7] - G-network with signals with random delay.

In this paper, we describe in more detail the other behaviors of negative customers, which arrive to the network systems. Let at time t in the system S_i there are $k_i \geq B_i$ positive customers, where B_i - integer random value. Negative customer arriving to the some system of the network instantly destroys (removes from the network) destroyed immediately B_i of positive customers. If $k_i < B_i$, the system S_i completely is devastated (all positive customers, which are in this QS at a given time are immediately destroyed). Thus, random value B_i, which effectively determines the maximum size of the destruction batch of customers in QS S_i, is subject to an arbitrary discrete distribution law:

$$P\{B_i = m\} = \pi_{im},\ m \geq 1. \tag{1}$$

Gelenbe studied such behavior of negative customers considering signals in the paper [8], but only at stationary regime.

As previously explained, in each network system only positive customers are serviced. A positive customer serviced in the QS S_i, with probability p_{ij}^{+} sent to the QS S_j as a positive customer, with a probability p_{ij}^{-} - as a negative customer, and with probability $p_{i0} = 1 - \sum_{j=1}^{n}\left(p_{ij}^{+} + p_{ij}^{-}\right)$ leaving from the network to the external environment (QS S_0), $i, j = \overline{1,n}$.

The state of the network at time t meaning the vector

$$k(t) = (k,t) = ((k_1,t),(k_2,t),\ldots,(k_n,t)), \tag{2}$$

which forms a Markov random process with a countable number of states, where state (k_i,t) means that at time t in QS S_i there are k_i of positive customers, $i=\overline{1,n}$.

2. The system of the difference-differential Kolmogorov equations for the network state probabilities

We introduce some notations. Let's $P(k,t)$ - probability of the network state k at time t; $u(x)$ - Heaviside function, $u(x) = \begin{cases} 1, x > 0; \\ 0, x \leq 0. \end{cases}$; I_i - a vector of dimension

n, consisting of zeros, except for the component with number of i, which is equal to 1, $i=\overline{1,n}$. Let's allows

$$k_i^{+}=k+I_i=(k_1,k_2,...,k_i+1,...,k_n),$$
$$k_i^{+m}=k+mI_i=(k_1,k_2,...,k_i+m,...,k_n),\ m\geq 1,$$
$$k_i^{-}=k-I_i=(k_1,k_2,...,k_i-1,...,k_n),$$
$$k_{ij}^{+-}=k+I_i-I_j=(k_1,k_2,...,k_i+1,...,k_j-1,...,k_n),$$
$$k_{ij}^{++m}=k+I_i+mI_j=(k_1,k_2,...,k_i+1,...,k_j+m,...,k_n),\ m\geq 1. \quad (3)$$

There are following transitions of a random process $k(t)$ to the state (k,t) during time Δt:

1) from the state (k_i^{-},t), in this case, during time Δt a positive customer arriving to the QS S_i with probability $\lambda_{0i}^{+}\,u(k_i(t))\Delta t+o(\Delta t)$;
2) from the state (k_i^{+},t), wherein the positive customer leaves the network or in an external environment passes into QS S_j as a negative customer, if it had no customers; the probability of such an event equals $\left(\mu_i p_{i0}+\mu_i p_{ij}^{-}\left(1-u(k_j(t))\right)\right)\Delta t+o(\Delta t)$;
3) from the states (k_i^{+m},t), in this case a negative customer arrives to the QS S_i and destroys a random batch of positive customers; the probability of such event is equal to $\lambda_{0i}^{-}(1-u(k_i(t)))\sum_{m=1}^{\infty}\pi_{im}\Delta t+o(\Delta t)$;
4) from the state (k_i^{+-},t), in this case, after the servicing of a positive customer in the QS S_i it goes to the QS S_j; the probability of such an event is equal to $\mu_i p_{ij}^{+}u(k_j(t))\Delta t+o(\Delta t)$;
5) from the states in this case, after the servicing of a positive customer in the QS S_i it goes to the QS S_j as a negative customer, which destroys in it random batch of positive customers; the probability of such an event is equal to $\mu_i p_{ij}^{-}(1-u(k_j(t)))\sum_{m=1}^{\infty}\pi_{im}\Delta t+o(\Delta t)$;
6) from the state (k,t), while in each QS S_i, $i=\overline{1,n}$, there are no positive customers nor any negative customers arriving, and which during Δt was not to serviced by any customer; the probability of such event is equal to $1-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i\right]\Delta t+o(\Delta t)$, $i=\overline{1,n}$;
7) of the remaining states with a probability $o(\Delta t)$.

