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FINITE DIFFERENCE METHOD FOR THE FRACTIONAL ORDER PSEUDO TELEGRAPH INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The main goal of this paper is to investigate the numerical solution of the fractional order pseudo telegraph integro-differential equation. We establish the first order finite difference scheme. Then for the stability analysis of the constructed difference scheme, we give theoretical statements and prove them. We also support our theoretical statements by performing numerical experiments for some fractions of order α.

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Keywords: *pseudo telegraph equation, integro-differential equation, finite difference scheme, stability analysis*

1. Introduction

Fractional calculus has become a valuable tool in mathematical modeling recently [1–9]. Although there are some different generalizations of the differential operator in the literature, Riemann-Liouville and Caputo fractional derivatives are the most commonly used definitions. The former type of fractional derivative is closely related to the latter type of fractional derivative. The definition of the Riemann-Liouville fractional derivative is

$$D_{\xi}^{\alpha}u(\xi,\nu)=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial\xi^n}\int_0^{\xi}\frac{1}{(\xi-s)^{\alpha-n+1}}u(s,\nu)ds,\ (n-1<\alpha\leq n). \tag{1}$$

The Caputo fractional derivative $D_{\xi}^{\alpha}u(\xi,\nu)$ of order α with respect to time is defined as:

$$D_{\xi}^{\alpha}u(\xi,\nu)=\frac{1}{\Gamma(n-\alpha)}\int_0^{\xi}\frac{1}{(\xi-s)^{\alpha-n+1}}\frac{\partial^n u(s,\nu)}{\partial s^n}ds,\ (n-1<\alpha\leq n). \tag{2}$$

We consider both the Riemann-Liouville and Caputo type fractional order of pseudo telegraph integro-differential equation. An arbitrary function does not have to be continuous at the origin or differentiable to use the Riemann-Liouville fractional derivative. On the other side, one of the major advantages of the Caputo fractional derivative is that it allows for the inclusion of traditional initial and boundary conditions in the formulation of the problem. When dealing with real-world situations, the Caputo fractional derivative additionally allows the application of the initial and boundary conditions. When modeling real-world problems, the Caputo derivative is the best fractional operator to use.

Several numerical and analytical solution techniques are applied to fractional order telegraph equations [10–16]. Pseudo types of this equation contain a mixed partial derivative with respect to variables of time and space. In [17], the pseudo partial differential equation is used to model signal propagation along a neuristor which is a one-dimensional channel through which signals can flow and which is utilized to generate all digital logic functions. Also, the line for transmitting active pulses is extended to an active surface and modeled by pseudo equations. Therefore, these types of equations have a significant role in applied science, and it is important to obtain exact and approximate numerical solutions for either integer or non-integer order. In the literature there are several methods studied to solve these equations both analytically and numerically [18–25]. For solving the time-fractional Burger--Huxley equation inside the Caputo type fractional derivative, a simple and powerful numerical technique was provided [26]. In [27], to get solutions to the time fractional Advection-Diffusion equation, a powerful technique was devised. The fractional Boussinesq-like equation with the β derivative, which explains the propagation of tiny amplitude long capillary-gravity waves on the surface of shallow water, was used in [28]. They proposed new solutions to the fractional-order Korteweg-de Vries problem by combining the benefits of fictitious time integration with group preservation methods [29]. In recent years, many studies have been done on the finite difference method [30–32].

We consider:

$$\begin{cases} u_{\xi\xi}(\xi,\nu)+u_{\xi}(\xi,\nu)+u(\xi,\nu)+\displaystyle\int_0^{\xi}\gamma(s)D_s^{\alpha}u(s,\nu)ds = u_{\xi\nu\nu}+u_{\nu\nu}(\xi,\nu)+f(\xi,\nu), \\ u(0,\nu)=u_{\xi}(0,\nu)=0, \quad 0<\nu<L, \\ u(\xi,0)=u(\xi,L)=0, \quad 0<\xi<T, \\ 0<\alpha<1 \end{cases} \tag{3}$$

where $\gamma(\xi)$ is continuous on $0<\xi\leq T$ with constraint

$$|\gamma(s)| \leq \frac{K}{s^{1-\alpha}}.$$

In the literature, there is no study about the approximate solution of this problem. For this reason, it is important to investigate the exact and approximate solutions of this problem. Obtaining difference schemes and making stability estimations for this problem make this study different.

