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ON A RECURRENCE FOR PERMANENTS OF A SEQUENCE OF 3-TRIDIAGONAL MATRICES

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Abstract. This is a corrigendum of the paper: Küçük, A. Z. & Düz, M. (2017). Relationships between the permanents of a certain type of k-tridiagonal symmetric Toeplitz and the Chebyshev polynomials. *Journal of Applied Mathematics and Computational Mechanics*, *16*, 75-86. We will show that **Remark 9**, on page 84, does not hold, what is the consequence of the incorrect proof, which authors formulated there.

MSC 2010: *15A15, 11B05*
Keywords: *permanent, k-tridiagonal matrix, Toeplitz matrix, recurrence relation, Chebyshev polynomial of the second kind*

1. Introduction

The so-called k-tridiagonal matrices (this name was introduced by El-Mikkawy and Sogabe [1]) were first studied by Egerváry and Szász in [2]. Perhaps the most important non-trivial case is due to Losonczi [3]. A very recent and important survey in this topic can be found in da Fonseca and Kowalenko [4].

The k-tridiagonal matrices $\mathbf{T}_n^{(k)}(\mathbf{D}_{-k}, \mathbf{D}_0, \mathbf{D}_k)$ are defined by the following way

$$\begin{pmatrix} d_1 & 0 & \cdots & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & a_{n-k} \\ 0 & \ddots & \ddots & \ddots & d_k & 0 & \ddots & \ddots & 0 \\ b_{k+1} & \ddots & \ddots & \ddots & 0 & d_{k+1} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_n & 0 & \cdots & \cdots & 0 & d_n \end{pmatrix}_{n\times n} \tag{1}$$

where sequences $\{d_j\}_{j=1}^{n}$, $\{a_j\}_{j=1}^{n-k}$ and $\{b_j\}_{j=k+1}^{n}$ create the main diagonal $\mathbf{D}_0$, the k-th superdiagonal $\mathbf{D}_k$ and the k-th subdiagonal $\mathbf{D}_{-k}$, respectively.[1] Thus, for the general k-tridiagonal matrix we use notation $\mathbf{T}_n^{(k)}(\mathbf{D}_{-k}, \mathbf{D}_0, \mathbf{D}_k)$ or directly

$$\mathbf{T}_n^{(k)}(\{b_j\}_{j=k+1}^{n}, \{d_j\}_{j=1}^{n}, \{a_j\}_{j=1}^{n-k})$$

but for the k-tridiagonal Toeplitz matrix we can write shortly $\mathbf{T}_n^{(k)}(b,d,a)$, since for diagonals of matrix (1) hold

$$\{d_j = d\}_{j=1}^{n}, \{a_j = a\}_{j=1}^{n-k}, \text{ and } \{b_j = b\}_{j=k+1}^{n}$$

Küçük, Düz [5] studied, recursive relations between the Chebyshev polynomials of the second kind (for more information, see [6]), which can be defined for $n > 2$ by the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with initial values $U_0(x) = 1$ and $U_1(x) = 2x$, and the permanents (the definition and many properties of permanents you can find in [7]) of a special type of matrix (1), namely k-tridiagonal symmetric Toeplitz matrix $\mathbf{T}_n^{(k)}(i, 2x, i)$, where i is the imaginary unit, i. e., the matrix with entries

$$t_{jm}^{(k)} = \begin{cases} 2x, & j = m; \\ i, & j = m \pm k; \\ 0, & \text{otherwise} \end{cases}$$

where $1 \leq j, m \leq n$.

To prove [5, Conjecture 8] first da Fonseca in [8] showed that the permanent of the matrix $\mathbf{T}_n^{(k)}(i, 2x, i)$ is equal to the permanent of the matrix $\mathbf{T}_n^{(k)}(-1, 2x, 1)$, with respect to the fact, that the permanent of a square matrix equals the sum of the weights of all cycle-covers of its underlying directed graph. Then, he used a result on *convertible matrices* from his paper [10] (some generalizations can be found in [11]) to show that the permanent of matrix $\mathbf{T}_n^{(k)}(-1, 2x, 1)$ is equal to the determinant of the matrix $\mathbf{T}_n^{(k)}(1, 2x, 1)$. Thus, he derived that

$$\operatorname{per} \mathbf{T}_n^{(k)}(i, 2x, i) = \det \mathbf{T}_n^{(k)}(1, 2x, 1) \tag{2}$$

Borowska et al. [12–14] dealt with determinants of some pentagonal and heptadiagonal symmetric Toeplitz matrices. Inter alia, they paid attention to the determinant of the following heptadiagonal matrix

[1] Here we use the notation for the numbering diagonals, which can be found, e.g., in [9].

$$\mathbf{A}_n = \begin{pmatrix} a & b & c & d & & & & & & \\ b & a & b & c & d & & & & & \\ c & b & a & b & c & d & & & & \\ d & c & b & a & b & c & d & & & \\ & d & c & b & a & b & c & d & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & b & c & d \\ & & & & d & c & b & a & b & c \\ & & & & & d & c & b & a & b \\ & & & & & & d & c & b & a \end{pmatrix}_{n\times n} \tag{3}$$