Then, using the formula of total probability, we shall have

$$
\begin{aligned}
P(k,t+\Delta t) = \sum_{i=1}^{n} \Big[& \left(\lambda_{0i}^{+}\Delta t + o(\Delta t)\right) P\left(k_i^{-},t\right) u(k_i(t)) + \\
& + \lambda_{0i}^{-} \sum_{m=1}^{\infty} \left(\pi_{im}\Delta t + o(\Delta t)\right) P\left(k_i^{+m},t\right) + \\
& + \lambda_{0i}^{-} \sum_{m=1}^{\infty} \pi_{im} \sum_{r=0}^{m-1} P\left(k_i^{+r},t\right)\left(1-u(k_i(t))\right)\Delta t + o(\Delta t) + \\
& + \sum_{j=1}^{n} \Big\{ \left(\mu_i p_{ij}^{+}\Delta t + o(\Delta t)\right) P\left(k_{ij}^{+-},t\right) u\left(k_j(t)\right) + \\
& + \sum_{m=1}^{\infty} \pi_{im} \left(\mu_i p_{ij}^{-}\Delta t + o(\Delta t)\right) P\left(k_{ij}^{++m},t\right) + \\
& + \sum_{m=1}^{\infty} \pi_{im} \sum_{r=0}^{m-1} P\left(k_{ij}^{++r},t\right)\left(1-u\left(k_j(t)\right)\right)\Delta t + o(\Delta t) + \\
& + \left(\mu_i p_{ij}^{-}\Delta t + o(\Delta t)\right) P\left(k_i^{+},t\right)\left(1-u\left(k_j(t)\right)\right) \Big\} + \\
& + \left(\mu_i p_{i0}\Delta t + o(\Delta t)\right) P\left(k_i^{+},t\right) + \left(1-\left(\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i\right)\Delta t + o(\Delta t)\right) P(k,t) \Big]. \qquad (4)
\end{aligned}
$$

Dividing both sides of this relation by Δt and passing to the limit $\Delta t \to 0$, we obtain that the non-stationary states probabilities of the considered network satisfy the following system of difference-differential equations

$$
\begin{aligned}
\frac{dP(k,t)}{dt} = & -\sum_{i=1}^{n} \left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i\right] P(k,t) + \sum_{i=1}^{n} \lambda_{0i}^{+} u(k_i(t)) P\left(k_i^{-},t\right) + \\
& + \sum_{i=1}^{n} \left[\mu_i \left(p_{i0} + \sum_{j=1}^{n} p_{ij}^{-}\left(1-u\left(k_j(t)\right)\right) \right) \right] P\left(k_i^{+},t\right) + \\
& + \sum_{i=1}^{n} \lambda_{0i}^{-} \left\{ \sum_{m=1}^{\infty} \pi_{im} \left[P\left(k_i^{+m},t\right) + \left(1-u(k_i(t))\right) \sum_{r=0}^{m-1} P\left(k_i^{+r},t\right) \right] \right\} + \qquad (5) \\
& + \sum_{i,j=1}^{n} \left\{ \mu_i p_{ij}^{+} u\left(k_j(t)\right) P\left(k_{ij}^{+-},t\right) + \sum_{m=1}^{\infty} \pi_{im} \left[\mu_i p_{ij}^{-} P\left(k_{ij}^{++m},t\right) + \left(1-u\left(k_j(t)\right)\right) \sum_{r=0}^{m-1} P\left(k_{ij}^{++r},t\right) \right] \right\}.
\end{aligned}
$$