The considered telegraph equation has an integral component, making it an integro-differential equation with fractional derivative. Due to the difficulty of solving these equations analytically, an efficient approximation solution is necessary. This study presents a numerical solution of the problem (3). In order to calculate approximate numerical solutions, we begin with constructing the first order finite difference scheme. Then for the stability analysis of the constructed difference scheme, in the next section we give theoretical statements and prove them. We also support our theoretical statements by performing numerical experiments for some fractions of order α. Error analysis for the numerical solutions are presented.

2. Matrix stability for finite difference method

We assume that $h = \dfrac{L}{M}$ for x-axis and $\tau = \dfrac{T}{N}$ for t-axis grid mess. Thus, we get

$$\nu_n = nh,\, n = 1,2,...,M, \quad \xi_k = k\tau,\, k = 1,2,...,N.$$

For the fractional pseudo telegraph integro-differential equation (3), we establish the difference schemes by the finite difference method as:

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_n^{k+1} - u_n^k}{\tau} + u_n^{k+1} + \displaystyle\sum_{l=1}^{k-1}\sum_{m=1}^{l} \dfrac{\Gamma(l-m+1-\alpha)(u_n^{m+1} - u_n^m)}{\Gamma(1-\alpha)(l-m)!\,l^{1-\alpha}} \\[2ex] = \dfrac{1}{\tau}\left(\dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} - \dfrac{u_{n+1}^{k} - 2u_n^{k} + u_{n-1}^{k}}{h^2}\right) + \dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} + f_n^k, \\[2ex] u_n^0 = \varphi(\nu_n), \quad \dfrac{u_n^1 - u_n^0}{\tau} = \psi(\nu_n). \end{cases} \tag{4}$$

Then, we get

$$
\begin{cases}
\left[(-\frac{1}{\tau h^2}-\frac{1}{h^2})u_{n-1}^{k+1}+(\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{\tau h^2}+\frac{2}{h^2})u_n^{k+1}+(-\frac{1}{\tau h^2}-\frac{1}{h^2})u_{n+1}^{k+1}\right] \\
+\left[(\frac{1}{\tau h^2})u_{n-1}^{k}+(-\frac{2}{\tau^2}-\frac{1}{\tau}-\frac{2}{\tau h^2})u_n^{k}+(\frac{1}{\tau h^2})u_{n+1}^{k}\right]+\left[(\frac{1}{\tau^2})u_n^{k-1}\right] \\
=f_n^k-\sum_{l=1}^{k-1}\sum_{m=1}^{l}\frac{\Gamma(l-m+1-\alpha)(u_n^{m+1}-u_n^m)}{\Gamma(1-\alpha)(l-m)!l^{1-\alpha}}, \\
u_n^0=\varphi(\nu_n), \quad \frac{u_n^1-u_n^0}{\tau}=\psi(\nu_n),\ 1\le n\le M, \\
u_0^k=u_M^k=0,\ 0\le k\le N.
\end{cases}
\tag{5}
$$

Then, we obtain

$$
\begin{cases}
u^1=u^0+\tau\psi, \\
Au^{k+1}=Bu^k+Cu^{k-1}+\varphi_n^k-\sum_{l=1}^{k-1}\sum_{m=1}^{l}\frac{\Gamma(l-m+1-\alpha)(u^{m+1}-u^m)}{\Gamma(1-\alpha)(l-m)!l^{1-\alpha}}, \\
u_0^k=u_M^k=0,\ 0\le k\le N,
\end{cases}
\tag{6}
$$

where $\varphi_n^k=\left[\varphi_0^k,\varphi_1^k,...,\varphi_M^k\right]$, $\varphi_n^0=\varphi(\nu_n)$, $\varphi_n^k=f_n^k=f(\xi_k,\nu_n)$, $1\le n\le M$, $1\le k\le N$ and $u^k=\left[u_0^k,u_1^k,...,u_N^k\right]^T$. Where A and B are symmetric tridiagonal matrices and C is a diagonal matrix.

