

AN APPROACH TO EXPONENTIATION WITH INTERVAL-VALUED POWER

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Abstract. The main aim of the work is introducing an operation of raising intuitionistic fuzzy values to intuitionistic fuzzy power, which not requiring to conversion of intuitionistic fuzzy values. Introducing an operation of raising intuitionistic fuzzy values to intuitionistic fuzzy power, which does not require conversion of intuitionistic fuzzy values is the main aim of the work. It is known that, in the classical intuitionistic fuzzy sets theory, the use of all aggregation modes is not always possible because of the lack of definition of raising intuitionistic fuzzy values to intuitionistic fuzzy power. Therefore, the specific aim of the work is to present the heuristic method of raising intuitionistic fuzzy values to intuitionistic fuzzy power, and the consideration of its properties.

Keywords: *IVFS, DST, Dempster-Shafer theory, interval-values fuzzy sets, operations on IVFS's*

1. Introduction

It is known that different variants of the aggregation of local criteria give rise to different results. It follows from this fact that the validity of the stage of formulation of a global criterion as an aggregation of local criteria is dominant. It is obvious that the evaluation of the validity of the criteria is not essential in some optimization processes and sometimes all local criteria have the same validity (weight) for decision-makers. What's more, it is sometimes impossible to define the weight by using real numbers. Therefore, it is more proper to use the transformation of verbal terms to interval or fuzzy values applied to various types of aggregation modes. Many aggregation modes are available to use to make decisions. Not only real numbers [1-12] can be used to describe them.

One of the most common generalizations of the fuzzy sets theory is the one introduced by Atanassov [3], which is mainly used for solving MCDM [13-19] and group MCDM [5, 6, 20-25] problems in cases when intuitionistic fuzzy values are the value of the local criteria of alternatives and/or their weight.

Disadvantages of classical operations on intuitionistic fuzzy values are the base of some issues with intuitionistic fuzzy uncertainty in MCDM framework. A few limitations of traditional operations on fuzzy values were conferred in [26]. In [27], appropriate critical examples can be read about.

An approach based on the intuitionistic fuzzy matrix and the relations between the elements of the matrix, or the intuitionistic fuzzy sets, were introduced in [28]. In [27] it was confirmed that correct results cannot always be obtained by the method of comparing the intuitionistic fuzzy numbers introduced in [28].

The lack of a definition of raising intuitionistic fuzzy values to intuitionistic fuzzy power is another issue which can be experienced with the classical intuitionistic fuzzy sets theory. It is noteworthy that the lack of such a definition severely curtails the number of aggregation modes which can be applied in MCDM problems under the circumstances of fuzzy uncertainty.

In paper [27], it was demonstrated that the use of the Dempster-Shafer theory based on converting intuitionistic fuzzy values to belief intervals allows one to achieve results which are more reliable, as well as facilitates the calculations in the solution of the MCDM problem. However, when using the conversion of the intuitionistic fuzzy values is not appropriate, using the heuristic method of raising intuitionistic fuzzy values to intuitionistic fuzzy power is advisable.

The remaining part of the paper is organized in the following way. The basic definitions of the classical intuitionistic fuzzy sets theory are given in section 2. In section 3, we present the extension of the intuitionistic fuzzy set theory in the framework of the Dempster-Shafer theory. Section 4 is devoted to an operation on the intuitionistic fuzzy sets in the framework of the Dempster-Shafer theory, and new operators of raising intuitionistic fuzzy values to intuitionistic fuzzy power, as well as the Intuitionistic Fuzzy Weighted Geometric operator with weights presented by intuitionistic fuzzy values are introduced. In section 5, a few properties of the exponentiation operation achieved by the transformation to belief intervals and the heuristic IFV method were proved.

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2. Basic definitions

One of the most common generalizations of the fuzzy sets is the intuitionistic fuzzy sets theory which was introduced by Atanassov [3] which is mainly used for solving issues with MCDM [7, 14-19] and group MCDM [5, 6, 20-25] under the circumstances when the intuitionistic fuzzy values are the value of the local criteria of alternatives and/or their weight.

The base of the definition of the intuitionistic fuzzy set are the considerations of membership function μ and non-membership function ν of element x to a set X , where $0 \leq \mu(x) + \nu(x) \leq 1$. After that, the following set can be constructed

$\{\langle x, \mu(x), \nu(x) \rangle \mid x \in E\}$, where $0 \leq \nu(x) + \mu(x) \leq 1$. For constant $x \in E$, pair $\langle \mu(x), \nu(x) \rangle$ is called the intuitionistic fuzzy value (IFV) or the intuitionistic fuzzy number.

