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ON A CERTAIN EMBEDDING IN THE SPACE OF MEASURES

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Abstract. We take under consideration Young measures – objects that can be interpreted as generalized solutions of a class of certain nonconvex optimization problems arising among others in nonlinear elasticity or micromagnetics. They can be looked at from several points of view. We look at Young measures as at a class of weak* measurable, measure-valued mappings and consider the basic existence theorem for them. On the basis of this theorem, an imbedding of the set of bounded Borel functions into the set of Young measures is defined. Using the weak* denseness of the set of Young measures associated with simple functions in the set of Young measures, it is shown that this imbedding assigns the Young measure associated with any bounded Borel function.

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1. Introduction

Many problems arising in technology and engineering can be modelled by functionals acting on an appropriate function space. Solutions of such problems can be resolved into minimizing those functionals.

More precisely, denote by Ω an elastic body which is ”represented” by an open, connected subset of $\mathbb{R}^3$ of positive Lebesgue measure M. We assume also that a boundary $\partial\Omega$ of Ω is sufficiently smooth. The body is now subject to a deformation denoted by u. It is a function on Ω with values in $\mathbb{R}^3$. We impose on u some regularity conditions, as they are usually summarized in the requirement that u is an element of an appropriate function space V. It should also be one-to-one, with the possible exception on $\partial\Omega$, and preserving an orientation that is the determinant of a matrix $\nabla u(x)$ is positive for almost all, with respect to the Lebesgue measure, $x \in \Omega$. The 3×3 matrix $\nabla u(x)$ is called the deformation matrix and can be regarded as a measure of a local strain.

We want to minimize the energy functional

$$\mathscr{J}: V \ni u \to \mathscr{J}(u) := \int\limits_{\Omega} f(x, u(x), \nabla u(x)) dx \in \mathbb{R},$$

where f denotes the energy density. It usually depends on some constant physical parameters. As to the regularity properties, the most common for f are the Carathéodory ones: measurability with respect to the first and continuity with respect to the second and third variable. To ensure the boundedness of the minimizing sequences some growth conditions on the energy density should be imposed. Generally, $\mathscr{J}$ should be coercive, that is the equality $\lim\limits_{|u| \to \infty} \mathscr{J}(u) = +\infty$ should be satisfied (here $|\cdot|$ denotes the norm in the function space V). This guarantees that minimizing sequences for $\mathscr{J}$ are always bounded. The growth conditions are different for different materials, see for example [1] and the references there. The solutions of such variational problems correlate with the equilibrium state of considered material.

Since the energy functional is bounded from below "by nature", the direct method of finding the minimizers can be applied. However, it turns out that energy functionals of, among others, some elastic crystals, for example In-Th, Cu-Al-Ni (see for instance [2]) do not admit minimizers: the infimum is not attained. A closer look at the internal stucture of these kinds of materials reveals presence of a *microstructure* – a phenomenon that lies between the micro- and macroscopic level. It becomes apparent that the existence of the microstructure is typical for materials whose energy functionals do not attain their infima. In this case, the minimizing sequences are divergent in the norm topology of V, but they are weakly (or weakly* when V is not reflexive) convergent. The elements of these sequences are the functions oscillating rapidly around their weak limits. Another difficulty appears here, namely the sequence whose elements are compositions of such functions, even with a continuous function φ, is in general not weakly convergent to the composition of the weak limit with φ. It is illustrated by the following cogent example.

Example 1 ([1], p. 13) Consider a function sequence $(\sin nx)$, $n \in \mathbb{N}$, defined on an interval $\left(0, \frac{\pi}{2}\right)$.

On the one hand, we have

$$\int\limits_0^{\frac{\pi}{2}} \sin(nx) dx \to \int\limits_0^{\frac{\pi}{2}} 0 dx, \text{ as } n \to \infty,$$

while on the other

$$\int\limits_0^{\frac{\pi}{2}} \sin^2(nx) dx \to \tfrac{\pi}{4} \neq \int\limits_0^{\frac{\pi}{2}} 0^2 dx.$$

Therefore it can be seen, that if (u_n) is a weakly convergent to u_0 minimizing sequence for $\mathscr{J}$, we can usually not conclude that u_0 is the minimizer. The absence of classical minimizers raises the need of looking for the generalized ones. The existence of such minimizers has been discovered by L.C. Young in 1937. Today they are called the Young measures. Their mathematical theory is intricate and it is based on measure theoretic and functional analytic methods. It can be seen for instance in books [1–3] devoted in large part to the precise mathematical description of problems arising in engineering.