$$
A=\begin{bmatrix}
a & b & 0 & \cdots & 0 & 0 & 0 \\
b & a & b & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b & a & b \\
0 & 0 & 0 & \cdots & 0 & b & a
\end{bmatrix}_{(N-1)\times(N-1)},
$$

where $a=\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{\tau h^2}+\frac{2}{h^2}$, $b=-\frac{1}{\tau h^2}-\frac{1}{h^2}$,

$$
B=\begin{bmatrix}
k & p & 0 & \cdots & 0 & 0 & 0 \\
p & k & p & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p & k & p \\
0 & 0 & 0 & \cdots & 0 & p & k
\end{bmatrix}_{(N-1)\times(N-1)},
$$

where $k = \dfrac{2}{\tau^2} + \dfrac{1}{\tau} + \dfrac{2}{\tau h^2}, \quad p = -\dfrac{1}{\tau h^2}$,

$$C = \begin{bmatrix} c & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & c & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & c \end{bmatrix}_{(N-1)\times(N-1)},$$

where $c = -\dfrac{1}{\tau^2}$.

We express $\|A\| = \|A\|_\infty = \max\limits_{1\le j\le N-1} \sum\limits_{j=1}^{N-1} |a_{jm}|$ where $A = [a_{jm}]_{(N-1)\times(N-1)}$.

Lemma 1 *Let* $d = \sum\limits_{l=1}^{k-1}\sum\limits_{m=1}^{l} \dfrac{\Gamma(l-m+1-\alpha)}{\Gamma(1-\alpha)(l-m)!l^{1-\alpha}}$. *If* $\dfrac{2}{\tau^2} + \dfrac{1}{\tau} + \dfrac{2}{\tau h^2} - d > 0$ *and* $\dfrac{1}{\tau^2} + \dfrac{4}{\tau h^2} - \dfrac{2}{h^2} - 1 - d < 0$, *then* $\|A^{-1}B\| \le 1$.

PROOF Let us assume $\dfrac{2}{\tau^2} + \dfrac{1}{\tau} + \dfrac{2}{\tau h^2} - d > 0$ and $\dfrac{1}{\tau^2} + \dfrac{4}{\tau h^2} - \dfrac{2}{h^2} - 1 - d < 0$, then

$$\begin{aligned} \|A^{-1}B\| \le \|A^{-1}\|\|B\| &\le \frac{1}{\min\limits_{1\le j\le N-1}\left\{|a_{jj}| - \sum\limits_{m\ne j, m=1}^{N-1} |a_{jm}|\right\}}\|B\| \\ &\le \frac{|\frac{2}{\tau^2}+\frac{1}{\tau}+\frac{2}{\tau h^2}-d| + |\frac{1}{\tau h^2}| + |\frac{1}{\tau h^2}|}{|\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{\tau h^2}+\frac{2}{h^2}| - |-\frac{1}{\tau h^2}| - |-\frac{1}{\tau h^2}|} \qquad (7) \\ &= \frac{\frac{2}{\tau^2}+\frac{1}{\tau}+\frac{2}{\tau h^2}-d+\frac{2}{\tau h^2}}{\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{\tau h^2}+\frac{2}{h^2}-\frac{2}{\tau h^2}} = \frac{\frac{2}{\tau^2}+\frac{1}{\tau}+\frac{4}{\tau h^2}-d}{\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{h^2}} \\ &\le 1 \end{aligned}$$

since $\dfrac{1}{\tau^2} + \dfrac{4}{\tau h^2} - \dfrac{2}{h^2} - 1 < d$.

Lemma 2 *Let* $d = \sum\limits_{l=1}^{k-1}\sum\limits_{m=1}^{l} \dfrac{\Gamma(l-m+1-\alpha)}{\Gamma(1-\alpha)(l-m)!l^{1-\alpha}}$. *If* $\dfrac{2}{\tau^2} + \dfrac{1}{\tau} + \dfrac{2}{\tau h^2} - d > 0$, *then* $\|A^{-1}C\| \le 1$.

