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LIE SYMMETRIES AND CONSERVED QUANTITIES OF DISCRETE CONSTRAINED HAMILTON SYSTEMS

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Abstract: In this paper, the Lie symmetry theory of discrete singular systems is studied in phase space. Firstly, the discrete canonical equations and the energy evolution equations of the constrained Hamilton systems are established based on the discrete difference variational principle. Secondly, the Lie point transformation of discrete group is applied to the difference equations and constraint restriction, and the Lie symmetry determination equations of the discrete constrained Hamilton systems are obtained; Meanwhile, the Lie symmetries of singular systems lead to the discrete Noehter type conserved quantities when the structure condition equations (discrete Noether identity) are established. Finally, an example is given to illustrate the application, the results show that the conservative constrained Hamilton systems also have the discrete energy conservation.

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1. Introduction

The systems described by the singular Lagrange function are called singular systems and expressed in Hamilton form in phase space. They are also referred to as constrained Hamilton systems [1]. Many important dynamic problems of mathematical physics and engineering technology are in accord with the model of constraint Hamilton systems, such as an electromagnetic field, light traversing phenomen, quantum electrodynamics, superstring theory etc. Symmetry theory [2] is a higher level rule in theoretical physics, engineering mathematics, modern mechanics and so on, and it is closely related to the conserved quantities, such as twins. In modern times, people seek conserved quantities through the symmetries that have: Noether symmetry [3], Lie symmetry [4] and Mei symmetry [5]. In recent years, the research on symmetries and conserved quantities of constrained

Hamilton systems are paid more and more attention, and have made some progress [6-10].

The theory of symmetry and conserved quantity for discrete mechanical systems has important theoretical and practical values for various numerical algorithms and integrability of differential equations in the fields such as computer visualization, quantum field theory, mathematical physics, economics and so on. In 1970, Cadzow [11] gave the discrete variational principle and the discrete Euler-Lagrange equations, which laid a foundation for the symmetry principle to be applied in discrete systems in the future. In 1980, on the basis of the discrete variational principle, Li Zhengdao [12] considered time as a dynamic variable to be discrete and obtained the motion equations of the discrete systems and the discrete form of energy conservation law. In 2002, Guo Hanying [13] proposed a new discrete variational method: the discrete difference variational method. This variational approach treats the difference as a geometric object and has the advantage of preserving the structure in dealing with discrete Hamilton systems. Fu Jingli and Shi Shenyang [14-16] studied the Noether symmetries and conserved quantities of discrete Lagrange-Maxwell electromechanical systems, discrete nonconservative and discrete nonholonomic constraint systems. Dorodnitsyn [17] firstly used Lie group transformation in differential equations and the lattice equations at the same time, he gave a conclusion that different Lie groups can lead to different conserved quantities. Fang Jianhui and Xia Lili [18-20] studied the Mei symmetries and conserved quantities of discrete constrained mechanical systems in form space and phase space. They were also compared with Noether symmetry and Lie symmetry. Generally speaking, the research on the symmetries and conserved quantities of discrete mechanical systems mainly focuses on the form space and nonsingular systems. There has been practically nothing written in literature about the discretization and the symmetry theory of the constrained Hamilton systems. According to the research methods about symmetries and conserved quantities of nonholonomic discrete mechanical systems, in this paper, the intrinsic constraints caused by singularity are considered, the Lie symmetry of a discrete singular singular systems in phase space is given, and I derived the conditions and form of discrete Noether conserved quantities by Lie symmetries.

