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RECURRENCE RELATIONS FOR TWO-CHANNEL CLOSED QUEUEING SYSTEMS WITH ERLANGIAN SERVICE TIMES

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Abstract. This paper proposes a method for determining the steady-state characteristics of two-channel closed queueing systems with an exponential distribution of the time generation of service requests and the Erlang distributions of the service times. Recurrence relations for computing the steady-state distribution of the number of customers in the system are deduced. The obtained algorithms are tested on examples using simulation models in the GPSS World environment.

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1. Introduction

Closed queueing systems are widely used as models to evaluate characteristics of the information systems, data networks and queueing processes in production, transport, trade, logistics and service systems [1]. The closed system is also called the system with a finite number of sources or the Engset system.

Suppose that a two-channel queueing system receives service requests from m identical sources. Each source is alternately on and off. A source is off when it has a service request being served, otherwise the source is on. A source in the on-state generates a new service request after an exponentially distributed time (the generation time) with mean $1/\lambda$. The sources act independently of each other. The service time of a service request has the Erlang distribution. A service request, that is generated when two channels are occupied, waits in the queue.

To investigate the systems with Erlangian service times, in particular the $M/E_n/1/\infty$ system [2], the method of fictitious phases, developed by A.K. Erlang [3], was applied. The Erlangian service times of the order n means that each customer runs sequentially n service phases, the duration of which is distributed exponentially with parameters $\mu_1, \mu_2, \ldots, \mu_n$ respectively.

The objective of this work is the construction with the help of fictitious phase method recursive algorithms for computing the steady-state distribution of the number of customers in two-channel closed queueing systems with an exponential distribution of the time generation of service requests and Erlangian service times of the order $n = 2$, $n = 3$ and $n \geq 4$. A similar approach is used in [4, 5], where recursive algorithms are developed for the systems $M/E_2/2/m$, $M/E_2/2/\infty$, $M/E_2/3/m$ and $M/E_2/3/\infty$ as well as for the systems of the same types with threshold and hysteretic strategies of the random dropping of customers.

2. The system with Erlangian service times of the second order

Suppose that the service time of each customer is distributed under the generalized Erlang law of the order n, that is, the service time is the sum of n independent random variables exponentially distributed with parameters $\mu_1, \mu_2, \ldots, \mu_n$ respectively.

Let n_c denote the number of customers in the system and let n_{bc} be the number of busy channels. In accordance with the method of phases, let us enumerate the system's states as follows: s_0 corresponds to the empty system; $s_{1(0j)}$ is the state, when $n_c = n_{bc} = 1$ and the service occurs in the phase j $(1 \leq j \leq n)$; $s_{k(ij)}$ is the state, when $n_c = k$ $(2 \leq k \leq m)$, $n_{bc} = 2$ and the services occur in the phase i and j $(1 \leq i \leq n, i \leq j \leq n)$ respectively. We denote by $p_0, p_{1(0j)}$ and $p_{k(ij)}$ respectively, steady-state probabilities that the system is in the each of these states. Assuming that $p_{1(0j)} = p_{1(1j)}$ $(1 \leq j \leq n)$ and $f_\lambda(k) = (m-k)\lambda$ $(0 \leq k \leq m-1)$, to calculate the steady-state probabilities, in the case of $n = 2$ we obtain the system of equations:

$$\begin{aligned}
&-f_\lambda(0)p_0 + \mu_2 p_{1(02)} = 0,\\
&-\left(f_\lambda(1) + \mu_2\right)p_{1(02)} + \mu_1 p_{1(01)} + 2\mu_2 p_{2(22)} = 0,\\
&-\left(f_\lambda(2) + 2\mu_2\right)p_{2(22)} + \mu_1 p_{2(12)} = 0,\\
&-\left(f_\lambda(1) + \mu_1\right)p_{1(01)} + m\lambda p_0 + \mu_2 p_{2(12)} = 0;
\end{aligned} \tag{1}$$

$$\begin{aligned}
&-\left(f_\lambda(k) + 2\mu_1\right)p_{k(11)} + f_\lambda(k-1)p_{k-1(11)} + \mu_2 p_{k+1(12)} = 0, \quad 2 \leq k \leq m-1;\\
&-\left(f_\lambda(k) + \mu_1 + \mu_2\right)p_{k(12)} + f_\lambda(k-1)p_{k-1(12)} + 2\mu_1 p_{k(11)} + 2\mu_2 p_{k+1(22)} = 0, \quad 2 \leq k \leq m-1;\\
&-\left(f_\lambda(k) + 2\mu_2\right)p_{k(22)} + f_\lambda(k-1)p_{k-1(22)} + \mu_1 p_{k(12)} = 0, \quad 3 \leq k \leq m-1;
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&-2\mu_1 p_{m(11)} + f_\lambda(m-1)p_{m-1(11)} = 0,\\
&-(\mu_1+\mu_2)p_{m(12)} + f_\lambda(m-1)p_{m-1(12)} + 2\mu_1 p_{m(11)} = 0,\\
&-2\mu_2 p_{m(22)} + f_\lambda(m-1)p_{m-1(22)} + \mu_1 p_{m(12)} = 0;
\end{aligned}
\tag{3}
$$

$$
p_0 + p_{1(01)} + p_{1(02)} + \sum_{k=2}^{m}(p_{k(11)} + p_{k(12)} + p_{k(22)}) = 1. \tag{4}
$$