PROOF

$$\begin{aligned}\|A^{-1}C\| \leq \|A^{-1}\|\|C\| &\leq \frac{1}{\min\limits_{1\leq j\leq N-1}\left\{|a_{jj}| - \sum\limits_{m\neq j, m=1}^{N-1}|a_{jm}|\right\}}\|C\| \\ &\leq \frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{h^2}} \\ &\leq 1.\end{aligned} \tag{8}$$

Theorem 1 *If* $\dfrac{2}{\tau^2}+\dfrac{1}{\tau}+\dfrac{2}{\tau h^2}-d>0$ *and* $\dfrac{1}{\tau^2}+\dfrac{4}{\tau h^2}-\dfrac{2}{h^2}-1-d<0$, *then the equation* (6) *is stable.* □

PROOF Utilizing a similar procedure in [10], and implementing Lemma 1 and 2, the proof of the theorem is completed. ■

3. Computational examples

3.1. Example 1

We consider

$$\begin{cases} u_{\xi\xi}(\xi,\nu)+u_{\xi}(\xi,\nu)+u(\xi,\nu)+\int_0^{\xi}\gamma(s)D_s^{\alpha}u(s,\nu)ds = u_{\xi\nu\nu}(\xi,\nu)+u_{\nu\nu}(\xi,\nu)+f(\xi,\nu), \\ f(\xi,\nu) = (2+2\xi+(1+\pi^2)\xi^2+2\pi^2\xi+\dfrac{\xi^2}{\Gamma(3-\alpha)})sin(\pi\nu), 0<\nu<1, 0<\xi<1, \\ u(0,\nu)=u_{\xi}(0,\nu)=0, \quad 0\leq\nu\leq 1, \\ u(\xi,0)=u(\xi,1)=0, \quad 0\leq\xi\leq 1, \\ 0<\alpha\leq 1. \end{cases} \tag{9}$$

The exact solution is given as $u(\xi,\nu)=\xi^2 sin(\pi\nu)$.

To solve this problem numerically, the established first order difference scheme is as follows:

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^k}{\tau}+u_n^{k+1}+\displaystyle\sum_{l=1}^{k-1}\sum_{m=1}^{l}\dfrac{\Gamma(l-m+1-\alpha)(u_n^{m+1}-u_n^m)}{\Gamma(1-\alpha)(l-m)!l^{1-\alpha}} \\ =\dfrac{1}{\tau}\left(\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}-\dfrac{u_{n+1}^{k}-2u_n^{k}+u_{n-1}^{k}}{h^2}\right)+\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+f_n^k, \\ \nu_n=nh, \xi_k=k\tau, 1\le k\le N-1, \quad 1\le n\le M-1, \\ u_n^0=0, \quad \dfrac{u_n^1-u_n^0}{\tau}=0, \quad 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N. \end{cases} \tag{10}$$

Then, writing the system in the matrix form, we have

$$Au_{n+1}+Bu_n+Cu_{n-1}=D\varphi_n \tag{11}$$

where A, B and C are $(N+1)\times(N+1)$ square matrices and D is identity matrix.

To solve this resulting difference equation, the modified Gauss elimination method is applied. The derivation of the Gaussian elimination method was given in [11]. In Table 1, the approximate solutions for N and M are calculated for each α with $N^2=M$ (or $h=\tau^2$). The maximum norm error of the approximate solution is calculated by

$$\varepsilon=\max_{\substack{1\le k\le N\\1\le n\le M}}|u(\xi,\nu)-u(\xi_k,\nu_n)|,$$

where $u(\xi,\nu)$ and $u(\xi_k,\nu_n)$ are exact and approximate solutions respectively. Then, we give the error plot of the difference scheme (10) in Figure 1 while changing time.

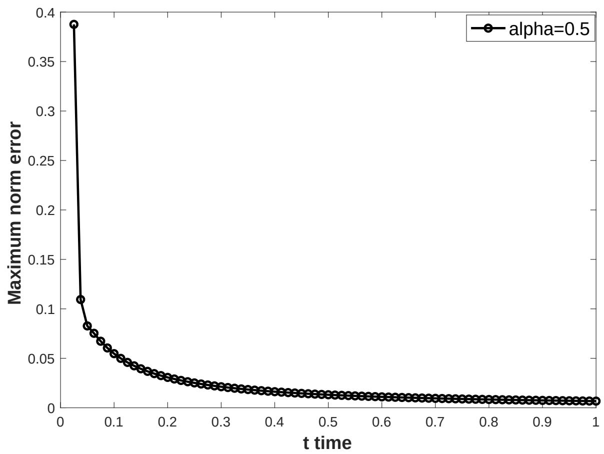

Fig. 1. Result of maximum norm errors of the difference scheme (10) for $\alpha=0.5$. As we vary the time values, it can be observed that errors keep decreasing

To make the error computations more obvious, we present the error table for various values of α.