2. Motion equations of discrete constrained Hamilton systems

The form of mechanical systems is determined by n generalized coordinates $q_s(s=1,...,n)$ and the Lagrange function of the systems is $L(t,\mathbf{q},\dot{\mathbf{q}})=T-V$, the generalized momentum is $p_s=\dfrac{\partial L}{\partial \dot{q}_s}(s=1,...,n)$, for constrained Hamilton systems, since the rank of L's Hess matrix $\left[\dfrac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right]$ is $r<n$, when the Legendre transformation $H=p_i\dot{q}_i-L(t,\mathbf{q},\dot{\mathbf{q}})$ is used to transfer the Lagrange systems to the Hamilton

systems description, there are some constraints between the regular variables in phase space:

$$\Phi_j(t,\mathbf{p},\mathbf{q}) = 0 \; (j = 1,...,n-r) \tag{1}$$

The constraints also must satisfy the restrictive conditions of virtual displacement and time variation:

$$\frac{\partial \Phi_j}{\partial q_s}\delta q_s + \frac{\partial \Phi_j}{\partial p_s}\delta p_s = 0 \tag{2}$$

The Hamilton functional action of the system is:

$$S = \int_{t_1}^{t_2} (p_s \dot{q}_s - H)dt \tag{3}$$

In the discrete mechanics systems, the variational principle is divided into two kinds: the discrete variational method and the discrete difference variational method. The former only uses the difference instead of the derivative and does not preserve the difference form, the latter regards difference as a geometric object (independent variable), generally speaking, the discrete difference variational principle is more practical and superior in discrete Hamilton mechanics systems. According to the discrete difference variational method, time is discretized into a point sequence $\{t_k\}(k = 1,2,...,N)$, the regular variables $q_s(t)$ and $p_s(t)$ are replaced by a discrete difference sequence, and the difference of time and regular variables are expressed as:

$$\Delta t_k = t_{k+1} - t_k,\; \Delta q_s^k = \frac{q_s^{k+1} - q_s^k}{t_{k+1} - t_k},\; \Delta p_s^k = \frac{p_s^{k+1} - p_s^k}{t_{k+1} - t_k} \tag{4}$$

The corresponding discrete form of the Hamilton function is:

$$\begin{aligned} &H_D^k = H_D(t_k, q_s^k, p_s^k) = p_s^k \Delta q_s^k - L_D^k(t_k, q_s^k, \Delta q_s^k) \\ &p_s^k = \frac{\partial L_D^k}{\partial \Delta q_s^k} \end{aligned} \tag{5}$$

The formula (3) is discretized:

$$S_D = \sum_k (t_{k+1} - t_k)(p_s^k \Delta q_s^k - H_D^k) \tag{6}$$

The δ_t is all variational operators, δ is isochronous variational operators, Δ is discrete derivative operators, there are the following important relationships:

$$
\begin{aligned}
&\delta_t \Delta t_k = \Delta(\delta_t t_k)\cdot(t_{k+1}-t_k)\\
&\delta_t q_s^k = \delta q_s^k + \Delta q_s^k \cdot \delta_t t_k,\ \ \delta_t p_s^k = \delta p_s^k + \Delta p_s^k \cdot \delta_t t_k\\
&\delta_t \Delta q_s^k = \Delta \delta_t q_s^k - \Delta q_s^k \cdot \Delta \delta_t t_k,\ \ \delta_t \Delta p_s^k = \Delta \delta_t p_s^k - \Delta p_s^k \cdot \Delta \delta_t t_k
\end{aligned}
\tag{7}
$$

The R_+ is the recursion operator, the discrete Leibniz rule can be expressed as under the variable time step:

$$
\begin{aligned}
&\Delta(f_k g_k) = \Delta f_k \cdot g_k + R_+ f_k \cdot \Delta g_k\\
&R_\pm f(z_k) = f(z_{k\pm 1})
\end{aligned}
\tag{8}
$$