Since in the problems requiring seeking the generalized infima, elements of the minimizing sequences of considered functionals are of highly oscillatory nature, one should be certain that there do exist Young measures associated with such functions. Fortunately, it can be proved much more, statement *Let* ν *be a Young measure associated with the elements of a sequence . . .* makes good sense for quite a large class of functions. The relevant existence theorems and their proofs, namely Theorem 3.1.6 in [4] and Theorem 2.2 in [1] are, as it is stated in this second book, *of a highly technical nature*.

In the first of the theorems mentioned above, embedding a bounded Borel function into the space of measures by assigning to it an appropriate Dirac measure is an important step. As a conclusion, it can be said that for such a function there exists a Young measure associated with it. On this basis one can look for Young measures associated with specific kinds of functions. However, looking at the examples in the literature, it seems that the existence theorems serve rather as a justification for the methods of finding Young measures in particular cases. The purpose of this article is to show that having the Theorem 3.1.6 in [4] as a foundation, it is possible to define an embedding into the space of measures in such a way that the form of the Young measure associated with particular function is more apparent. The change seems to provide a more convincing argument of the existence of the desired Young measure than assigning just a Dirac measure to a function. Moreover, it also connects the Young measure, which is a probability measure on a codomain of the considered function with the Lebesgue measure on the domain of definition of this function.

In the next section, after a brief mention of the sequential approach to Young measures, we sketch the basics of our approach to the Young measures theory. We closely follow the contents of the relevant parts of chapter three in [4]. This outline culminates with the basic existence Theorem 1. All the results are presented without proofs, for which we refer the reader to [4], where they are presented carefully and in detail. The second subsection is the main part of the article. We formulate and prove inthe second subsection the central result, the Theorem 2. The article concludes with the third section containing final remarks and additional bibliographical references.

2. Young measures associated with functions

As it has been observed, L.C. Young was the first to notice that a sequence of compositions of a continuous function with violently oscillating bounded functions has a subsequence (not relabelled) which is weakly (or weakly*) convergent, usually to a measure. In the article [5] published in 1937, he proved that if (u_n) is a sequence of functions defined on a bounded open interval $I \subset \mathbb{R}$, taking values in a compact set $K \subset \mathbb{R}$, φ – a continuous real function on $\mathbb{R}$, then the weak* limit of the (sub)sequence $(\varphi(u_n))$ can be expressed as

$$\int_I \overline{\varphi}(x)w(x)dx.$$

Here, w is any integrable real function on I, while $\overline{\varphi}$ is the function on I of the form

$$\overline{\varphi}(x) = \int_K \varphi(s)\nu_x(ds).$$

The family $(\nu_x)_{x\in\Omega}$ of probability measures on K is called the *Young measure associated with the sequence* (u_n) or a *parametrized measure associated with the sequence* (u_n).

This is one of the possible approaches to Young measures: parametrized measures are looked at as generalized limits. This approach is studied in detail in [7]. One can also check Appendix D1 in [3] and the references given there. In this article the understanding of a Young measure as a map is taken under consideration. The basic text is [4], especially chapter three, section 3.1. It should be noted however that these two ways are not disjoint, see for example a paragraph 4.3 in [2]. Various approaches to Young measures are described in book [6].

2.1. A functional approach to Young measures

Here we describe the outline of the Young measures theory as it is done in [4], where the interested reader can find proofs of the results and the references. The main theorem of this section is also the theorem of interest of this article.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of positive Lebesgue measure M, let a set $K \subset \mathbb{R}^l$ be nonempty and compact and denote by $\mathscr{U}$ the set of all measurable functions on Ω with values in K. For the purpose of this section, the space of integrands is the space $Car(\Omega,K;\mathbb{R})$ of real functions defined on the product $\Omega \times K$. They satisfy the Carathéodory conditions. This means that for any $h \in Car(\Omega,K;\mathbb{R})$, the function $h(\cdot,k)$, $k \in K$, is measurable, while the function $h(x,\cdot)$, $x \in \Omega$, is continuous. We equip this space with a norm given by the formula: for all $h \in Car(\Omega,K;\mathbb{R})$

$$||h||_{Car} := \int_\Omega \sup_{k\in K} |h(x,k)|dx.$$

Let us denote by $L^1(\Omega, C(K))$ the space of those functions denoted on $\Omega \subset \mathbb{R}^d$ with values in a vector space $C(K)$, which are integrable in a Bochner sense. It can be proved that this space is isometrically isomorphic to the space $Car(\Omega, K; \mathbb{R})$, so these spaces can be identified.