Table 1. Error calculations

$\tau = \frac{1}{N}, h = \frac{1}{M}, h = \tau^2$					
The example (9).					
N, M	N = 20, M = 20^2	N = 40, M = 40^2	N = 80, M = 80^2	N = 100, M = 100^2	N = 150, M = 150^2
$\alpha = 0.01$	0.0244	0.0128	0.0066	0.0053	0.0035
$\alpha = 0.5$	0.0251	0.0132	0.0067	0.0054	0.0036
$\alpha = 0.99$	0.0263	0.0137	0.0070	0.0056	0.0038

The accuracy of the developed scheme is confirmed by Table 1, which shows that as the grid points increase, the maximum norm error decreases. We then exhibit precise and numerical solutions for each α to demonstrate how similar the solutions are.

The exact solution and the numerical solution obtained for $\alpha = 0.01$ with the step sizes $h = \dfrac{1}{6400}$ and $\tau = \dfrac{1}{80}$ are shown in Figure 2.

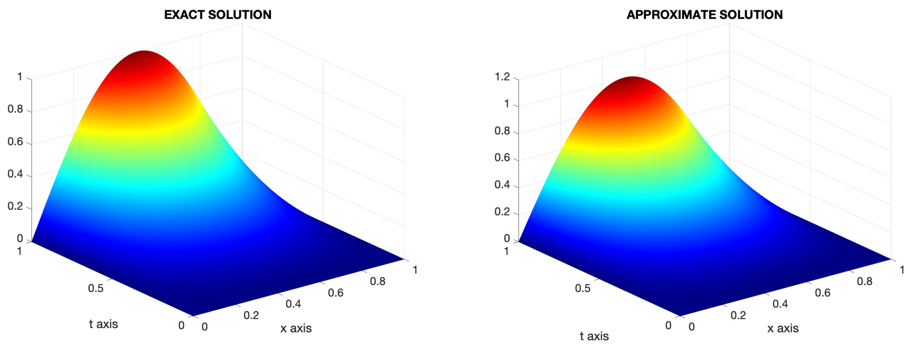

Fig. 2. Left: exact solution. Right: numerical simulation of the problem (9) for N = 80, M = 6400 and $\alpha = 0.01$

For $\alpha = 0.5$ with the same step sizes as in the previous figure, the exact and numerical solutions obtained for the problem (9) are presented in Figure 3.

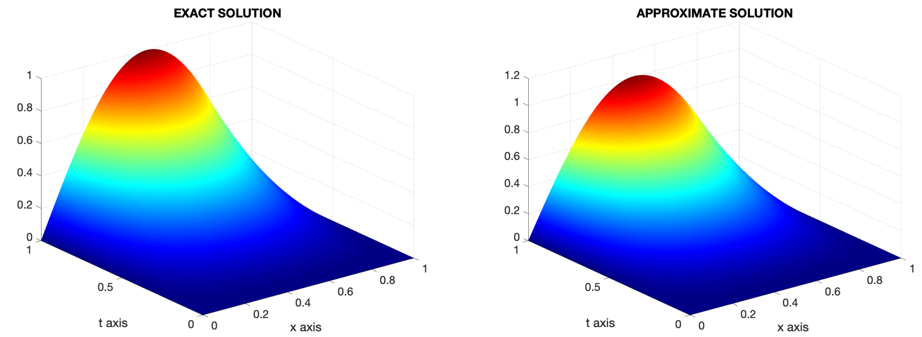

Fig. 3. Left: exact solution. Right: numerical simulation of the problem (9) for N = 80, M = 6400 and $\alpha = 0.5$

Lastly, for $\alpha = 0.99$, the exact and numerical solutions of the problem (9) are presented in Figure 4.