Suppose that all systems of the network operating in a heavy traffic regime, i.e. $k_i(t) > 0,\ \forall t > 0,\ i = \overline{1,n}$. Taking this assumption into account, the system (5) takes the form

$$\frac{dP(k,t)}{dt} = -\sum_{i=1}^{n}\left[\lambda_{0i}^{+} + \lambda_{0i}^{-} + \mu_i\right]P(k,t) +$$

$$\sum_{i=1}^{n}\lambda_{0i}^{+}P(k_i^{-},t) + \sum_{i=1}^{n}\mu_i p_{i0}P(k_i^{+},t) + \sum_{i=1}^{n}\lambda_{0i}^{-}\sum_{m=1}^{\infty}\pi_{im}P(k_i^{+m},t) +$$

$$+\sum_{i,j=1}^{n}\left[\mu_i p_{ij}^{+}P(k_{ij}^{+-},t) + \sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}P(k_{ij}^{++m},t)\right]. \qquad (6)$$

3. Finding network state probabilities using the generating function

Denote by $\Psi_n(z,t)$, where $z=(z_1,z_2,...,z_n)$, $|z|<1$, n-dimensional generating function:

$$\Psi_n(z,t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_1,..,k_n,t)\,z_1^{k_1}z_2^{k_2}\cdot...\cdot z_n^{k_n} = \sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k,t)\prod_{i=1}^{n}z_i^{k_i} \qquad (7)$$

Multiplied (6) on $\prod_{l=1}^{n}z_l^{k_l}$ and adding together all possible values k_l from 0 to $+\infty$, $l=\overline{1,n}$, obtain:

$$\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\frac{dP(k,t)}{dt}\prod_{l=1}^{n}z_l^{k_l} = -\sum_{i=1}^{n}\left(\lambda_{0i}^{+} + \lambda_{0i}^{-} + \mu_i\right)\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l} +$$

$$+\sum_{i=1}^{n}\lambda_{0i}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_i^{-},t)\prod_{l=1}^{n}z_l^{k_l} + \sum_{i=1}^{n}\mu_i p_{i0}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_i^{+},t)\prod_{l=1}^{n}z_l^{k_l} +$$

$$+\sum_{i=1}^{n}\lambda_{0i}^{-}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{m=1}^{\infty}\pi_{im}P(k_i^{+m},t)\prod_{l=1}^{n}z_l^{k_l} + \sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_{ij}^{+-},t)\prod_{l=1}^{n}z_l^{k_l} +$$

$$+\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_{ij}^{++m},t)\prod_{l=1}^{n}z_l^{k_l}. \qquad (8)$$

Consider some sums, contained on the right side of (8). Let's allows

$$\sum\nolimits_1(z,t) = \sum_{i=1}^{n}\lambda_{0i}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_i^{-},t)\prod_{l=1}^{n}z_l^{k_l},$$

then we can show that

$$\sum\nolimits_1(z,t) = \sum_{i=1}^{n}\lambda_{0i}^{+}z_i\sum_{\substack{k_j=0\\ j=\overline{1,n}}}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l} = \sum_{i=1}^{n}\lambda_{0i}^{+}z_i\,\Psi_n(z,t).$$

For the sum of $\sum_2(z,t)=\sum_{i=1}^{n}\mu_i p_{i0}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P\left(k_i^{+},t\right)\prod_{l=1}^{n}z_l^{k_l}$ we have:

$$\sum_2(z,t)=\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}\Psi_n(z,t)-\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}\sum_{\substack{k_j=0\\ j=\overline{1,n},\, j\neq i}}^{\infty}P(k_1,...,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

For the sum of $\sum_3(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{-}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{m=1}^{\infty}\pi_{im}P\left(k_i^{+m},t\right)\prod_{l=1}^{n}z_l^{k_l}$ we will have:

$$\sum_3(z,t)=\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}\Psi_n(z,t)-\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}\sum_{\substack{k_j=0\\ j=\overline{1,n},\, j\neq i}}^{\infty}P(k_1,...,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

For the sum of $\sum_4(z,t)=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P\left(k+I_i-I_j,t\right)\prod_{l=1}^{n}z_l^{k_l}$ we obtain:

$$\sum_4(z,t)=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\Psi_n(z,t)-$$
$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}P\left(k_1,...,k_{i-1},0,k_{i+1},...,k_n,t\right)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

And finally, for the last sum

$$\sum_5(z,t)=\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P\left(k_{ij}^{++m},t\right)\prod_{l=1}^{n}z_l^{k_l}$$

we shall have:

$$\sum_5(z,t)=\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}\Psi_n(z,t)-$$
$$-\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}\sum_{\substack{k_s=0\\ s=\overline{1,n},s\neq j}}^{\infty}P(k_1,...,k_{i-1},0,k_{i+1},...,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