To find a recurrence relation for determinants of matrix $\mathbf{A}_n$ they introduced the following two auxiliary heptadiagonal matrices

$$\mathbf{A}_n = \begin{pmatrix} a & b & c & d & & & & & & \\ b & a & b & c & d & & & & & \\ c & b & a & b & c & d & & & & \\ d & c & b & a & b & c & d & & & \\ & d & c & b & a & b & c & d & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & b & c & d \\ & & & & d & c & b & a & b & c \\ & & & & & d & c & b & a & b \\ & & & & & & \mathbf{0} & d & c & b \end{pmatrix}_{n\times n}$$

and

$$\widehat{\mathbf{A}}_n = \begin{pmatrix} a & b & c & d & & & & & & \\ b & a & b & c & d & & & & & \\ c & b & a & b & c & d & & & & \\ d & c & b & a & b & c & d & & & \\ & d & c & b & a & b & c & d & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & b & c & \mathbf{0} \\ & & & & d & c & b & a & b & d \\ & & & & & d & c & b & a & c \\ & & & & & & \mathbf{0} & d & c & a \end{pmatrix}_{n\times n}$$

They denoted determinants of matrices $\mathbf{A}_n$, $\overline{A}_n$, and $\widehat{\mathbf{A}}_n$ by W_n, $\overline{W}_n$, and $\widehat{W}_n$, respectively, and derived the following system of linear recurrence relations (see formulae (4) and (5) in [14], where all the needed initial conditions can be found too)

$$\begin{aligned}
W_{n+7} &= aW_{n+6} + bd(bd-2c^2)W_{n+3} + d^2(2c^3-4bcd+b^2c+ad^2)W_{n+2} \\
&\quad + d^3(2c^2d+b^2d-bc^2-d^3)W_{n+1} - bcd^5W_n - b\overline{W}_{n+6} + bc\overline{W}_{n+5} \\
&\quad + d(2ac-b^2)\overline{W}_{n+4} + bd^2(2c-a)\overline{W}_{n+3} + d^3(2bd-b^2-c^2)\overline{W}_{n+2} \\
&\quad + cd^4(b-2d)\overline{W}_{n+1} + bd^6\overline{W}_n - c^2\widehat{W}_{n+5} + d(bc-ad)\widehat{W}_{n+4}\,, \qquad (4) \\
\overline{W}_{n+6} &= bW_{n+5} - bcd^2W_{n+2} + d^3(c^2-bd)W_{n+1} + cd^5W_n - c\overline{W}_{n+5} \\
&\quad + bd\overline{W}_{n+4} + ad^2\overline{W}_{n+3} + bd^3\overline{W}_{n+2} - cd^4\overline{W}_{n+1} - d^6\overline{W}_n - cd\widehat{W}_{n+4}\,, \\
\widehat{W}_{n+2} &= aW_{n+1} - c^2W_n + 2cd\overline{W}_n - d^2\widehat{W}_n
\end{aligned}$$

2. Main result

Küçük, Düz [5] formulated the following proposition (we have made a small technical textual modification, that does not change their assertion, to avoid copying the whole text above this proposition)

Remark 1

$$\operatorname{per}\mathbf{T}_n^{(3)}(i,2x,i),\ \operatorname{per}\mathbf{T}_n^{(4)}(i,2x,i),\ \operatorname{per}\mathbf{T}_n^{(5)}(i,2x,i),\ldots \qquad (5)$$

cannot be written in terms of themselves, thus as a self-recurrence for every of these permanents individually. □

Küçük, Düz formulated the proof of this Remark 1 for the case $\operatorname{per}\mathbf{T}_n^{(3)}(i,2x,i)$, but the idea of this proof is incorrect, what we show by proving that there is a self-recurrence for $\operatorname{per}\mathbf{T}_n^{(3)}(i,2x,i)$.

For the simplification of notation, we will use for permanent of matrix $\mathbf{T}_n^{(3)}(i,2x,i)$ the following denotation

$$p_n := \operatorname{per}\mathbf{T}_n^{(3)}(i,2x,i) \qquad (6)$$

where n is a positive integer.

Theorem 1 *Let n be any positive integer. The sequence $\{p_n\}$, defined by* (6)*, satisfies the following recurrence relation for $n > 8$*

$$p_n = 2xp_{n-1} - p_{n-2} + 2xp_{n-3} - 4x^2p_{n-4} + 2xp_{n-5} - p_{n-6} + 2xp_{n-7} - p_{n-8} \qquad (7)$$

with the initial values

$$\begin{aligned}
&p_1 = 2x,\ p_2 = 4x^2,\ p_3 = 8x^3,\\
&p_4 = 4x^2(4x^2-1),\ p_5 = 2x(4x^2-1)^2,\\
&p_6 = (4x^2-1)^3,\ p_7 = 4x(2x^2-1)(4x^2-1)^2,\\
&p_8 = (4x)^2(2x^2-1)^2(4x^2-1)
\end{aligned} \tag{8}$$