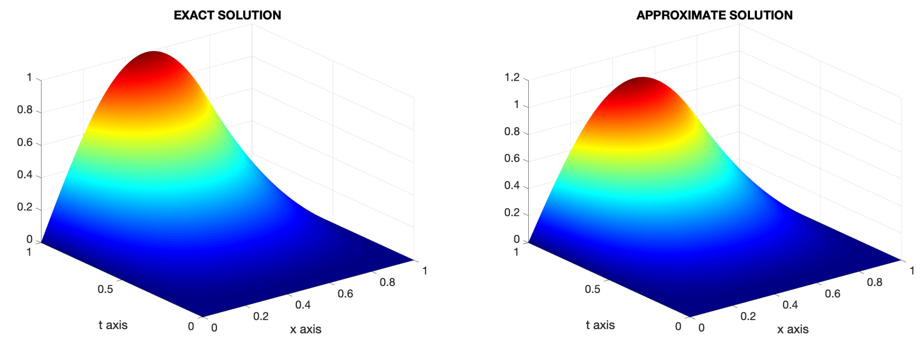

Fig. 4. Left: exact solution. Right: numerical simulation of the problem (9) for N = 80, M = 6400 and $\alpha = 0.99$

From Figures 2-4, we conclude that numerical results are consistent with the theoretical results and the constructed difference scheme is accurate and effective for the considered problem.

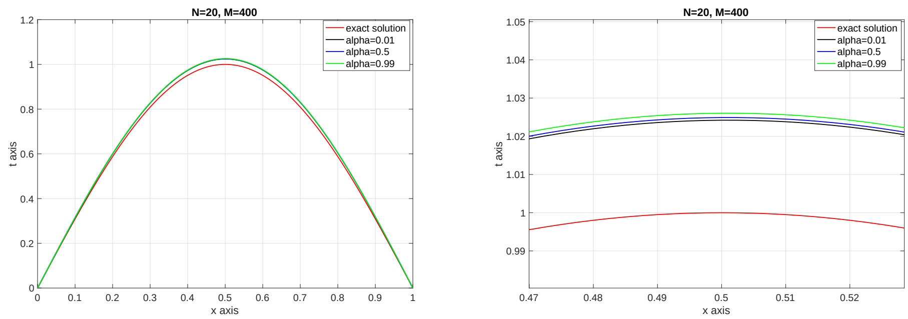

Fig. 5. Left: numerical simulations of the problem (9) for N = 20, M = 400. Right: zoom in near (0.5,1) to see how, as fractional order approaches zero, the approximate and precise solutions have an almost identical match

We also compare the precise and numerical solutions of (9) in Figure 5 by varying α between 0 and 1 for the same grid number.

3.2. Example 2

We give another example to show the accuracy of the proposed method. We consider the following problem

$$\begin{cases} u_{\xi\xi}(\xi,\nu)+u_{\xi}(\xi,\nu)+u(\xi,\nu)+\int_0^{\xi}\gamma(s)D_s^{\alpha}u(s,\nu)ds = u_{\xi\nu\nu}(\xi,\nu)+u_{\nu\nu}(\xi,\nu)+f(\xi,\nu), \\ f(\xi,\nu)=(2+2\xi+\xi^2+\dfrac{\xi^2}{\Gamma(3-\alpha)})(\nu-\nu^2)+4\xi+2\xi^2, 0<\nu<1, 0<\xi<1, \\ u(0,\nu)=u_{\xi}(0,\nu)=0, \quad 0\le\nu\le 1, \\ u(\xi,0)=u(\xi,1)=0, \quad 0\le\xi\le 1, \\ 0<\alpha\le 1. \end{cases} \tag{12}$$

The exact solution of this problem is $u(\xi,\nu)=\xi^2(\nu-\nu^2)$. Then, we give the error analysis graph of the problem (12) in Figure 6 while changing time. In addition to that, we present the following error table for various values of α to help illustrate the error computations.

Table 2. Error analysis

$\tau=\frac{1}{N}, h=\frac{1}{M}, h=\tau^2$					
The example (12).					
N, M	N = 20, M = 20^2	N = 40, M = 40^2	N = 80, M = 80^2	N = 100, M = 100^2	N = 150, M = 150^2
$\alpha=0.01$	0.0060	0.0032	0.0016	0.0013	0.0009
$\alpha=0.5$	0.0062	0.0033	0.0017	0.0013	0.0009
$\alpha=0.99$	0.0065	0.0034	0.0017	0.0014	0.0010

We then present precise and approximate solutions for each α to demonstrate how similar the solutions are.