Introducing the notation

$$
\alpha_i = \frac{\lambda}{\mu_i},\ 1 \le i \le n; \quad \eta_i = \frac{\mu_i}{\mu_1},\ 1 \le i \le n; \quad g_\alpha(k) = (m-k)\alpha_1,\ 0 \le k \le m-1;
$$

$$
\tilde{p}_{k(ij)} = \frac{p_{k(ij)}}{p_0},\ 1 \le k \le m,\ 1 \le i \le n,\ i \le j \le n,
$$

and using equations (1), we find:

$$
\begin{aligned}
&\tilde{p}_{1(02)} = m\alpha_2, \quad \tilde{p}_{2(22)} = \frac{m\big(\alpha_2\left(g_\alpha(1)+1\right)\left(g_\alpha(1)+\eta_2\right)-\alpha_1\big)}{\eta_2\left((3m-4)\alpha_1+\eta_2+2\right)},\\
&\tilde{p}_{1(01)} = m\alpha_2\left(g_\alpha(1)+\eta_2\right) - 2\eta_2\tilde{p}_{2(22)}, \quad \tilde{p}_{2(12)} = (g_\alpha(2)+\eta_2)\tilde{p}_{2(22)}.
\end{aligned}
\tag{5}
$$

From equations (2), we obtain the recurrence relations:

$$
\begin{aligned}
&\tilde{p}_{k(22)} = f_{22}(\tilde{p}_{k-1(22)}, \tilde{p}_{k-2(11)}, \tilde{p}_{k-2(12)}, \tilde{p}_{k-1(12)}), \quad 3 \le k \le m-1;\\
&\tilde{p}_{k(12)} = f_{12}(\tilde{p}_{k22}, \tilde{p}_{k-1,22}), \quad 3 \le k \le m-1;\\
&\tilde{p}_{k(11)} = f_{11}(\tilde{p}_{k(12)}, \tilde{p}_{k-1(12)}, \tilde{p}_{k+1(22)}), \quad 2 \le k \le m-1,
\end{aligned}
\tag{6}
$$

where:

$$
\begin{aligned}
&f_{22}(x,y,z,u) = \frac{g(x,y,z,u)}{2\eta_2\left(g_\alpha(k)+2\eta_2+g_\alpha(k-1)+2\right)},\\
&g(x,y,z,u) = 2\eta_2 g_\alpha(k-1)x - 2g_\alpha(k-2)y -\\
&\qquad -\left(g_\alpha(k-1)+2\right)\left(g_\alpha(k-2)z - \left(g_\alpha(k-1)+\eta_2+1\right)u\right);\\
&f_{12}(x,y) = \left(g_\alpha(k)+2\eta_2\right)x - g_\alpha(k-1)y;\\
&f_{11}(x,y,z) = \frac{g_\alpha(k)+\eta_2+1}{2}x - \frac{g_\alpha(k-1)}{2}y - \eta_2 z.
\end{aligned}
$$

Using equations (3) we find:

$$\tilde{p}_{m(12)} = \frac{\alpha_1}{\alpha_1 + 2\eta_2 + 2}\Big((\alpha_1 + \eta_2 + 3)\tilde{p}_{m-1(12)} - \alpha_1(\tilde{p}_{m-1(22)} + 2\tilde{p}_{m-2(12)})\Big),$$
$$\tilde{p}_{m(22)} = \frac{1}{2\eta_2}\Big(\alpha_1\tilde{p}_{m-1(22)} + \tilde{p}_{m(12)}\Big), \qquad \tilde{p}_{m(11)} = \frac{\alpha_1}{2}\tilde{p}_{m-1(11)}. \tag{7}$$

Recurrence relations (5)-(7) allow us to consistently calculate $\tilde{p}_{1(02)}$, $\tilde{p}_{2(22)}$, $\tilde{p}_{1(01)}$, $\tilde{p}_{2(12)}$; $\tilde{p}_{k(22)}$, $\tilde{p}_{k(12)}$, $\tilde{p}_{k-1(11)}$ $(3 \le k \le m-1)$, $\tilde{p}_{m(12)}$, $\tilde{p}_{m(22)}$, $\tilde{p}_{m-1(11)}$ and $\tilde{p}_{m(11)}$. Using the normalization condition (4), we find steady-state probabilities by the formulas

$$p_0 = \left(1 + \sum_{j=1}^{n}\tilde{p}_{1(0j)} + \sum_{k=2}^{m}\sum_{i=1}^{n}\sum_{j=i}^{n}\tilde{p}_{k(ij)}\right)^{-1}; \quad p_k = p_0\tilde{p}_k, \quad 1 \le k \le m;$$
$$\tilde{p}_1 = \sum_{j=1}^{n}\tilde{p}_{1(0j)}; \quad \tilde{p}_k = \sum_{i=1}^{n}\sum_{j=i}^{n}\tilde{p}_{k(ij)}, \quad 2 \le k \le m. \tag{8}$$