A few operators based on the synthesis of intuitionistic fuzzy sets and the Dempster-Shafer Theory (DST) were presented in sections [29] and [30]. It can easily be seen that the operators which base on the Choquet integral [30] are advantageous under circumstances where the aggregate weight of the assessments have some correlation with one other.

The strong connection between the intuitionistic fuzzy sets and DST was presented in [31]. This connection allows one to directly apply Dempster's rule of combination in MCDM issues in order to aggregate local criteria with intuitionistic fuzzy values. This connection, or link, was also presented in [32, 33].

In [3] Atanassov gives the following definition of intuitionistic fuzzy set:

Definition 1. Let $X = \{x_1, x_2, \dots, x_n\}$ be finite universal set. An object A in X is called intuitionistic fuzzy set (IFS) if it has the following form: $A = \{\langle x_j, \mu_A(x_j), \nu_A(x_j) \rangle \mid x_j \in X\}$, where functions $\mu_A: X \rightarrow [0, 1]$, $x_j \in X \rightarrow \mu_A(x_j) \in [0, 1]$ and $\nu_A: X \rightarrow [0, 1]$, $x_j \in X \rightarrow \nu_A(x_j) \in [0, 1]$ determines the degree of membership and non-membership of element $x_j \in X$ of $A \subseteq X$ respectively, and, for each $x_j \in X$, the inequality $0 \leq \mu_A(x_j) + \nu_A(x_j) \leq 1$ holds.

Parameter $\pi_A(x) = 1 - \mu_A(x_j) - \nu_A(x_j)$ is called an intuitionistic index (or the hesitation degree) of the element x_j in A [3].

Of course, for each $x_j \in X$, we have $0 \leq \pi_A(x) \leq 1$.

Being that an intuitionistic set is the generalization of a typical fuzzy one, all of the regular results from the classical fuzzy set theory may be converted in the framework of the intuitionistic fuzzy sets theory (IFST), and all of the studies which base on regular fuzzy sets can be expressed by IFSs. However, IFST does not only contain operations which are compatible on fuzzy sets, but also such operations which cannot be expressed in the framework of the regular fuzzy set theory [34].

The operations of addition \oplus and multiplication \otimes on IFV are defined by Atanassov [4] as follows. Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be IFV's. Then

$$A \oplus B = \langle \mu_A + \mu_B - \mu_A \mu_B, \nu_A \nu_B \rangle, \tag{1}$$

$$A \otimes B = \langle \mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B \rangle. \tag{2}$$

These operations were constructed in such a way that the result of their use is IFV. It is easy to prove that $0 \leq \mu_A + \mu_B - \mu_A \mu_B \leq 1$ and $0 \leq \nu_A + \nu_B - \nu_A \nu_B \leq 1$.

Based on operations (1) and (2) the following expressions were received in [35] for each integer $n = 1, 2, \dots$:

$$nA = A \oplus \dots \oplus A = \langle 1 - (1 - \mu_A)^n, \nu_A^n \rangle, \quad A^n = A \otimes \dots \otimes A = \langle \mu_A^n, 1 - (1 - \nu_A^n) \rangle.$$

It was shown that these operations can be used not only for integer values, but also for the real values $\lambda > 0$, i.e.:

$$\lambda A = \langle 1 - (1 - \mu_A)^\lambda, \nu_A^\lambda \rangle, \quad (3)$$

$$A^\lambda = \langle \mu_A^\lambda, 1 - (1 - \nu_A^\lambda) \rangle. \quad (4)$$

The operations (1)-(4) have the following algebraic properties [36]:

Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be IFV's. Then

$$A \oplus B = B \oplus A, \quad (5)$$

$$A \otimes B = B \otimes A, \quad (6)$$

$$\lambda(A \oplus B) = \lambda A \oplus \lambda B, \quad (7)$$

$$(A \otimes B)^\lambda = A^\lambda \otimes B^\lambda, \quad (8)$$

$$\lambda_1 A \oplus \lambda_2 A = (\lambda_1 \oplus \lambda_2) A, \quad \lambda_1, \lambda_2 > 0, \quad (9)$$

$$A^{\lambda_1} \otimes A^{\lambda_2} = A^{\lambda_1 + \lambda_2}, \quad \lambda_1, \lambda_2 > 0. \quad (10)$$

Operations (1)-(4) are used for aggregation of local criteria in the case of solving the MCDM problems in terms of fuzzy intuitionistic uncertainty.