According to the variational principle, the total variation of formula (6) should be as follows:

$$
\begin{aligned}
\delta_t S_D &= \delta_t \sum_k \left(t_{k+1}-t_k\right)\left(p_s^k \Delta q_s^k - H_D^k\right)\\
&= \sum_k \left[\delta_t\left(t_{k+1}-t_k\right)\cdot\left(p_s^k \Delta q_s^k - H_D^k\right) + \left(t_{k+1}-t_k\right)\cdot \delta_t\left(p_s^k \Delta q_s^k - H_D^k\right)\right]\\
&= \sum_k \left(t_{k+1}-t_k\right)\left[\Delta\left(\delta_t t_k\right)\cdot\left(p_s^k \Delta q_s^k - H_D^k\right) + \delta_t\left(p_s^k \Delta q_s^k - H_D^k\right)\right]\\
&= \sum_k \left(t_{k+1}-t_k\right)\left[p_s^k \Delta q_s^k \Delta\left(\delta_t t_k\right) - H_D^k \Delta\left(\delta_t t_k\right) + \delta_t p_s^k \Delta q_s^k + p_s^k \delta_t \Delta q_s^k - \delta_t H_D^k\right]\\
&= \sum_k \left(t_{k+1}-t_k\right)\left(p_s^k \Delta \delta_t q_s^k - H_D^k \Delta\left(\delta_t t_k\right) + \delta_t p_s^k \Delta q_s^k - \delta_t H_D^k\right)\\
&= \sum_k \begin{array}{l} \left(t_{k+1}-t_k\right)\Big[\Delta\left(p_s^{k-1}\delta_t q_s^k\right) - \Delta p_s^{k-1}\delta_t q_s^k - \Delta\left(H_D^{k-1}\delta_t t_k\right) + \Delta H_D^{k-1}\delta_t t_k +\\ \quad \delta_t p_s^k \Delta q_s^k - \dfrac{\partial H_D^k}{\partial t_k}\delta_t t_k - \dfrac{\partial H_D^k}{\partial q_s^k}\delta_t q_s^k - \dfrac{\partial H_D^k}{\partial p_s^k}\delta_t p_s^k \Big] \end{array}\\
&= \sum_k \begin{array}{l} \left(t_{k+1}-t_k\right)\Bigg[\left(\Delta H_D^{k-1} - \dfrac{\partial H_D^k}{\partial t_k}\right)\delta_t t_k - \left(\Delta p_s^{k-1} + \dfrac{\partial H_D^k}{\partial q_s^k}\right)\delta_t q_s^k + \left(\Delta q_s^k - \dfrac{\partial H_D^k}{\partial p_s^k}\right)\delta_t p_s^k\\ \quad + \Delta\left(p_s^{k-1}\delta_t q_s^k - H_D^{k-1}\delta_t t_k\right)\Bigg] \end{array}\\
&= 0
\end{aligned}
\tag{9}
$$

The discrete-time forms of internal constraints (1) and restrictive conditions (2) resulting from the singularity of the constrained Hamilton systems are:

$$
\begin{aligned}
&\Phi_j^k(t, q_s^k, p_s^k) = 0\\
&\frac{\partial \Phi_j^k}{\partial q_s^k}\delta q_s^k + \frac{\partial \Phi_j^k}{\partial p_s^k}\delta p_s^k = 0
\end{aligned}
\tag{10}
$$

For independence and independence q_s^k, p_s^k, as long as we choose the discrete constraint multiplier $\lambda_j^k(t_k, q_s^k, p_s^k)$ rationally and appropriately, it multiplies the second equation of formula (10) and takes the sum in the interval $(t_{k+1}-t_k)$, and then adds it to the formula (9), we can get the following result:

$$
\begin{aligned}
&(t_{k+1}-t_k)[(\Delta H_D^{k-1} - \frac{\partial H_D^k}{\partial t_k} + \lambda_j^k \frac{\partial \Phi_j^k}{\partial q_s^k}\Delta q_s^k + \lambda_j^k \frac{\partial \Phi_j^k}{\partial p_s^k}\Delta p_s^k)\delta_t t_k - \\
&\sum_k (\Delta p_s^{k-1} + \frac{\partial H_D^k}{\partial q_s^k} + \lambda_j^k \frac{\partial \Phi_j^k}{\partial q_s^k})\delta_t q_s^k + (\Delta q_s^k - \frac{\partial H_D^k}{\partial p_s^k} - \lambda_j^k \frac{\partial \Phi_j^k}{\partial p_s^k})\delta_t p_s^k + \\
&\Delta(p_s^{k-1}\delta_t q_s^k - H_D^{k-1}\delta_t t_k)] \\
&= 0
\end{aligned} \tag{11}
$$