Here $n = 2$ and p_k is the steady-state probability that $n_c = k$. We calculate the steady-state characteristics - the average number of customers in the system $\mathbf{E}(n_c)$, the average queue length $\mathbf{E}(Q)$ and average waiting time $\mathbf{E}(W)$ - by the formulas

$$\mathbf{E}(n_c) = \sum_{k=1}^{m} kp_k, \qquad \mathbf{E}(Q) = \sum_{k=1}^{m-2} kp_{k+2}, \qquad \mathbf{E}(W) = \frac{\mathbf{E}(Q)}{\lambda_{\text{av}}}.$$

Here λ_{av} is a steady-state value of the arrival rate of customers, defined by the equality

$$\lambda_{\text{av}} = \lambda\sum_{k=0}^{m-1}(m-k)p_k.$$

The parameter λ_{av} is a characteristic of the system capacity, because for the steady-state regime we have the equality of the intensities of flows of customers arriving and served.

3. The system with Erlangian service times of the third order

To calculate the steady-state probabilities, in the case of $n = 3$ we obtain the system of equations:

$$
\begin{aligned}
&-f_\lambda(0)p_0+\mu_3 p_{1(03)}=0;\\
&-(f_\lambda(1)+\mu_1)p_{1(01)}+f_\lambda(0)p_0+\mu_3 p_{2(13)}=0;\\
&-(f_\lambda(1)+\mu_2)p_{1(02)}+\mu_3 p_{2(23)}+\mu_1 p_{1(01)}=0;\\
&-(f_\lambda(1)+\mu_3)p_{1(03)}+2\mu_3 p_{2(33)}+\mu_2 p_{1(02)}=0;\\
&-(f_\lambda(k)+2\mu_1)p_{k(11)}+f_\lambda(k-1)p_{k-1(11)}+\mu_3 p_{k+1(13)}=0,\quad 2\le k\le m-1;\\
&-2\mu_1 p_{m(11)}+f_\lambda(m-1)p_{m-1(11)}=0;\\
&-(f_\lambda(k)+\mu_1+\mu_2)p_{k(12)}+f_\lambda(k-1)p_{k-1(12)}+\mu_3 p_{k+1(23)}+2\mu_1 p_{k(11)}=0,\\
&\qquad\qquad 2\le k\le m-1;\\
&-(\mu_1+\mu_2)p_{m(12)}+f_\lambda(m-1)p_{m-1(12)}+2\mu_1 p_{m(11)}=0;\\
&-(f_\lambda(k)+\mu_1+\mu_3)p_{k(13)}+f_\lambda(k-1)p_{k-1(13)}+2\mu_3 p_{k+1(33)}+\mu_2 p_{k(12)}=0,\\
&\qquad\qquad 2\le k\le m-1;\\
&-(\mu_1+\mu_3)p_{m(13)}+f_\lambda(m-1)p_{m-1(13)}+\mu_2 p_{m(12)}=0;\\
&-(f_\lambda(2)+2\mu_i)p_{2(ii)}+\mu_{i-1}p_{2(i-1,i)}=0,\quad i=2,3;\\
&-(f_\lambda(2)+\mu_2+\mu_3)p_{2(23)}+2\mu_2 p_{2(22)}+\mu_1 p_{2(13)}=0;\\
&-(f_\lambda(k)+2\mu_i)p_{k(ii)}+f_\lambda(k-1)p_{k-1(ii)}+\mu_{i-1}p_{k(i-1,i)}=0,\\
&\qquad\qquad 3\le k\le m-1,\ i=2,3;\\
&-2\mu_i p_{m(ii)}+f_\lambda(m-1)p_{m-1(ii)}+\mu_{i-1}p_{m(i-1,i)}=0,\ i=2,3;\\
&-(f_\lambda(k)+\mu_2+\mu_3)p_{k(23)}+f_\lambda(k-1)p_{k-1(23)}+2\mu_2 p_{k(22)}+\mu_1 p_{k(13)}=0,\\
&\qquad\qquad 3\le k\le m-1;\\
&-(\mu_2+\mu_3)p_{m(23)}+f_\lambda(m-1)p_{m-1(23)}+2\mu_2 p_{m(22)}+\mu_1 p_{m(13)}=0;
\end{aligned}
\tag{9}
$$

$$p_0+\sum_{j=1}^{3}p_{1(0j)}+\sum_{k=2}^{m}\sum_{i=1}^{3}\sum_{j=i}^{3}p_{k(ij)}=1. \tag{10}$$