Consider a mapping

$$i\colon \mathscr{U} \to L^1(\Omega, C(K))^*$$

with values in a space conjugate to $L^1(\Omega, C(K))$, defined by the formula

$$\langle i(u), h \rangle := \int_{\Omega} h(x, u(x))dx$$

and take account of the image $i(\mathscr{U}) \subset L^1(\Omega, C(K))^*$ of the set $\mathscr{U}$ under the mapping i. The weak* closure of $i(\mathscr{U})$ in $L^1(\Omega, C(K))^*$ will be denoted by $Y(\Omega, K)$. Since the weak* topology is not metrizable, we have to consider generalized sequences indexed by the indices $\alpha \in D$, where D is a directed set. Thus

$$Y(\Omega, K) := \Big\{ \eta \in L^1(\Omega, C(K))^* : \exists (u_\alpha) \subset \mathscr{U} : i(u_\alpha) \xrightarrow[\alpha]{w^*} \eta \Big\}.$$

Thus by definition the elements of the set $Y(\Omega, K)$ are the limits of the generalized sequences of compositions of the mapping i with elements from $\mathscr{U}$. Since the set $\mathbb{N}$ is directed by the relation $\leq$, the set $Y(\Omega, K)$ also contains the limits of those sequences of such compositions, whose elements are indexed with natural numbers.

The following spaces will be of importance in further considerations:

- $rca(K)$ – the space of regular, countably additive scalar measures on K, equipped with the norm $\|m\|_{rca(K)} := |m|(\Omega)$, where $|\cdot|$ stands in this case for the total variation of the measure m. With this norm, $rca(K)$ is a Banach space;
- $rca^1(K)$ – the subset of $rca(K)$ with elements being probability measures on K;
- $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$ – the set of the weakly* measurable mappings

  $$\nu\colon \Omega \ni x \to \nu(x) \in rca(K),$$

  assigning to the points from the domain of definition of $u \in \mathscr{U}$ the measures on the range of u and such that

  $$\operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\} < +\infty.$$

  By the fact that ν is a weakly* measurable mapping, we mean that for all $x \in \Omega$ the function $x \mapsto \int_K g(k)(\nu(x))(dk) = \langle \nu(x), g \rangle$, $g \in C(K)$, is Borel measurable.

  We equip this set with the norm

  $$\|\nu\|_{L^\infty_{w^*}(\Omega, \mathrm{rca}(K))} := \operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\}.$$

By the Dunford-Pettis theorem this space is isometrically isomorphic with the space $L^1(\Omega, C(K))^*$.

Remark 1 In [4] weakly* measurable mappings are defined as *weakly* measurable mappings, but since $rca(K)$ is a conjugate space, the term *weakly* measurable* is proper. □

It turns out that the space $(L^\infty_{w^*}(\Omega, \mathrm{rca}(K)), \|\cdot\|_{L^\infty_{w^*}(\Omega,\mathrm{rca}(K))})$ of weakly* measurable measure valued mappings is isometrically isomorphic to the space $L^1(\Omega, C(K))^*$. This means that there exists a mapping

$$\psi\colon L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) \to L^1(\Omega, C(K))^*$$

realizing this isomorphism. It assigns to any $\nu \in L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$ a functional $\eta \in (L^1(\Omega, C(K))^*$, acting on $L^1(\Omega, C(K))$ in the following way:

$$\eta : L^1(\Omega, C(K)) \ni h \to \langle \eta, h\rangle := \int\limits_\Omega \Big(\int\limits_K h(x,k)\, d\nu_x(k) \Big) dx.$$

Among the weakly* measurable mappings we distinguish the *Young measures*, that is mappings, whose values are probability measures on the set K. The set of the Young measures on the compact set $K \subset \mathbb{R}^l$ will be denoted by $\mathscr{Y}(\Omega, K)$:

$$\mathscr{Y}(\Omega, K) := \big\{ \nu = (\nu(x)) \in L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) : \nu_x \in \mathrm{rca}^1(K) \text{ for a.a } x \in \Omega \big\}.$$

We will write ν_x or $(\nu_x)_{x\in\Omega}$ instead of $\nu(x)$. To emphasize the Young measure associated with particular function u we will write ν^u_x. If the Young measure $(\nu_x)_{x\in\Omega}$ does not depend on the parameter $x \in \Omega$, it is called *homogeneous*. Most specific examples of Young measures are homogeneous Young measures, see for instance [1, 2, 6, 7]. In [8], it has been shown, that homogeneous Young measures are exactly the constant mappings from Ω to $\mathrm{rca}(K)$. It is also shown that homogeneous Young measure is associated with a Borel function if and only if it is an image of the normed to unity Lebesgue measure on Ω under this Borel function. The homogeneous Young measure will be denoted simply by ν (or ν^u respectively).