Therefore, the generating function is a fairly ordinary inhomogeneous linear ordinary differential equation

$$\frac{d\,\Psi_n(z,t)}{dt}=-\left[\sum_{i=1}^{n}\left(\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i\right)-\sum_{i=1}^{n}\lambda_{0i}^{+}z_i-\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}-\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}-\right.$$
$$\left.-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}-\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}\right]\Psi_n(z,t)-$$
$$-\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}\sum_{\substack{k_j=0\\ j=\overline{1,n},\, j\neq i}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}-$$
$$-\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}\sum_{\substack{k_j=0\\ j=\overline{1,n},\, j\neq i}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}-$$
$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},\, m\neq j}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}-$$
$$-\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}\sum_{\substack{k_s=0\\ s=\overline{1,n},\, s\neq j}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}. \quad (9)$$

Since all of the whole QS network operate under high load conditions, the last two expressions in the form of the sums in equation (9) will be zero, and it becomes homogeneous:

$$\frac{d\,\Psi_n(z,t)}{dt}=-\left[\sum_{i=1}^{n}\left(\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i\right)-\sum_{i=1}^{n}\lambda_{0i}^{+}z_i-\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}-\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}-\right.$$
$$\left.-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}-\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}\right]\Psi_n(z,t).$$

Its general solution has the form

$$\Psi_n(z,t)=C_n\exp\left\{-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i-\lambda_{0i}^{+}z_i-\frac{1}{z_i}\left(\mu_i p_{i0}+\sum_{m=1}^{\infty}\frac{\pi_{im}\lambda_{0i}^{-}}{z_i^{m-1}}-\right.\right.\right.$$
$$\left.\left.\left.-\mu_i\sum_{j=1}^{n}p_{ij}^{+}z_j-\mu_i\sum_{j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_j^m}\right)\right]t\right\}. \quad (10)$$

We assume that at the initial moment of time, the network is in state $(\alpha_1,\alpha_2,\ldots,\alpha_n,0)$, $\alpha_i>0$, $P(\alpha_1,\alpha_2,\ldots,\alpha_n,0)=1$, $P(k_1,k_2,\ldots,k_n,0)=0$, $\forall\alpha_i\neq k_i$, $i=\overline{1,n}$. Then the initial condition for the last equation (10) will be

$\Psi_n(z,0) = P(\alpha_1,\alpha_2,...,\alpha_n,0)\prod_{l=1}^{n} z_l^{\alpha_l} = \prod_{l=1}^{n} z_l^{\alpha_l}$. Using it, we obtain $C_n = 1$. Thus, the expression for the generating function $\Psi_n(z,t)$ has the form

$$\Psi_n(z,t) = a_0(t)\exp\left\{\sum_{i=1}^{n}\lambda_{0i}^{+}z_i t\right\}\exp\left\{\sum_{i=1}^{n}\frac{\mu_i p_{i0}}{z_i}t\right\}\exp\left\{\sum_{i=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\frac{\lambda_{0i}^{-}}{z_i^m}t\right\}\times$$
$$\times\exp\left\{\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}t\right\}\exp\left\{\sum_{i,j=1}^{n}\sum_{m=1}^{\infty}\pi_{im}\mu_i p_{ij}^{-}\frac{1}{z_i z_j^m}t\right\}\prod_{l=1}^{n} z_l^{\alpha_l}, \tag{11}$$

where

$$a_0(t) = \exp\left\{-\sum_{i=1}^{n}\left(\lambda_{0i}^{+} + \lambda_{0i}^{-} + \mu_i\right)t\right\}.$$

To find the probability of the network states, we transform (11) to a convenient form by expanding its member exhibitors in a Maclaurin series. We can show that the statement is true.