Introducing the notation $\tilde{p}_{2(33)}=p$ and using equations (9) we find:

$$\tilde{p}_{1(03)}=m\alpha_3,\quad \tilde{p}_{1(02)}=\frac{1}{\eta_2}\Big((g_\alpha(1)+\eta_3)\tilde{p}_{1(03)}-2\eta_3 p\Big),\quad \tilde{p}_{2(23)}=\frac{g_\alpha(2)+2\eta_3}{\eta_2}p;$$

$$\tilde{p}_{1(01)}=(g_\alpha(1)+\eta_2)\tilde{p}_{1(02)}-\eta_3\tilde{p}_{2(23)},\quad \tilde{p}_{2(13)}=\frac{1}{\eta_3}\Big((g_\alpha(1)+1)\tilde{p}_{1(01)}-\alpha_1\Big);$$

$$\tilde{p}_{2(22)}=\frac{1}{2\eta_2}\Big((g_\alpha(2)+\eta_2+\eta_3)\tilde{p}_{2(23)}-\tilde{p}_{2(13)}\Big),\quad \tilde{p}_{2(12)}=\big(g_\alpha(2)+2\eta_2\big)\tilde{p}_{2(22)};$$

$$\tilde{p}_{k(33)} = \frac{1}{2\eta_3}\Big((1 + g_\alpha(k-1) + \eta_3)\tilde{p}_{k-1(13)} - g_\alpha(k-2)\tilde{p}_{k-2(13)} - \eta_2\tilde{p}_{k-1(12)}\Big), \quad 3 \le k \le m;$$

$$\tilde{p}_{k(23)} = \frac{1}{\eta_2}\Big((g_\alpha(k) + 2\eta_3)\tilde{p}_{k(33)} - g_\alpha(k-1)\tilde{p}_{k-1(33)}\Big), \quad 3 \le k \le m-1;$$

$$\tilde{p}_{m(23)} = \frac{1}{\eta_2}\Big(2\eta_3\tilde{p}_{m(33)} - g_\alpha(m-1)\tilde{p}_{m-1(33)}\Big);$$

$$\tilde{p}_{k(11)} = \frac{1}{2}\Big((1 + g_\alpha(k) + \eta_2)\tilde{p}_{k(12)} - g_\alpha(k-1)\tilde{p}_{k-1(12)} - \eta_3\tilde{p}_{k+1(23)}\Big), \quad 2 \le k \le m-1;$$

$$\tilde{p}_{m(11)} = \frac{g_\alpha(m-1)}{2}\tilde{p}_{m-1(11)};$$

$$\tilde{p}_{k(13)} = \frac{1}{\eta_3}\Big((g_\alpha(k-1) + 2)\tilde{p}_{k-1(11)} - g_\alpha(k-2)\tilde{p}_{k-2(11)}\Big), \quad 3 \le k \le m;$$

$$\tilde{p}_{k(22)} = \frac{1}{2\eta_2}\Big((\eta_2 + g_\alpha(k) + \eta_3)\tilde{p}_{k(23)} - g_\alpha(k-1)\tilde{p}_{k-1(23)} - \tilde{p}_{k(13)}\Big), \quad 3 \le k \le m-1; \tag{11}$$

$$\tilde{p}_{m(22)} = \frac{1}{2\eta_2}\Big((\eta_2 + \eta_3)\tilde{p}_{m(23)} - g_\alpha(m-1)\tilde{p}_{m-1(23)} - \tilde{p}_{m(13)}\Big);$$

$$\tilde{p}_{k(12)} = (2\eta_2 + g_\alpha(k))\tilde{p}_{k(22)} - g_\alpha(k-1))\tilde{p}_{k-1(22)}, \quad 3 \le k \le m-1;$$

$$\tilde{p}_{m(12)} = 2\eta_2\tilde{p}_{m(22)} - g_\alpha(m-1))\tilde{p}_{m-1(22)}.$$

Recurrence relations (11) allow us to calculate $\tilde{p}_{1(03)}$ and consistently obtain the expression for $\tilde{p}_{1(03)}$, $\tilde{p}_{1(02)}$, $\tilde{p}_{2(23)}$, $\tilde{p}_{1(01)}$, $\tilde{p}_{2(13)}$, $\tilde{p}_{2(22)}$, $\tilde{p}_{2(12)}$; $\tilde{p}_{k(33)}$, $\tilde{p}_{k(23)}$, $\tilde{p}_{k-1(11)}$, $\tilde{p}_{k(13)}$, $\tilde{p}_{k(22)}$, $\tilde{p}_{k(12)}$, $3 \le k \le m,$ and $\tilde{p}_{m(11)}$ as linear functions of the unknown parameter $p = \tilde{p}_{2(33)}$. To find $p,$ we use any of the equations

$$\begin{aligned} &-(\mu_1 + \mu_2)p_{m(12)} + f_\lambda(m-1)p_{m-1(12)} + 2\mu_1 p_{m(11)} = 0, \\ &-(\mu_1 + \mu_3)p_{m(13)} + f_\lambda(m-1)p_{m-1(13)} + \mu_2 p_{m(12)} = 0. \end{aligned} \tag{12}$$

Equations (12) were not involved in obtaining (11). Using the normalization condition (10), we find the steady-state probabilities by (8).