Let A_1, \dots, A_n be an IFV's of local criteria and w_1, \dots, w_n ($\sum_{i=1}^n w_i = 1$) be a weight of this criteria. Then the Intuitionistic Weighted Arithmetic Mean (IWAM) may be specified by using the operation (1) and (3) as follows [31]:

$$IWAM = w_1 A_1 \oplus w_2 A_2 \oplus \dots \oplus w_n A_n = \left\langle 1 - \prod_{i=1}^n (1 - \mu_{A_i})^{w_i}, \prod_{i=1}^n \nu_{A_i}^{w_i} \right\rangle. \quad (11)$$

The result is obtained by the aggregation operator (11) in the IFV for and its idempotent. It is this aggregation operator that is the most common in MCDM problem solving under the circumstances of intuitionistic fuzzy uncertainty.

It is also worth noting that there are no problems with the idempotent Intuitionistic Fuzzy Weighted Geometric operator (IFWG), which can be obtained directly from (2) and (4):

$$IFWG = A_1^{w_1} \otimes A_2^{w_2} \otimes \dots \otimes A_n^{w_n} = \left\langle \prod_{i=1}^n \mu_{A_i}^{w_i}, 1 - \prod_{i=1}^n (1 - \nu_{A_i})^{w_i} \right\rangle. \quad (12)$$

3. An extension of intuitionistic fuzzy set theory in the framework of DST

The close connection between the intuitionistic fuzzy sets and DST was proved in paper [31]. This connection allows one to directly apply the Dempster's rule of combination in MCDM issues in order to aggregate local criteria with intuitionistic fuzzy values.

Furthermore, in paper [31], the possibility of transforming intuitionistic fuzzy values to Belief Intervals (BI), based on the extension of intuitionistic fuzzy sets theory in the context of DST was also presented. The presentation of mathematical operations on the IFVs as operations on BI is allowed by this fact.

In [37] Shafer introduced several measures. The belief measure is a mapping $Bel: 2^X \rightarrow [0,1]$, such that for any subset B from X occurring the expression [31]:

$$Bel(B) = \sum_{i=1}^n m(A_i), \quad A_i \subseteq B, \quad i = 1 \text{ to } n. \quad (13)$$

The next measure proposed by Shafer is a measure of plausibility, which is a mapping $Pl: 2^X \rightarrow [0,1]$, such that for any subset B from X the relation [31]

$$Pl(B) = \sum_{i=1}^n m(A_i), \quad A_i \cap B, \quad i = 1 \text{ to } n. \quad (14)$$

holds.

It can easily be seen that $Bel(B) \leq Pl(B)$. A clear measure of ignorance about the opportunity B and its completion \bar{B} as the length of the interval $[Bel(B), Pl(B)]$ is allowed to be shown by a DST. This interval, which is called the belief interval (BI), can be depicted as the inaccuracy of the probability of opportunity B [31] as well. In [14], Hong and Choi proposed an interval representation $[\mu_A(x_j), 1 - \nu_A(x_j)]$ of IFS A on X instead of a pair $\langle \mu_A(x_j), \nu_A(x_j) \rangle$ in framework of MCDM problems.

The fact that the expression $[\mu_A(x_j), 1 - \nu_A(x_j)]$ represents the authentic interval with its right bound being no smaller than the left one (due to the rule $0 \leq \mu_A(x_j) + \nu_A(x_j) \leq 1$) is the first obvious asset.

The consideration of the basic definition of the intuitionistic fuzzy sets theory with regards to the DST is the second asset.

The following definition was referred to in [31].