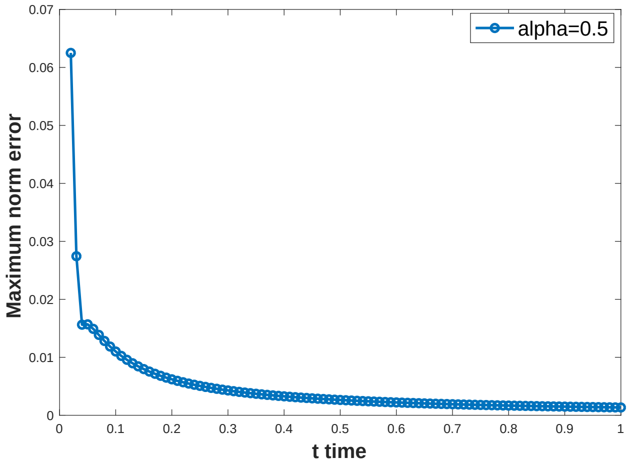

Fig. 6. Result of maximum norm errors of the difference scheme (12) for $\alpha = 0.5$.
As we vary the time values, it can be observed that errors keep decreasing

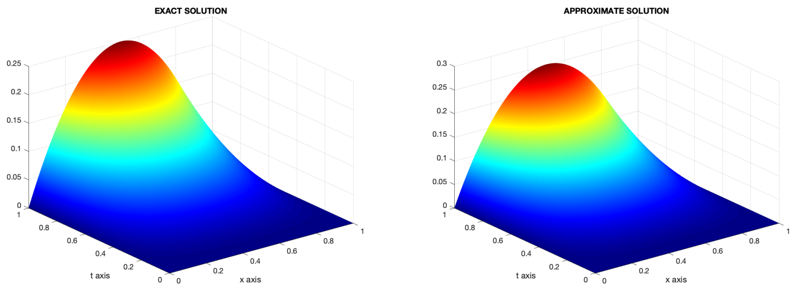

Fig. 7. Left: exact solution. Right: numerical simulation of the problem (12) for N = 80, M = 6400 and $\alpha = 0.01$

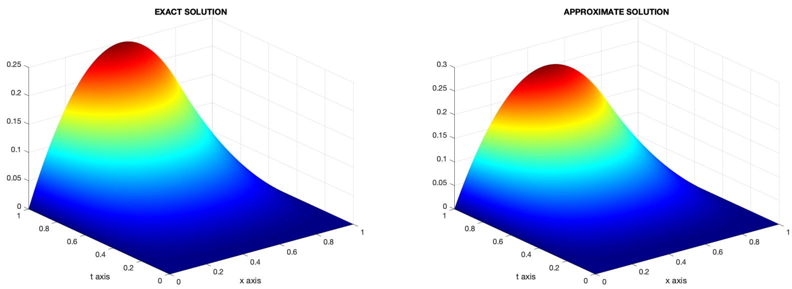

Fig. 8. Left: exact solution. Right: numerical simulation of the problem (12) for N = 80, M = 6400 and $\alpha = 0.5$

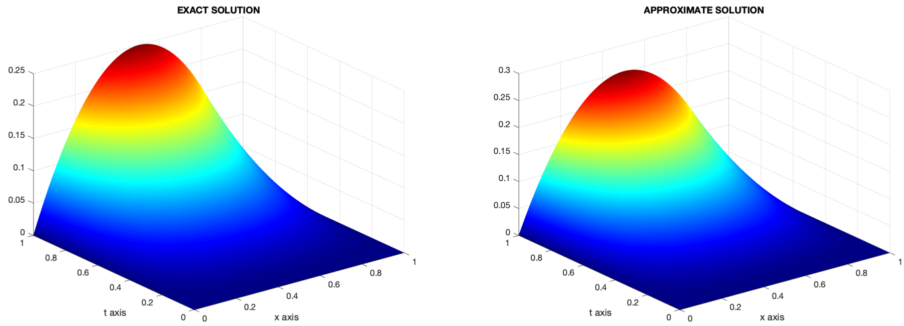

Fig. 9. Left: exact solution. Right: numerical simulation of the problem (12) for N = 80, M = 6400 and $\alpha = 0.99$

4. Conclusion

In the present work, we have studied an initial-boundary value problem for the fractional order pseudo telegraph integro-differential equation. Stability estimates of the constructed difference scheme for the problem were presented.Numerical solutions for different fractional values were computed and plotted. Outputs show that the maximum norm error is decreasing while the grid points are increasing. We have provided numerical solutions of three different fractional values in order to demonstrate the efficiency and high accuracy of the difference scheme.

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