4. The system with Erlangian service times of *n*-th order

To calculate the steady-state probabilities in the case of $n \ge 4$, we obtain the system of equations:

$$
\begin{aligned}
&-f_\lambda(0)p_0+\mu_n p_{1(0n)}=0;\\
&-(f_\lambda(1)+\mu_1)p_{1(01)}+f_\lambda(0)p_0+\mu_n p_{2(1n)}=0;\\
&-(f_\lambda(1)+\mu_j)p_{1(0j)}+\mu_n p_{2(jn)}+\mu_{j-1}p_{1(0,j-1)}=0,\quad 2\le j\le n-1;\\
&-(f_\lambda(1)+\mu_n)p_{1(0n)}+2\mu_n p_{2(nn)}+\mu_{n-1}p_{1(0,n-1)}=0;\\
&-(f_\lambda(k)+2\mu_1)p_{k(11)}+f_\lambda(k-1)p_{k-1(11)}+\mu_n p_{k+1(1n)}=0,\quad 2\le k\le m-1;\\
&-2\mu_1 p_{m(11)}+f_\lambda(m-1)p_{m-1(11)}=0;\\
&-(f_\lambda(k)+\mu_1+\mu_2)p_{k(12)}+f_\lambda(k-1)p_{k-1(12)}+\mu_n p_{k+1(2n)}+2\mu_1 p_{k(11)}=0,\\
&\qquad 2\le k\le m-1;\\
&-(\mu_1+\mu_2)p_{m(12)}+f_\lambda(m-1)p_{m-1(12)}+2\mu_1 p_{m(11)}=0;\\
&-(f_\lambda(k)+\mu_1+\mu_j)p_{k(1j)}+f_\lambda(k-1)p_{k-1(1j)}+\mu_n p_{k+1(jn)}+\mu_{j-1}p_{k(1,j-1)}=0,\\
&\qquad 2\le k\le m-1,\ 3\le j\le n-1;\\
&-(\mu_1+\mu_j)p_{m(1j)}+f_\lambda(m-1)p_{m-1(1j)}+\mu_{j-1}p_{m(1,j-1)}=0,\quad 3\le j\le n;\\
&-(f_\lambda(k)+\mu_1+\mu_n)p_{k(1n)}+f_\lambda(k-1)p_{k-1(1n)}+2\mu_n p_{k+1(nn)}+\mu_{n-1}p_{k(1,n-1)}=0,\\
&\qquad 2\le k\le m-1;\\
&-(\mu_1+\mu_n)p_{m(1n)}+f_\lambda(m-1)p_{m-1(1n)}+\mu_{n-1}p_{m(1,n-1)}=0;\\
&-(\lambda+2\mu_i)p_{2(ii)}+\mu_{i-1}p_{2(i-1,i)}=0,\quad 2\le i\le n;\\
&-(\lambda+\mu_i+\mu_{i+1})p_{2(i,i+1)}+2\mu_i p_{2(ii)}+\mu_{i-1}p_{2(i-1,i+1)}=0,\quad 2\le i\le n-1;\\
&-(\lambda+\mu_i+\mu_j)p_{2(ij)}+\mu_{i-1}p_{2(i-1,j)}+\mu_{j-1}p_{2(i,j-1)}=0,\quad 2\le i\le n-2,\\
&\qquad i+2\le j\le n;\\
&-(f_\lambda(k)+2\mu_i)p_{k(ii)}+f_\lambda(k-1)p_{k-1(ii)}+\mu_{i-1}p_{k(i-1,i)}=0,\quad 3\le k\le m-1,\\
&\qquad 2\le i\le n;\\
&-2\mu_i p_{m(ii)}+f_\lambda(m-1)p_{m-1(ii)}+\mu_{i-1}p_{m(i-1,i)}=0,\ 2\le i\le n;\\
&-(f_\lambda(k)+\mu_i+\mu_{i+1})p_{k(i,i+1)}+f_\lambda(k-1)p_{k-1(i,i+1)}+2\mu_i p_{k(ii)}+\mu_{i-1}p_{k(i-1,i+1)}=0,\\
&\qquad 3\le k\le m-1,\ 2\le i\le n-1;\\
&-(\mu_i+\mu_{i+1})p_{m(i,i+1)}+f_\lambda(m-1)p_{m-1(i,i+1)}+2\mu_i p_{m(ii)}+\mu_{i-1}p_{m(i-1,i+1)}=0,\\
&\qquad 2\le i\le n-1;\\
&-(f_\lambda(k)+\mu_i+\mu_j)p_{k(ij)}+f_\lambda(k-1)p_{k-1(ij)}+\mu_{i-1}p_{k(i-1,j)}+\mu_{j-1}p_{k(i,j-1)}=0,\\
&\qquad 3\le k\le m-1,\ 2\le i\le n-2,\ i+2\le j\le n;\\
&-(\mu_i+\mu_j)p_{m(ij)}+f_\lambda(m-1)p_{m-1(ij)}+\mu_{i-1}p_{m(i-1,j)}+\mu_{j-1}p_{m(i,j-1)}=0,\\
&\qquad 2\le i\le n-2,\ i+2\le j\le n;
\end{aligned}
\tag{13}
$$