Definition 2. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal finite set and x_j be an element from X described by functions $\mu_A(x_j), \nu_A(x_j)$, representing the membership

and non-membership of element $x_j \in X$ to $A \subseteq X$, properly, such as $\mu_A : X \rightarrow [0,1]$, $x_j \in X \rightarrow \mu_A(x_j) \in [0,1]$ and $\nu_A : X \rightarrow [0,1]$, $x_j \in X \rightarrow \nu_A(x_j) \in [0,1]$, and for any $x_j \in X$, $0 \leq \mu_A(x_j) + \nu_A(x_j) \leq 1$. Intuitionistic fuzzy set A in X is an object of the following form: $A = \{ \langle x_j, BI_A(x_j) \mid x_j \in X \rangle \}$, where $BI_A(x_j) = [Bel_A(x_j), Pl_A(x_j)]$ is a belief interval $Bel_A(x_j) = \mu_A(x_j)$ and $Pl_A(x_j) = 1 - \nu_A(x_j)$ are the belief and plausibility functions of $x_j \in X$ belonging to a set $A \subseteq X$.

When first examined, the definition of 1 expresses a simple re-definition of IFS as an interval fuzzy set, but the use of the DST semantic allows the increasing of the reliability of the calculations when dealing with operations on the IFVs and MCDM issues. Specifically, such an approach allows one to aggregate the local criteria determined by IFVs and the development of the MCDM method avoiding defuzzification during the time in which the local criteria and their weights are expressed by IFVs. Correspondingly, a final assessment in the form of the belief interval [31] is obtained.

4. Operations on IFVs in framework of DST

Two approaches of formulating the operation on belief intervals are suggested in paper [27]. The first approach is based on a probability interpretation of belief intervals. The second approach is based on a non-probability interpretation. In [27], the fact that the operations based on the non-probability interpretation of belief intervals have superior algebraic priorities to those based on the probability approach was proved. It is essential to indicate that arithmetic operators which have properties superior to algebraic operations done within the framework of the classical intuitionistic fuzzy sets theory are generated by each of the approaches. For this reason alone, operations defined in [27] based on the non-probability interpretation of belief intervals will be used by us.

Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be the IFVs represented by belief intervals $BI(A) = [Bel(A), Pl(A)]$, $BI(B) = [Bel(B), Pl(B)]$, where $Bel(A) = \mu_A$, $Pl(A) = 1 - \nu_A$ and $Bel(B) = \mu_B$, $Pl(B) = 1 - \nu_B$, respectively. In this case, $Bel(A)$ and $Pl(A)$ are measures of belief and plausibility such as element $x_j \in X$ belongs to a set $A \subseteq X$. The belief interval $BI(A) = [Bel(A), Pl(A)]$ can be treated as an interval belonging a true power of ascertainment (argument, proposition, hypothesis etc.).

The additional and multiplication operators on belief intervals are shown in [27]. This becomes possible when we define an additional operator \oplus_{BNP} of belief intervals as follows:

$$BI(A) \oplus_{BNP} BI(B) = \left[\frac{Bel(A) + Bel(B)}{2}, \frac{Pl(A) + Pl(B)}{2} \right].$$

So, if we have n different ascertainties represented by belief intervals $BI(A_i)$, then their sum \oplus_{BNP} can be defined as follows:

$$BI(A_1) \oplus_{BNP} BI(A_2) \oplus_{BNP} \dots \oplus_{BNP} BI(A_n) = \left[\frac{1}{n} \sum_{i=1}^n Bel(A_i), \frac{1}{n} \sum_{i=1}^n Pl(A_i) \right]. \quad (14)$$

The multiplication operation \otimes_{BNP} of belief intervals we can define as follows [27]:

$$BI(A) \otimes_{BNP} BI(B) = [Bel(A)Bel(B), Pl(A)Pl(B)]. \quad (15)$$

It can easily be seen that this multiplication operator is the same as the one used in the conventional interval arithmetic [37].

The scalar multiplication is defined in [27] as follows:

$$\lambda BI(A) = [\lambda Bel(A), \lambda Pl(A)], \quad (16)$$

where λ is a real value, while $\lambda \in [0, 1]$, because for $\lambda > 1$ this operator does not always lead to the real belief intervals. This restriction is justified by the fact that we can define operations on belief intervals for MCDM problems, where λ usually represents the weight of local criteria, which are smaller than one.

The exponentiation operation is defined in [27] as

$$BI(A)^\lambda = [Bel(A)^\lambda, Pl(A)^\lambda], \quad (17)$$

and it leads to a real belief interval for all $\lambda \geq 0$.