Now we want to embed the set $\mathscr{U}$ into the set $\mathscr{Y}$ of Young measures. This will be done by the Dirac measure. Recall that if A is any set, then the Dirac measure δ_b concentrated at point b is defined as follows:

$$\delta_b(A) := \begin{cases} 1, & \text{if } b \in A \\ 0, & \text{if } b \notin A. \end{cases}$$

We define a Dirac mapping δ assigning to $u \in \mathscr{U}$ a mapping $\delta(u)$ defined by

$$\Omega \ni x \to (\delta(u))(x) := \delta_{u(x)}(\cdot). \tag{1}$$

We thus have for any Borel $A \subseteq K$

$$\delta_{u(x)}(A) = \begin{cases} 1, & \text{if } u(x) \in A \\ 0, & \text{if } u(x) \notin A. \end{cases}$$

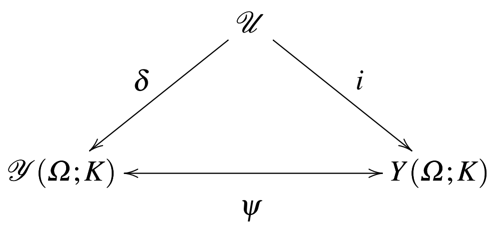
Observe that for any $g \in C(K)$ we have

$$\int_K g(k)(\delta_{u(x)})(dk) = g(u(x)),$$

so from the fact that the composition of a continuous function with a measurable one (the continuous function being the outer function), it follows that the mapping (1) is weakly* measurable, that is $\delta(u) \in \mathscr{Y}(\Omega, K)$.

The following theorem shows mutual relations between the set of Borel measurable bounded functions $\mathscr{U}$, the set probabilty measure valued weakly* measurable mappings $\mathscr{Y}$ and the set of limits of the generalized sequences of compositions Y.

Theorem 1 (Theorem 3.1.6 in [4]) *The diagram*



is commutative. □

Corollary 1 *(see Corollary 3.1.7 in [4]) The set $\mathscr{Y}(\Omega;K)$ of all Young measures is a convex and weakly*: compact, sequentially compact set in which $\delta(U)$ is dense.* □

Recalling that a sequence $(\nu_\alpha)_{\alpha \in D}$ of bounded measures on a compact set $K \subset \mathbb{R}^l$ converges weakly* to a measure ν_0, if $\forall g \in C(K, \mathbb{R})$, there holds

$$\lim_\alpha \int_K g(k) d\nu_\alpha(k) = \int_K g(k) d\nu_0(k),$$

we have the following corollary.

Corollary 2 *For any $u \in \mathscr{U}$ there exists a Young measure associated with it. Moreover, Young measures from the set $\delta(\mathscr{U})$ can be regarded as weak* limits of certain sequences of compositions.* □

2.2. Assigning a Borel function its Young measure

On the basis of the Theorem 1 and Corollary 2, we will define an embedding assigning to a Borel function $u\colon \Omega \to K$ a Young measure associated with it. We begin with some preliminary remarks.

Denote by μ the normalized Lebesgue measure on Ω: $d\mu(x) := \frac{1}{M}dx$ with a d-dimensional Lebesgue measure dx. Let $\{\Omega_i\}_{i=1}^n$ be an open partition of Ω into pairwise disjoint subsets Ω_i with Lebesgue measure $m_i > 0$. The elements of the partition satisfy the relation $\bigcup_{i=1}^{n} cl(\Omega_i) = cl(\Omega)$, where '$cl(A)$' means closure of the set A (in a suitable topology). By $\mathbf{1}_A$ we denote the characteristic function of the set A.

Choose and fix points $p_i \in \mathbb{R}^l$, $i = 1,2,\ldots,n$ and let f be a simple function, that is a function taking a finite number of values:

$$f := \sum_{i=1}^{n} p_i \mathbf{1}_{\Omega_i}. \tag{2}$$

It turns out that the Young measure associated with f is homogoneous and is a convex combination of Dirac measures concentrated at the values of f.

Proposition 1 *(see e.g. [9]) The Young measure associated with f is homogeneous and is of the form*

$$\nu^f = \frac{1}{M}\sum_{i=1}^{n} m_i \delta_{p_i}.$$

Observe the particular case of the above for f being constant on Ω.