$$p_0 + \sum_{j=1}^{n} p_{1(0j)} + \sum_{k=2}^{m}\sum_{i=1}^{n}\sum_{j=i}^{n} p_{k(ij)} = 1. \tag{14}$$

Introducing the notation $\tilde{p}_{k(in)} = p_{ki}, 2 \le k \le m, 1 \le i \le n,$ and using equations (13) we find:

$$\tilde{p}_{1(0n)} = m\alpha_n, \quad \tilde{p}_{1(01)} = \frac{1}{g_\alpha(1)+1}(g_\alpha(0) + \eta_n p_{21}),$$

$$\tilde{p}_{1(0j)} = \frac{1}{g_\alpha(1)+\eta_j}\left(\eta_{j-1}\tilde{p}_{1(0,j-1)} + \eta_n p_{2j}\right), \quad 2 \le j \le n-1;$$

$$\tilde{p}_{k(11)} = \frac{1}{g_\alpha(k)+2}\left(g_\alpha(k-1)\tilde{p}_{k-1(11)} + \eta_n p_{k+1,1}\right), \quad 2 \le k \le m-1;$$

$$\tilde{p}_{m(11)} = \frac{g_\alpha(m-1)}{2}\tilde{p}_{m-1(11)};$$

$$\tilde{p}_{k(12)} = \frac{1}{g_\alpha(k)+\eta_2+1}\left(g_\alpha(k-1)\tilde{p}_{k-1(12)} + 2\tilde{p}_{k(11)} + \eta_n p_{k+1,2}\right),$$
$$2 \le k \le m-1;$$

$$\tilde{p}_{m(12)} = \frac{1}{\eta_2+1}\left(g_\alpha(m-1)\tilde{p}_{m-1(12)} + 2\tilde{p}_{m(11)}\right);$$

$$\tilde{p}_{k(1j)} = \frac{1}{g_\alpha(k)+\eta_j+1}\left(g_\alpha(k-1)\tilde{p}_{k-1(1j)} + \eta_{j-1}\tilde{p}_{k(1,j-1)} + \eta_n p_{k+1,j}\right),$$
$$2 \le k \le m-1, \; 3 \le j \le n-1;$$

$$\tilde{p}_{m(1j)} = \frac{1}{\eta_j+1}\left(g_\alpha(m-1)\tilde{p}_{m-1(1j)} + \eta_{j-1}\tilde{p}_{m(1,j-1)}\right), \; 3 \le j \le n-1;$$

$$\tilde{p}_{2(ii)} = \frac{\eta_{i-1}}{g_\alpha(2)+2\eta_i}\tilde{p}_{2(i-1,i)}, \quad 2 \le i \le n-1;$$

$$\tilde{p}_{k(ii)} = \frac{1}{g_\alpha(k)+2\eta_i}\left(g_\alpha(k-1)\tilde{p}_{k-1(ii)} + \eta_{i-1}\tilde{p}_{k(i-1,i)}\right),$$
$$3 \le k \le m-1, \; 2 \le i \le n-1;$$

$$\tilde{p}_{m(ii)} = \frac{1}{2\eta_i}\left(g_\alpha(m-1)\tilde{p}_{m-1(ii)} + \eta_{i-1}\tilde{p}_{m(i-1,i)}\right), \; 2 \le i \le n-1;$$

$$\tilde{p}_{2(i,i+1)} = \frac{1}{g_\alpha(2)+\eta_i+\eta_{i+1}}\left(2\eta_i\tilde{p}_{2(ii)} + \eta_{i-1}\tilde{p}_{2(i-1,i+1)}\right), \; 2 \le i \le n-2;$$