Using the conventional rules of interval arithmetic [38], we obtain:

$$BI(A)^{BI(B)} = [\min\{Bel(A)^{Bel(B)}, Pl(A)^{Bel(B)}, Bel(A)^{Pl(B)}, Pl(A)^{Pl(B)}\}, \max\{Bel(A)^{Bel(B)}, Pl(A)^{Bel(B)}, Bel(A)^{Pl(B)}, Pl(A)^{Pl(B)}\}].$$

Taking into account the properties of the belief intervals, we can lead these expressions to a following form [27]

$$BI(A)^{BI(B)} = [Bel(A)^{Pl(B)}, Pl(A)^{Bel(B)}]. \quad (18)$$

The operators defined in that way have good algebraic properties (the same as in the case of the conventional theory of IFSs, see (5)-(10)). It can be directly inferred from expressions (14)-(17):

$$\begin{aligned}
BI(A) \oplus_{BNP} BI(B) &= BI(B) \oplus_{BNP} BI(A), \\
BI(A) \otimes_{BNP} BI(B) &= BI(B) \otimes_{BNP} BI(A), \\
(BI(A) \otimes_{BNP} BI(B))^\lambda &= BI(A)^\lambda \otimes_{BNP} BI(B)^\lambda, \\
BI(A)^{\lambda_1} \otimes_{BNP} BI(A)^{\lambda_2} &= BI(A)^{\lambda_1 + \lambda_2}, \\
\lambda BI(A) \oplus_{BNP} \lambda BI(B) &= \lambda(BI(A) \oplus_{BNP} BI(B)), \\
\lambda_1 BI(A) \oplus_{BNP} \lambda_2 BI(A) &= (\lambda_1 + \lambda_2)BI(A).
\end{aligned}$$

Using expressions (14) and (16) we get following Intuitionistic Weighted Arithmetic Mean (IWAM):

$$IWAM_{DSTNP}(A_1, A_2, \dots, A_n) = \left[\frac{1}{n} \sum_{i=1}^n w_i Bel_{A_i}, \frac{1}{n} \sum_{i=1}^n w_i Pl_{A_i} \right]. \quad (19)$$

This operator is not idempotent [27]. However, a small modification of (19) (multiplication by n) allows one to obtain an idempotent operator:

$$IWAMI_{DSTNP}(A_1, A_2, \dots, A_n) = \left[\sum_{i=1}^n w_i Bel_{A_i}, \sum_{i=1}^n w_i Pl_{A_i} \right]. \quad (20)$$

The Intuitionistic Fuzzy Weighted Geometric operator ($IFWG_{DSTP}$) obtained directly from (12) and (17) has the form [33]:

$$IFWG_{DSTP}(A_1, A_2, \dots, A_n) = \left[\prod_{i=1}^n Bel_{A_i}^{w_i}, \prod_{i=1}^n Pl_{A_i}^{w_i} \right]. \quad (21)$$

It is easy to see that the operator (21) is idempotent.

The Intuitionistic Fuzzy Weighted Geometric operator with weights presented by belief intervals ($IFWGB_{DSTP}$) $BI_i = [Bel_i, Pl_i]$, $i = 1, \dots, n$, obtained directly from (15) and (17) has the form [33]:

$$IFWGB_{DSTP}(A_1, A_2, \dots, A_n) = \left[\prod_{i=1}^n Bel_{A_i}^{Pl_i}, \prod_{i=1}^n Pl_{A_i}^{Bel_i} \right]. \quad (22)$$

It showed in [33] that the result obtained by means of this operator has the form of belief intervals.

This operator is not idempotent. Of course, the idempotence of operator (21) is guaranteed by the normalization of weight value in the form of the real numbers, or $\sum_{i=1}^n w_i = 1$. Given that in (22) weights are have a belief interval's form $BI_i = [Bel_i, Pl_i]$. Observe that we have a problem with their normalization [27].

Using the proposed approach (15) and (20), we get IWAM in the case when the local criteria and their weights are IFVs.

Let $BI_i = [Bel_i, Pl_i]$, $i = 1, \dots, n$, be belief intervals corresponding to the intuitionistic fuzzy weights of the local criteria A_i , $i = 1, \dots, n$ presented by belief intervals $BI(A_i) = [Bel_{A_i}, Pl_{A_i}]$.