By using the fixed boundary conditions and the arbitrariness of the sum k interval, the regular equations of the discrete constrained Hamilton systems are as follows:

$$
\begin{cases}
\Delta q_s^k = \dfrac{\partial H_D^k}{\partial p_s^k} + \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial p_s^k} \\
\Delta p_s^{k-1} = -\dfrac{\partial H_D^k}{\partial q_s^k} - \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial q_s^k}
\end{cases} \tag{12}
$$

The energy evolution equations are as follows:

$$
\Delta H_D^{k-1} - \frac{\partial H_D^k}{\partial t_k} + \lambda_j^k \frac{\partial \Phi_j^k}{\partial q_s^k}\Delta q_s^k + \lambda_j^k \frac{\partial \Phi_j^k}{\partial p_s^k}\Delta p_s^k = 0 \tag{13}
$$

In this case, this article only considers the constraint formula (1) as the second class constraint, namely $\det\left|\{\Phi_i, \Phi_j\}\right|_{\mathbf{\Phi}=0} \neq 0 \ (i \neq j; i, j = 1, \ldots, n-r)$, then all constrained multipliers λ_j can be completely determined by the self-consistent stability condition of constraints [21]: $\lambda_j = \lambda_j(t, q_s, p_s)$.

3. Lie symmetries of discrete constrained Hamilton systems

The infinitesimal transformation of discrete time t_k, discrete generalized coordinates q_s^k and discrete generalized momentum p_s^k are:

$$
\begin{aligned}
&t_k^* = t_k + \varepsilon\tau_k(t_k, q_s^k, p_s^k) = t_k + \delta_t t_k, q_s^{k^*} = q_s^k + \varepsilon\xi_s^k(t_k, q_s^k, p_s^k) = q_s^k + \delta_t q_s^k, \\
&p_s^{k^*} = p_s^k + \varepsilon\eta_s^k(t_k, q_s^k, p_s^k) = p_s^k + \delta_t p_s^k
\end{aligned} \tag{14}
$$

The vector of the infinitesimal generating element is:

$$\Pr \overline{X} = \tau_k \frac{\partial}{\partial t_k} + \xi_s^k \frac{\partial}{\partial q_s^k} + \eta_s^k \frac{\partial}{\partial p_s^k} \tag{15}$$

It is important to note that the motion difference equations (12) and (13) involve variables $t_{k-1}, t_{k+1}, q_s^{k-1}, p_s^{k-1}, q_s^{k+1}, p_s^{k+1}$, but we regard the difference as a whole variable, the vector field (15) does not need to expand two discrete points and three discrete points.

According to the Lie symmetry definition of the mechanical systems, the invariance of the differential motion equations of the discrete constrained Hamilton systems under the infinitesimal transformation (14) is reduced to the following discrete deterministic equations:

$$\begin{cases} \Pr \overline{X}\left(\Delta q_s^k - \dfrac{\partial H_D^k}{\partial p_s^k} - \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial p_s^k} \right) = 0 \\ \Pr \overline{X}\left(\Delta p_s^{k-1} + \dfrac{\partial H_D^k}{\partial q_s^k} + \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial q_s^k} \right) = 0 \\ \Pr \overline{X}\left(\Delta H_D^{k-1} - \dfrac{\partial H_D^k}{\partial t_k} + \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial q_s^k} \Delta q_s^k + \lambda_j^k \dfrac{\partial \Phi_j^k}{\partial p_s^k} \Delta p_s^k \right) = 0 \end{cases} \tag{16}$$

The discrete restriction equations with invariance of the intrinsic constraint equations (1) under the infinitesimal transformation (14) are:

$$\Pr \overline{X}\left(\Phi_j^k (t_k, q_s^k, p_s^k) \right) = 0 \tag{17}$$

Considering the export process of differential equations, then the infinitesimal generating element needed to meet the discrete additional restriction equations:

$$\frac{\partial \Phi_j^k}{\partial q_s^k}\left(\xi_s^k - \Delta q_s^k \tau_k \right) + \frac{\partial \Phi_j^k}{\partial p_s^k}\left(\eta_s^k - \Delta p_s^k \tau_k \right) = 0 \tag{18}$$

Definition 1: If the discrete infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ satisfies the deterministic equations (16), the corresponding symmetry is the Lie symmetry of the discrete-time constrained Hamilton systems corresponding to the discrete-time free Hamilton system.