Theorem. *The expression for the generating function can be represented in the form*

$$\Psi_n(z,t) = a_0(t)\sum_{l_1=0}^{\infty}\cdots\sum_{l_n=0}^{\infty}\sum_{q_1=0}^{\infty}\cdots\sum_{q_n=0}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\sum_{u_1=0}^{\infty}\cdots\sum_{u_n=0}^{\infty}\sum_{w_1=0}^{\infty}\cdots\sum_{w_n=0}^{\infty} t^{\sum_{i=1}^{n}(l_i+q_i+r_i+u_i+w_i)}$$
$$\times\prod_{i=1}^{n}\left[\frac{\left(\lambda_{0i}^{+}\right)^{l_i}\left(\mu_i p_{i0}\right)^{q_i}\left(\lambda_{0i}^{-}\prod_{m=1}^{\infty}\pi_{im}\right)^{r_i}\mu_i^{u_i+w_i}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{u_i}\left(\prod_{j=1}^{n}\prod_{m=1}^{\infty}\pi_{im}p_{ij}^{-}\right)^{w_i}}{l_i!\,q_i!\,r_i!\,u_i!\,w_i!}\times\right.$$
$$\left.\times z_i^{\alpha_i+l_i-q_i-mr_i-u_i-U-mw_i-mW}\right], \tag{12}$$

where $U = \sum_{i=1}^{n} u_i$, $W = \sum_{i=1}^{n} w_i$.

The probability $P(k_1,k_2,...,k_n,t)$ can be found as the coefficient of $z_1^{k_1}z_2^{k_2}...z_n^{k_n}$ in the expansion of function $\Psi_n(z,t)$ in multiple series (12), and the mean number of customers in the system S_i in the form $N_i(t) = \left.\frac{\partial\Psi_n(z,t)}{\partial z_i}\right|_{z=(1,1,...,1)}$.

4. Model example

In this example the count of QS $n=5$ lets arriving probabilities of customers to i-th QS be respectively equal $p_{01}^{+}=0.3$, $p_{02}^{+}=0.1$, $p_{03}^{+}=p_{04}^{+}=p_{05}^{+}=0.2$, $p_{01}^{-}=p_{02}^{-}=p_{03}^{-}=p_{04}^{-}=p_{05}^{-}=0.2$, and $\sum_{i=1}^{5} p_{0i}^{+}=1$, $\sum_{i=1}^{5} p_{0i}^{-}=1$.

Consider the time interval $[0;T]$. As a time unit we take 1 hour, then $T=24$ h. Let the arriving rates of positive and negative customers be respectively equal $\lambda^{+}=100$ and $\lambda^{-}=90$. Let's arriving rates of positive and negative customers to each QS λ_{0i}^{+} and λ_{0i}^{-}, taking into account that $\lambda_{0i}^{+}=\lambda^{+}p_{0i}^{+}$, $\lambda_{0i}^{-}=\lambda^{-}p_{0i}^{-}$, $i=\overline{1,5}$, respectively will be equal: $\lambda_{01}^{+}=30$, $\lambda_{02}^{+}=10$, $\lambda_{03}^{+}=20$, $\lambda_{04}^{+}=20$, $\lambda_{05}^{+}=20$, $\lambda_{01}^{-}=15$, $\lambda_{02}^{-}=15$, $\lambda_{03}^{-}=15$, $\lambda_{04}^{-}=15$, $\lambda_{05}^{-}=15$. Suppose, that service rates of customers in QS are equal: $\mu_1=50$, $\mu_2=50$, $\mu_3=50$, $\mu_4=50, \mu_5=50$.

Let us also transition probabilities of positive and negative customers between QS be equal $p_{11}^{+}=0.1$, $p_{12}^{+}=0.3$, $p_{12}^{-}=0.1$, $p_{1i}^{+}=0.05$, $p_{1i}^{-}=0.05$, $i=\overline{3,5}$, $p_{10}=1-\sum_{j=1}^{5}\left(p_{1j}^{+}+p_{1j}^{-}\right)=0.1$; $p_{21}^{+}=0.1$, $p_{21}^{-}=0.1$, $p_{22}^{+}=0.2$, $p_{22}^{-}=0.1$, $p_{2i}^{-}=0.05$, $p_{2i}^{+}=0.05$, $i=\overline{3,5}$, $p_{20}=1-\sum_{j=1}^{5}\left(p_{2j}^{+}+p_{2j}^{-}\right)=0.2$, $p_{ji}^{-}=0.1$, $p_{ji}^{+}=0.1$, $i=\overline{1,3}$, $p_{ji}^{-}=0.05, p_{ji}^{+}=0.05$, $i=\overline{4,5}$, $j=\overline{3,5}$, $p_{i0}=1-\sum_{j=1}^{5}\left(p_{ij}^{+}+p_{ij}^{-}\right)=0.2, i=\overline{3,5}$.