$$\tilde{p}_{2(ij)} = \frac{1}{g_\alpha(2) + \eta_i + \eta_j}\left(\eta_{i-1}\tilde{p}_{2(i-1,j)} + \eta_{j-1}\tilde{p}_{2(i,j-1)}\right),$$
$$2 \le i \le n-3,\ i+2 \le j \le n-1;$$
$$\tilde{p}_{k(i,i+1)} = \frac{1}{g_\alpha(k) + \eta_i + \eta_{i+1}}\left(g_\alpha(k-1)\tilde{p}_{k-1(i,i+1)} + 2\eta_i\tilde{p}_{k(ii)} + \eta_{i-1}\tilde{p}_{k(i-1,i+1)}\right),$$
$$3 \le k \le m-1,\ 2 \le j \le n-2;$$
$$\tilde{p}_{m(i,i+1)} = \frac{1}{\eta_i + \eta_{i+1}}\left(g_\alpha(m-1)\tilde{p}_{m-1(i,i+1)} + 2\eta_i\tilde{p}_{m(ii)} + \eta_{i-1}\tilde{p}_{m(i-1,i+1)}\right), \quad (15)$$
$$2 \le i \le n-2;$$
$$\tilde{p}_{k(ij)} = \frac{1}{g_\alpha(k) + \eta_i + \eta_j}\left(g_\alpha(k-1)\tilde{p}_{k-1(ij)} + \eta_{i-1}\tilde{p}_{k(i-1,j)} + \eta_{j-1}\tilde{p}_{k(i,j-1)}\right),$$
$$3 \le k \le m-1,\ 2 \le i \le n-3,\ i+2 \le j \le n-1;$$
$$\tilde{p}_{m(ij)} = \frac{1}{\eta_i + \eta_j}\left(g_\alpha(m-1)\tilde{p}_{m-1(ij)} + \eta_{i-1}\tilde{p}_{m(i-1,j)} + \eta_{j-1}\tilde{p}_{m(i,j-1)}\right),$$
$$2 \le i \le n-3,\ i+2 \le j \le n-1.$$

Recurrence relations (15) allow us to calculate $\tilde{p}_{1(0n)}$ and consistently obtain the expression for

$\tilde{p}_{1(0n)};\ \tilde{p}_{1(0j)},\ 1 \le j \le n-1;\quad \tilde{p}_{k(11)},\ 2 \le k \le m;\quad \tilde{p}_{k(12)},\ 2 \le k \le m;$
$\tilde{p}_{k(13)}, \tilde{p}_{k(14)}, \ldots, \tilde{p}_{k(1,n-1)},\ 2 \le k \le m-1;\quad \tilde{p}_{m(1j)},\ 3 \le j \le n-1;$
$\tilde{p}_{k(22)},\ 2 \le k \le m;\quad \tilde{p}_{2(2j)},\ 3 \le j \le n-1;\quad \tilde{p}_{k(23)},\ 3 \le k \le m;$
$\tilde{p}_{k(24)}, \tilde{p}_{k(25)}, \ldots, \tilde{p}_{k(2,n-1)},\ 3 \le k \le m-1;\quad \tilde{p}_{m(2j)},\ 4 \le j \le n-1;$
$\tilde{p}_{k(33)},\ 2 \le k \le m;\quad \tilde{p}_{2(3j)},\ 4 \le j \le n-1;\quad \tilde{p}_{k(34)},\ 3 \le k \le m;$
$\tilde{p}_{k(35)}, \tilde{p}_{k(36)}, \ldots, \tilde{p}_{k(3,n-1)},\ 3 \le k \le m-1;\quad \tilde{p}_{m(3j)},\ 5 \le j \le n-1;$
……………………………………………………………
$\tilde{p}_{k(ii)},\ 2 \le k \le m;\quad \tilde{p}_{2(ij)},\ i+1 \le j \le n-1;\quad \tilde{p}_{k(i,i+1)},\ 3 \le k \le m;$
$\tilde{p}_{k(i,i+2)}, \tilde{p}_{k(i,i+3)}, \ldots, \tilde{p}_{k(i,n-1)},\ 3 \le k \le m-1;\quad \tilde{p}_{m(ij)},\ i+2 \le j \le n-1;$
……………………………………………………………
$\tilde{p}_{k(n-2,n-2)},\ 2 \le k \le m;\quad \tilde{p}_{k(n-2,n-1)},\ 3 \le k \le m;$
$\tilde{p}_{k(n-1,n-1)},\ 2 \le k \le m,$

as linear functions of the unknown parameters $p_{k\,i}$ $(2 \le k \le m,\ 1 \le i \le n)$. To determine $p_{k\,i}$, we use system of $n(m+1)$ equations