Definition 2: If the discrete infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ satisfies the deterministic equations and restriction equations (17), the corresponding symmetry is the weak Lie symmetry of the discrete-time constrained Hamilton systems.

Definition 3: If the discrete infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ satisfies the deterministic equations (16), restriction equations (17) and additional restriction equations (18), the corresponding symmetry is the strong Lie symmetry of the discrete-time constrained Hamilton systems.

4. Conserved quantities of discrete constrained Hamilton systems

For the constrained mechanical systems, the Lie symmetry of the system can lead to the conservation of the Noether type under certain conditions. The following theorem gives the conditions and forms about the Lie symmetries of the discrete constrained Hamilton systems leading to discrete Noether conserved quantities.

Theorem 1: According to differential discrete variational principle, if the infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ can make the deterministic equations (16) can be established, meanwhile, there are also normal functions $G^k = G(t_k, q_s^k, p_s^k)$ satisfying the following structural equations:

$$\begin{aligned} & p_s^k \Delta\xi_s^k - H_D^k \Delta\tau_k - \frac{\partial H_D^k}{\partial t_k}\tau_k - \frac{\partial H_D^k}{\partial q_s^k}\xi_s^k - \\ & \lambda_j^k \frac{\partial \Phi_j^k}{\partial q_s^k}(\xi_s^k - \Delta q_s^k \cdot \tau_k) + \lambda_j^k \frac{\partial \Phi_j^k}{\partial p_s^k}\Delta p_s^k \cdot \tau_k + \Delta G^k = 0 \end{aligned} \tag{19}$$

Then the Lie symmetry of the discrete constrained Hamilton systems leads to the discrete form of the Noether type conserved quantity:

$$I_D = p_s^{k-1}\xi_s^k - H_D^{k-1}\tau_k + G^k = \text{const} \tag{20}$$

At the same time, it can be seen from the above proof that the infinitesimal generating element that satisfies the structural condition equations are also the Noether quasi symmetric transformation of the discrete constrained Hamilton systems.

Theorem 2: According to differential discrete variational principle, if the infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ can make the deterministic equations (16) and restriction equations (17) can be established, meanwhile, there are also normal functions satisfying the following structural equations (19), then the weak Lie symmetry of discrete constrained Hamilton systems leads to the discrete conserved quantity of type (20).

Theorem 3: According to differential discrete variational principle, if the infinitesimal generating element $\tau_k, \xi_s^k, \eta_s^k$ can make the deterministic equations (16), restriction equations (17) and additional restriction equations (18) can be established, meanwhile, there are also normal functions satisfying the following structural equation (19), then the weak Lie symmetry of discrete constrained Hamilton systems leads to the discrete conserved quantity of type (20).

5. Example

The Lagrange function of the system is:

$$L = \dot{q}_1 q_2 - q_1 \dot{q}_2 + q_1^2 + q_2^2 \tag{21}$$

Try to investigate the Lie symmetries and conserved quantities of discrete systems in phase space.