Calculations have shown that $a_0(t)=e^{-425t}$.

Let's a random value B_i has a Poisson distribution with parameter $\lambda>0$ and let $m=5$, then (1) takes form $\pi_{im}=\frac{\lambda^m}{m!}e^{-\lambda}$, $m=\overline{0,5}$, $i=\overline{1,5}$. Let also $\lambda=5$.

We need to find, for example, state probability $P(1,2,3,4,5,t)$. It is the coefficient of $z_1^1 z_2^2 z_3^3 z_4^4 z_5^5$ in the expansion of $\Psi_5(z,t)$ in multiple series (12), so that the degree of z_i must satisfy the relation $\alpha_i+l_i-q_i-mr_i-u_i-U-mw_i-mW=k_i$, $i=\overline{1,5}$, where $k_1=1$, $k_2=5$, $k_3=3$, $k_4=4$, $k_5=5$. Hence, it follows that

$$q_i=\alpha_i+l_i-mr_i-u_i-U-mw_i-mW-k_i,\ i=\overline{1,5},$$

$$q_i=\alpha_i+l_i-m\left(r_i+w_i+\sum_{i=1}^{5}w_i\right)-\left(u_i+\sum_{i=1}^{5}u_i\right)-k_i,\ i=\overline{1,5},$$

$$l_i+q_i+r_i+u_i+w_i=\alpha_i+2l_i+r_i+u_i+w_i-m\left(r_i+w_i+\sum_{i=1}^{5}w_i\right)-\left(u_i+\sum_{i=1}^{5}u_i\right)-k_i,$$

$$l_i + q_i + r_i + u_i + w_i = \alpha_i + 2l_i + (1-m)(r_i + w_i) - m\sum_{i=1}^{5} w_i - \sum_{i=1}^{5} u_i - k_i,\ i = \overline{1,5},$$

$$\sum_{i=1}^{5}(l_i + q_i + r_i + u_i + w_i) =$$

$$= \sum_{i=1}^{5}(\alpha_i + 2l_i) + 5(1-m)\sum_{i=1}^{5}(r_i + w_i) - 4(mW + U + k_i)$$

Then from (12) we find that

$$P(1,2,3,4,5,t) = e^{-425t} \times$$

$$\times \sum_{\substack{l_i=0 \\ i=\overline{1,5}}}^{\infty} \sum_{\substack{r_i=0 \\ i=\overline{1,5}}}^{\infty} \sum_{\substack{u_i=0 \\ i=\overline{1,5}}}^{\infty} \sum_{\substack{w_i=0 \\ i=\overline{1,5}}}^{\infty} t^{\sum_{i=1}^{5}(\alpha_i + 2l_i) + 4(1-m)\sum_{i=1}^{5}(r_i + w_i) - 4(mW + U + k_i)} \times$$

$$\times \prod_{i=1}^{5} \frac{(\lambda_{0i}^{+})^{l_i} (\mu_i p_{i0})^{\alpha_i + l_i - m\left(r_i + w_i + \sum_{i=1}^{5} w_i\right) - \left(u_i + \sum_{i=1}^{5} u_i\right) - k_i} \left(\lambda_{0i}^{-} \prod_{m=0}^{5} \frac{\lambda^m}{m!} e^{-\lambda}\right)^{r_i}}{l_i!\left(\alpha_i + l_i - m\left(r_i + w_i + \sum_{i=1}^{5} w_i\right) - \left(u_i + \sum_{i=1}^{5} u_i\right) - k_i\right)!\ r_i!u_i!w_i!} \times$$

$$\times \prod_{j=1}^{5}\left[(\mu_j p_{ji}^{+})^{u_j} (\mu_i p_{4i}^{+})^{u_4} \left(\mu_j p_{ji}^{-} \prod_{m=0}^{5} \frac{\lambda^m}{m!} e^{-\lambda_j}\right)^{w_j}\right].$$

Figure 1 shows a chart of the state probability $P(1,2,3,4,5,t)$ for different t on the condition that at the initial time, the network is in a state of $\alpha_i = 0$, $i = \overline{1,5}$.