Then, from (15) and (19) we get [27]

$$IWAMB_{DSTNP}(A_1, A_2, \dots, A_n) = \frac{1}{n} \left[\sum_{i=1}^n Bel_i Bel_{A_i}, \sum_{i=1}^n Pl_i Pl_{A_i} \right]. \quad (23)$$

The simple modification of foregoing operator (multiplying by n) allows one to obtain a more handy operator [27]:

$$IWAMB_{DSTNP}(A_1, A_2, \dots, A_n) = \left[\sum_{i=1}^n Bel_i Bel_{A_i}, \sum_{i=1}^n Pl_i Pl_{A_i} \right]. \quad (24)$$

This operator is not idempotent. Of course, the idempotence of operator (24) is guaranteed by the normalization of weight value in the form of the real numbers, or $\sum_{i=1}^n w_i = 1$, while in (21) weights have a belief interval's form $BI_i = [Bel_i, Pl_i]$.

It seems that idempotence is notably important in MCDM issues among the basic properties of the aggregation operations (boundary conditions, monotonicity, continuity, symmetry, idempotence and others).

To conclude, it can be stated that the operators presented in the framework of the non-probability approach to the belief interval have their correspondents in the classical theory of IFSs [27].

It is important to remember that there is no definition of raising an IFV to intuitionistic fuzzy power in the classical intuitionistic fuzzy sets theory. So, considering the analysis of the raise to power converting IFSs to belief intervals, the following expression is obtained:

$$A^B = \langle \mu_A^{1-\nu_B}, 1 - (1 - \nu_A)^{\mu_B} \rangle. \quad (25)$$

Let us consider an example of A^B calculating using the convert to belief intervals and expression (29).

Example 5. Let $A = \langle 0.78, 0.2 \rangle$ and $B = \langle 0.44, 0.33 \rangle$. Then $BI(A) = [0.78, 0.8]$ and $BI(B) = [0.44, 0.67]$, so $BI(A)^{BI(B)} = [0.8467, 0.9065]$. Using expression (25), we get $A^B = \langle 0.8467, 0.0935 \rangle$. It is easy to see, that the results coincide qualitatively, and $BI(A^B) = [0.8467, 0.9065]$.

Similarly, their equivalent operator to $IFWGI_{DST}$ is an operator presented in the expression (26):

$$IFWGI = \left\langle \prod_{i=1}^n \mu_{A_i}^{(1-v_i)}, 1 - \prod_{i=1}^n (1 - v_{A_i})^{\mu_i} \right\rangle, \quad (26)$$

6. The properties of exponentiation operation realized by transformation to belief intervals and heuristic IFV method

Let us consider the basic properties of exponential functions (18) and (25).

Let $A = \langle \mu_A, \nu_A \rangle$, $B = \langle \mu_B, \nu_B \rangle$ and $C = \langle \mu_C, \nu_C \rangle$ be IFV's. Then, belief intervals $BI(A) = [Bel_A, Pl_A]$, $BI(B) = [Bel_B, Pl_B]$ and $BI(C) = [Bel_C, Pl_C]$ are representation of IFVs.

It follows from (14)-(18) that

$$BI(A)^{BI(B)+BI(C)} = BI(A)^{BI(B)} \otimes BI(A)^{BI(C)}. \quad (27)$$

$$\begin{aligned} \text{Proof. } BI(A)^{BI(B)} \otimes BI(A)^{BI(C)} &= [Bel_A^{Pl_B}, Pl_A^{Bel_B}] \otimes [Bel_A^{Pl_C}, Pl_A^{Bel_C}] = \\ &[Bel_A^{Pl_B} \cdot Bel_A^{Pl_C}, Pl_A^{Bel_B} \cdot Pl_A^{Bel_C}] = [Bel_A^{Pl_B+Pl_C}, Pl_A^{Bel_B+Bel_C}] = BI(A)^{BI(B)+BI(C)}. \end{aligned}$$

The next met property is

$$\left(BI(A)^{BI(B)} \right)^{BI(C)} = BI(A)^{BI(B) \otimes BI(C)}. \quad (28)$$

$$\begin{aligned} \text{Proof. } \left(BI(A)^{BI(B)} \right)^{BI(C)} &= \left([Bel_A^{Pl_B}, Pl_A^{Bel_B}] \right)^{[Bel_C, Pl_C]} = [(Bel_A^{Pl_B})^{Pl_C}, (Pl_A^{Bel_B})^{Bel_C}] = \\ &[Bel_A^{Pl_B \cdot Pl_C}, Pl_A^{Bel_B \cdot Bel_C}] = BI(A)^{BI(B) \cdot BI(C)}. \end{aligned}$$