It is obvious that the L's Hess matrix rank is $0 < 2$, so it is a constrained Hamilton system, thus there are two constraints in the regular variables:

$$\begin{aligned} \Phi_1(t, \boldsymbol{p}, \boldsymbol{q}) = p_1 - q_2 = 0 \\ \Phi_2(t, \boldsymbol{p}, \boldsymbol{q}) = p_2 + q_1 = 0 \end{aligned} \tag{22}$$

The compatibility conditions and stability conditions of the constraints are obtained:

$$\lambda_1 = -q_2, \ \lambda_2 = q_1 \tag{23}$$

The discrete Hamilton function of the system is as follows:

$$H_D^k = -(q_1^k)^2 - (q_2^k)^2 \tag{24}$$

According to formula (12) and (13), the motion difference equations of the system are obtained:

$$\begin{cases} \Delta q_1^k = -q_2^k, \ \Delta q_2^k = q_1^k \\ \Delta p_1^{k-1} = q_1^k, \ \Delta p_2^{k-1} = q_2^k \\ \Delta H_D^{k-1} + q_1^k \Delta q_1^k + q_2^k \Delta q_2^k - q_2^k \Delta p_1^k + q_1^k \Delta p_2^k = 0 \end{cases} \tag{25}$$

The deterministic equations (16) of Lie symmetry are given:

$$\begin{cases} \Pr \overline{X}(\Delta q_1^k + q_2^k) = 0, \Pr \overline{X}(\Delta q_2^k - q_1^k) = 0 \\ \Pr \overline{X}(\Delta p_1^{k-1} + q_1^k) = 0, \Pr \overline{X}(\Delta p_2^{k-1} - q_2^k) = 0 \\ \Pr \overline{X}(\Delta H_D^{k-1} + q_1^k \Delta q_1^k + q_2^k \Delta q_2^k - q_2^k \Delta p_1^k + q_1^k \Delta p_2^k) = 0 \end{cases} \tag{26}$$

For convenience, we take the standard lattice, that is, the time step is the equal step length $\Delta t_k = h$, then the equations (26) have a set of solutions as follows:

$$\tau_k = 1, \ \xi_1^k = 0, \ \xi_2 = 0, \ \eta_1^k = 0, \ \eta_2^k = 0 \tag{27}$$

The restriction equations (17) are given:

$$\begin{aligned} \eta_1^k - \xi_2^k &= 0 \\ \eta_2^k + \xi_1^k &= 0 \end{aligned} \tag{28}$$

The additional restriction equations (18) are given:

$$\begin{aligned} \eta_1^k - \xi_2^k + (\Delta q_2^k - \Delta p_1^k)\tau_k &= 0 \\ \eta_2^k + \xi_1^k - (\Delta q_1^k + \Delta p_2^k)\tau_k &= 0 \end{aligned} \tag{29}$$

The structural equations (19) give the norm function corresponding to the generating element and it is:

$$G^k = 0 \tag{30}$$

Therefore, the system has the conserved quantity of the corresponding form (20) is:

$$I_D = (q_1^{k-1})^2 + (q_2^{k-1})^2 = \text{const} \tag{31}$$

It is easy to see that the above results are correct, and formula (31) is the discretization of energy conservation in the conservative systems. At the same time, the generating element can be verified to satisfy the restriction equations (28) and the additional restriction equations (29). Therefore, the infinitesimal generating element corresponds to the strong Lie symmetry of the discrete constrained Hamilton systems.

6. Conclusion

The symmetry and conserved quantity of discrete singular mechanical systems have not yet appeared, in this paper, the Lie symmetry and the Noether conserved quantity of discrete constrained Hamilton systems are studied by the difference discrete variational. The main conclusions are: the differential motion equations (12) and (13) of discrete constrained Hamilton systems; Lie symmetry deterministic equations (16) and three definitions for discrete constrained Hamilton systems; the structural equations and form of the Lie symmetry leads to a conserved quantity. It is found that the discrete constrained Hamilton systems have similar forms and properties with the constrained Hamilton systems under continuous variation when the difference is regarded as an independent variable, in particular, it can keep the structure unchanged. An example shows that the results of this paper can be regarded as a natural generalization to the Lie symmetry of continuous constrained Hamilton systems $(h \to 0)$, the conservative discrete singular systems also have energy conservation. The method and content of this paper can be extended to study the symmetries and conserved quantities of the nonconservative and nonholonomic discrete singular mechanical systems in phase space.

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