$$
\begin{aligned}
&-(f_\lambda(k)+\mu_1+\mu_n)p_{k(1n)}+f_\lambda(k-1)p_{k-1(1n)}+2\mu_n p_{k+1(nn)}+\\
&\qquad\qquad +\mu_{n-1}p_{k(1,n-1)}=0, \quad 2\le k\le m-1;\\
&-(\mu_1+\mu_n)p_{m(1n)}+f_\lambda(m-1)p_{m-1(1n)}+\mu_{n-1}p_{m(1,n-1)}=0;\\
&-(f_\lambda(2)+2\mu_n)p_{2(nn)}+\mu_{n-1}p_{2(n-1,n)}=0;\\
&-(f_\lambda(2)+\mu_{n-1}+\mu_n)p_{2(n-1ns)}+2\mu_{n-1}p_{2(n-1,n-1)}+\mu_{n-2}p_{2(n-2,n)}=0;\\
&-(f_\lambda(2)+\mu_i+\mu_n)p_{2(in)}+\mu_{i-1}p_{2(i-1,n)}+\mu_{n-1}p_{2(i,n-1)}=0, \quad 2\le i\le n-2;\\
&-(f_\lambda(k)+2\mu_n)p_{k(nn)}+f_\lambda(k-1)p_{k-1(nn)}+\mu_{n-1}p_{k(n-1,n)}=0, \quad 3\le k\le m-1;\\
&-2\mu_n p_{m(nn)}+f_\lambda(m-1)p_{m-1(nn)}+\mu_{n-1}p_{m(n-1,n)}=0;\\
&-(f_\lambda(k)+\mu_{n-1}+\mu_n)p_{k(n-1,n)}+f_\lambda(k-1)p_{k-1(n-1,n)}+2\mu_{n-1}p_{k(n-1,n-1)}+\\
&\qquad\qquad +\mu_{n-2}p_{k(n-2,n)}=0, \quad 3\le k\le m-1;\\
&-(\mu_{n-1}+\mu_n)p_{m(n-1,n)}+f_\lambda(m-1)p_{m-1(n-1,n)}+2\mu_{n-1}p_{m(n-1,n-1)}+\\
&\qquad\qquad +\mu_{n-2}p_{m(n-2,n)}=0;\\
&-(f_\lambda(k)+\mu_i+\mu_n)p_{k(in)}+f_\lambda(k-1)p_{k-1(in)}+\mu_{i-1}p_{k(i-1,n)}+\\
&\qquad\qquad +\mu_{n-1}p_{k(i,n-1)}=0, \quad 3\le k\le m-1,\ 2\le i\le n-2;\\
&-(\mu_i+\mu_s)p_{m+2(is)}+f_\lambda(m+1)p_{m+1(is)}+\mu_{i-1}p_{m+2(i-1,s)}+\\
&\qquad\qquad +\mu_{s-1}p_{m+2(i,s-1)}=0, \quad 2\le i\le s-2.
\end{aligned} \tag{16}
$$

Equations (16) as well as the equation

$$-(f_\lambda(1)+\mu_n)p_{1(0n)}+2\mu_n p_{2(nn)}+\mu_{n-1}p_{1(0,n-1)}=0,$$

were not involved in obtaining (15). Using the normalization condition (14), we find the steady-state probabilities by (8).

5. Numerical example

Consider a two-channel closed queueing system with an exponential distribution of the time generation of service requests and Erlangian service times of the order $n=10$ for the following values of the parameters: $m=12$; $\lambda=1$; $\mu_i=20$, $1\le i\le 10$.

The values of the steady-state characteristics of the system, found using the recurrence relations obtained in this paper, are presented in Tables 1 and 2. In order to verify the obtained values, the tables contain the computing results obtained with the help of the GPSS World simulation system [6] for the time value $t=10^6$.

Table 1

Stationary distribution of the number of customers in the system

k	Values of the steady-state probabilities p_k	
	Recurrence method	GPSS World
0	0.000004	0.000004
1	0.000055	0.000053
2	0.000440	0.000461
3	0.002605	0.002620
4	0.011830	0.011963
5	0.040833	0.040495
6	0.105048	0.104882
7	0.196206	0.196322
8	0.256876	0.256441
9	0.224423	0.224365
10	0.121478	0.122119
11	0.035863	0.036016
12	0.004340	0.004259

Table 2

Stationary characteristics of the system

Method	$\mathbf{E}(n_c)$	$\mathbf{E}(Q)$	$\mathbf{E}(W)$	λ_{av}
Recurrence	8.000127	6.000191	1.500095	3.999873
GPSS World	8.002	6.001	1.500	–

6. Conclusions

The numerical algorithm for solving a system of equations for steady-state probabilities, developed in this article, is based on the presence of three or four unknowns in most equations. The constructed recurrence relations are used for the direct calculation of the steady-state probabilities, that allows us to reduce the amount of calculations in comparison with the case of application of the direct or iterative classical methods. Using the obtained recurrence relations makes it possible in the case of $n=2$ to directly calculate the steady-state probabilities, in the case of $n=3$ to reduce the system of equations for the steady-state probabilities for a single linear equation for the unknown parameter p, and in the case of $n \geq 4$ to reduce the number of solved equations from $(n+1)\big(2+n(m-1)\big)/2$ to $n(m-1)$.

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