PROOF Combining identities (2) and (6) we get $p_n = \det \mathbf{T}_n^{(3)}(1,2x,1)$, but this determinant is a special case of the determinant of the heptadiagonal matrix $\mathbf{A}_n$ in (3), when we set $a = 2x$, $b = c = 0$, and $d = 1$. Similarly, we denote determinants of matrices $\overline{\mathbf{A}}_n$ and $\widehat{A}_n$ by $\overline{p}_n$ and $\widehat{p}_n$, respectively. Then, from (4) we get the following system of three homogeneous linear recurrences for sequences $\{p_n\}$, $\{\overline{p}_n\}$ and $\{\widehat{p}_n\}$

$$\begin{aligned}
p_{n+6} &= 2x\,p_{n+5} + 2xp_{n+1} - p_n - 2x\,\widehat{p}_{n+3},\\
\overline{p}_{n+6} &= 2x\,\overline{p}_{n+3} - \overline{p}_n,\\
\widehat{p}_{n+2} &= 2x\,p_{n+1} - \widehat{p}_n
\end{aligned} \tag{9}$$

Since we are only interested in the sequence $\{p_n\}$, we can omit the second recurrence from the previous system to take the following system of two linear recurrences for sequences $\{p_n\}$ and $\{\widehat{p}_n\}$

$$\begin{aligned}
p_{n+6} &= 2x\,p_{n+5} + 2xp_{n+1} - p_n - 2x\,\widehat{p}_{n+3},\\
\widehat{p}_{n+2} &= 2x\,p_{n+1} - \widehat{p}_n
\end{aligned}$$

which can be easily reduced by substitution method to the self-recurrence (7) of the sequence $\{p_n\}$. Initial conditions (8) for p_i, $1 \le i \le 7$, we easily get as special cases of (5) in [14] and the initial condition for p_8 we can compute from (4) in [14]. Thus, the proof is complete.

3. Conclusions

In this article, our main purpose was to show that the statement in [5, Remark 9] is incorrect. For this purpose, we have found the self-recurrence for the sequence of permanents of the 3-tridiagonal Toeplitz matrix $\mathbf{T}_n^{(3)}(i,2x,i)$. Our derivation was based on two substantial previous results. First, we used da Fonseca [8], in which the author showed that the permanent of matrix $\mathbf{T}_n^{(k)}(i,2x,i)$, studied by Küçük and Düz [5], is equal to the determinant of the matrix $\mathbf{T}_n^{(k)}(1,2x,1)$. Subsequently, we used Borowska and Łacińska [14], in which authors found the recurrence system for calculating determinants of the heptadiagonal Toeplitz matrices.

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References

[1] El-Mikkawy, M.E.A., & Sogabe, T. (2010). A new family of k-Fibonacci numbers. *Appl. Math. Comput.*, *215*, 4456-4461.

[2] Egerváry, V.E., & Szász, O. (1928). Einige Extremalprobleme im Bereiche der trigonometrischen Polynome. *Math. Z.*, *27*(1), 641-652.

[3] Losonczi, L. (1992). Eigenvalues and eigenvectors of some tridiagonal matrices. *Acta Mathematica Hungarica*, *60*(3), 309-332.

[4] da Fonseca, C.M. & Kowalenko, V. (2019). Eigenpairs of a family of tridiagonal matrices: three decades later. *Acta Math. Hungar.*

[5] Küçük, A.Z., & Düz, M. (2017). Relationships between the permanents of a certain type of k-tridiagonal symmetric Toeplitz and the Chebyshev polynomials. *Journal of Applied Mathematics and Computational Mechanics*, *16*, 75-86.

[6] Mason, J.C., & Handscomb, D.C. (2003). *Chebyshev Polynomials*. Boca Raton, USA: Chapman and Hall/CRC Press Company.

[7] Minc, H. (1978). *Permanents*. Reading: Addison-Wesley.

[8] da Fonseca, C.M. (2018). On some conjectures regarding tridiagonal matrices. *Journal of Applied Mathematics and Computational Mechanics*, *17*(4), 13-17.

[9] https://www.ibm.com/support/knowledgecenter/en/SSFHY8_6.1/reference/am5gr_cdst.html

[10] da Fonseca, C.M. (2011). An identity between the determinant and the permanent of Hessenberg type-matrices. *Czechoslovak Mathematical Journal*, *61*(136), 917-921.

[11] da Cruz, H. F., Rodrigues, I. I., Serodio, R., Simoes, A., & Velhinho, J. (2017). Convertible Subspaces of Hessenberg-Type Matrices. *Mathematics*, *5*(4), Article number: 79.

[12] Borowska, A., Łacińska, L., & Rychlewska, J. (2013). On determinant of certain pentadiagonal matrix. *Journal of Applied Mathematics and Computational Mechanics*, *12*(3), 21-26.

[13] Borowska, A., Łacińska, L., & Rychlewska, J. (2014). A system of linear recurrence equations for determinant of pentadiagonal matrix. *Journal of Applied Mathematics and Computational Mechanics*, *13*(2), 5-12.

[14] Borowska, A., & Łacińska, L. (2014). Recurrence form for determinant of a heptadiagonal symmetric Toeplitz matrix. *Journal of Applied Mathematics and Computational Mechanics*, *13*, 19-26.