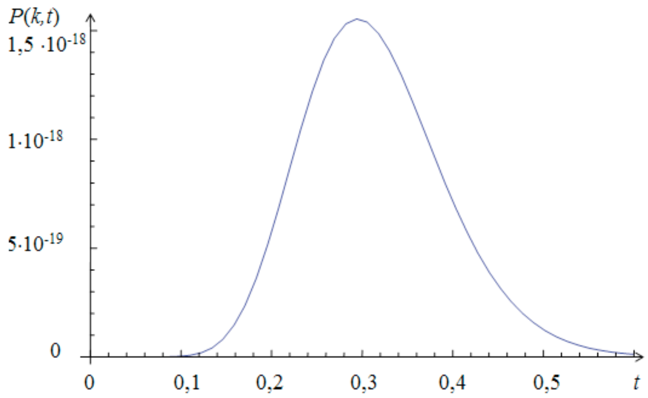

Fig. 1. The chart of the state probability $P(1,2,3,4,5,t)$

Calculating the partial derivative of the generating function (12) by variable z_i in dot $z=(1,1,\ldots,1)$, and carrying out a series of mathematical transformations, we find that the mean number of customers in the queue system S_i is calculated by the formula

$$N_i(t)=a_0(t)\sum_{k_1=0}^{\infty}\ldots\sum_{k_i=0}^{\infty}k_i\ldots\sum_{k_n=0}^{\infty}\ldots\sum_{r_1=0}^{\infty}\ldots\sum_{r_n=0}^{\infty}\sum_{u_1=0}^{\infty}\ldots\sum_{u_n=0}^{\infty}\sum_{w_1=0}^{\infty}\ldots\sum_{w_n=0}^{\infty}$$

$$\sum_{q_1=0}^{\alpha_1-mr_1-u_1-U-mw_1-mW}\ldots\sum_{q_n=0}^{\alpha_n-mr_n-u_n-U-mw_n-mW} t^{\sum_{i=1}^{n}(k_i-\alpha_i+2q_i+(m+1)(r_i+w_i)+2u_i+U+mW)}\times$$

$$\times\prod_{i=1}^{n}\frac{\left(\lambda_{0i}^{+}\right)^{k_i-\alpha_i+q_i+mr_i+u_i+U+mw_i+mW}(\mu_i p_{i0})^{q_i}\left(\lambda_{0i}^{-}\prod_{m=1}^{\infty}\pi_{im}\right)^{r_i}\mu_i^{u_i+w_i}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{u_i}\left(\prod_{j=1}^{n}\prod_{m=1}^{\infty}\pi_{im}p_{ij}^{-}\right)^{w_i}}{(k_i-\alpha_i+q_i+mr_i+u_i+U+mw_i+mW)!\,q_i!\,r_i!\,u_i!\,w_i!}.$$

Figure 2 shows a plot of changes of the mean number of customers in QS S_1.

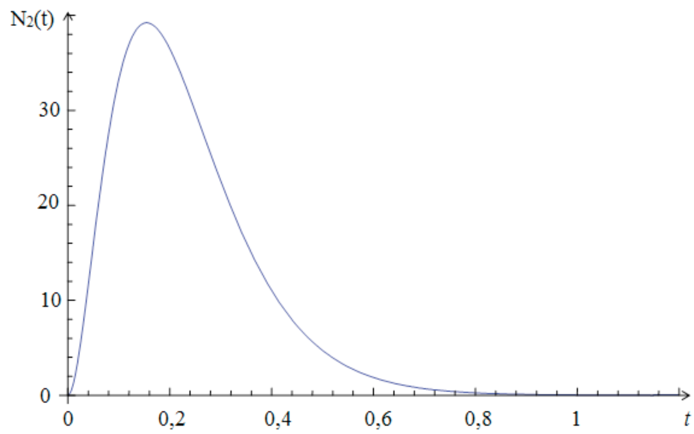

Fig. 2. The chat for changes of the mean number of positive customers $N_2(t)$ in QS S_2

5. Conclusions

In the article an investigation of Markov G-network was conducted in the case when a negative customer can destroy a random batch of positive customers. For such a network operating under a heavy traffic regime, finding a technique for non-stationary state probabilities and a mean number of customers in network systems was proposed using an apparatus based on the use of multivariate generating functions.

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