The next definition arises in a natural way.

Definition 1 The Young measure associated with simple function will be called a simple Young measure. □

The following result is a special case of a general denseness result, for which the reader can consult [10] and the references there.

Proposition 2 *The set of all simple Young measures is weak* dense in the set of the Young measures associated with functions from $\mathscr{U}$.* □

Let $u \in \mathscr{U}$, $A \in \mathscr{B}(K)$ and define a map $\varkappa$ as follows.

$$\Omega \ni x \to (\varkappa(u))(x) := \tfrac{m_A}{M}\delta_{u(x)}(\cdot), \tag{3}$$

where

$$m_A := \begin{cases} \mu(u^{-1}(A)), & \text{if } u(x) \in A \\ 0, & \text{if } u(x) \notin A. \end{cases}$$

We have $\varkappa(u)(x)(\emptyset) = 0$ and since $\mu(u^{-1}(K)) = M$ there is $\varkappa(u)(x)(K) = 1$ so $\varkappa$ is a probability measure on K. Moreover, for any $g \in C(K)$ there holds

$$x \mapsto \langle \varkappa(x), g\rangle = \int_K g(k)(\varkappa(u)(x))(dk) = \int_K g(k)\tfrac{\mu(u^{-1}(K))}{M}(\delta_{u(x)})(dk) = g(u(x)),$$

which means weak* measurability of $\varkappa$; therefore it is a Young measure.

Theorem 2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set of positive Lebesgue measure and smooth boundary. Denote by K a nonempty compact subset of $\mathbb{R}^l$ and consider a Borel function $u\colon \Omega \to K$. Then the Young measure associated with u is equal to the measure defined by (3).* □

PROOF First consider the case when u is a simple function having values $p_1, p_2, \dots, p_n \in K$. Denote by ν^u a Young measure associated with u. It is of the form as stated in Proposition 1. By definition, we have for any $x \in \Omega$

$$\varkappa^u(A) = \frac{\mu(u^{-1}(A))}{M}\delta_{u(x)}(A).$$

Let A be any open subset of K. If none of the points $p_1, p_2, \dots, p_n$ belongs to A, then $\varkappa^u(A) = 0 = \nu^u(A)$, otherwise we have $\delta_{u(x)}(A) = 1$ and

$$\mu(u^{-1}(A)) = \mu\Big(\bigcup_{\{i:p_i \in A\}} \Omega_i\Big) = \sum_{\{i:p_i \in A\}} m_i.$$

Thus, due to the arbitrariness of A, we have $\varkappa^u = \nu^u$. It follows that $\varkappa^u$ equals ν^u for any simple function u. The theorem statement follows now from Corollary 2 and the Lebesgue dominated convergence theorem. ■

3. Conclusions

The Theorem 2 states that it is possible to embed a bounded Borel function into the set of Young measures in such a way that value of this embedding is equal to the Young measure associated with this function. Obviously, before doing so, we have to be certain that the procedure makes sense, i.e. that the Young measure associated with this function does exist. It is the case, because the Theorem 1 guarantees the existence of such a measure, Proposition 3.5 in [9] gives an explicit form of the Young measure associated with simple functions, and the standard routine of the measure theory does the rest. This result also directly shows the connection between Young measure associated with a function with the Lebesgue measure on the σ-algebra of subsets of the domain of definition of this function.

The next step in the investigation seems to be checking if the measure defined by the equation (3) can replace the Dirac delta mapping in the Theorem 1.

As mentioned in the Introduction, Young measures have found applications in various branches of pure and applied sciences. The fact that they are rather abstract objects raises difficulties in calculating them in concrete cases. This leads to developing various techniques of calculating or approximating Young measures for specific problems. For example, in [11], a version of a finite element method in constructing an approximation of a (Young) measure-valued solution to some optimization problems arising in micromagnetics is presented.

Although very useful in capturing oscillation effects, Young measures are of little use when concentration effects occur. These problems appear when an appropriate sequence of L^p-integrable functions converges to zero in measure, with some additional condition. An example (from micromagnetics) of a concentration effect is given in [2] in a paragraph entitled *What Young measure cannot detect*. These effects also appear when studying measure-valued solutions of the conservation laws. Fortunately, some generalizations of the notion of a Young measure make it possible to analyze such problems. In the seminal article [12], the authors proposed a notion of a generalized Young measure, describing both oscillation and concentration effects. The literature concerning various generalizations of Young measures is large; an interested reader can look for instance at the Appendix D.2 in [3], section 3.2 in [4, 13, 14] and the references cited there.

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