So, in the same way, we get

$$\left(BI(A) \otimes BI(B) \right)^{BI(C)} = BI(A)^{BI(C)} \otimes BI(B)^{BI(C)}. \quad (29)$$

$$\begin{aligned} \text{Proof. } BI(A)^{BI(C)} \otimes BI(B)^{BI(C)} &= [Bel_A^{Pl_C}, Pl_A^{Bel_C}] \otimes [Bel_B^{Pl_C}, Pl_B^{Bel_C}] = \\ &[Bel_A^{Pl_C} Bel_B^{Pl_C}, Pl_A^{Bel_C} Pl_B^{Bel_C}] = [(Bel_A Bel_B)^{Pl_C}, (Pl_A Pl_B)^{Bel_C}] = (BI(A) \otimes BI(B))^{BI(C)}. \end{aligned}$$

However, the following properties are not met:

$$\begin{aligned} BI(A)^{BI(B)-BI(C)} &\neq \frac{BI(A)^{BI(B)}}{BI(A)^{BI(C)}}, \quad \left(\frac{BI(A)}{BI(B)} \right)^{BI(C)} \neq \frac{BI(A)^{BI(C)}}{BI(B)^{BI(C)}}, \\ BI(A)^{-BI(B)} &\neq \frac{1}{BI(A)^{BI(B)}}. \end{aligned}$$

In the case of expression (25) the following properties are met:

$$(A^B)^C = A^{B \otimes C}. \quad (30)$$

Proof. $(A^B)^C = \langle \mu_A^{1-\nu_B}, 1-(1-\nu_A)^{\mu_B} \rangle^C = \langle (\mu_A^{1-\nu_B})^{1-\nu_C}, 1-(1-(1-(1-\nu_A)^{\mu_B}))^{\mu_C} \rangle =$
 $\langle (\mu_A^{(1-\nu_B)(1-\nu_C)}), 1-(1-\nu_A)^{\mu_B \mu_C} \rangle = \langle (\mu_A^{1-\nu_B-\nu_C+\nu_B \nu_C}), 1-(1-\nu_A)^{\mu_B \mu_C} \rangle = A^{B \otimes C}.$

The following property is also met

$$(A \otimes B)^C = A^C \otimes B^C. \quad (31)$$

Proof. $A^C \otimes B^C = \langle \mu_A^{1-\nu_C}, 1-(1-\nu_A)^{\mu_C} \rangle \otimes \langle \mu_B^{1-\nu_C}, 1-(1-\nu_B)^{\mu_C} \rangle =$
 $\langle \mu_A^{1-\nu_C} \cdot \mu_B^{1-\nu_C}, 1-(1-\nu_A)^{\mu_C} + 1-(1-\nu_B)^{\mu_C} - 1 + (1-\nu_B)^{\mu_C} +$
 $\quad + (1-\nu_A)^{\mu_C} - (1-\nu_B)^{\mu_C} \cdot (1-\nu_A)^{\mu_C} \rangle =$
 $\langle (\mu_A \cdot \mu_B)^{1-\nu_C}, 1-(1-\nu_B)^{\mu_C} \cdot (1-\nu_A)^{\mu_C} \rangle = \langle (\mu_A \cdot \mu_B)^{1-\nu_C}, 1-(\nu_A + \nu_B - \nu_A \cdot \nu_B)^{\mu_C} \rangle =$
 $= (A \otimes B)^C.$

Despite the fact that expression (25) is similar to (18) the property similar to (27) is not met:

$$A^{B+C} \neq A^B \otimes A^C.$$

Due to the lack of a strict definition of subtraction of intuitionistic fuzzy sets, we have no way to conclusively determine whether the following properties are

preserved: $A^{B-C} = \frac{A^B}{A^C}$, $\left(\frac{A}{B}\right)^C = \frac{A^C}{B^C}$, $A^{-B} = \frac{1}{A^B}$.

7. Conclusions

The heuristic method to solve the problem of raising an intuitionistic fuzzy values to intuitionistic fuzzy power and Intuitionistic Fuzzy Weighted Geometric operator with weights presented by intuitionistic fuzzy values are proposed. The advantage of this heuristic method is lack of the requirement to convert intuitionistic fuzzy values. In addition, the properties of this method and method based on DST theory are proved.

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