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MATHEMATICAL ANALYSIS AND NUMERICAL SIMULATION OF TIME-FRACTIONAL HOST-PARASITOID SYSTEM WITH CAPUTO OPERATOR

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Abstract. In this paper, a time-fractional host-parasitoid dynamics which is regarded as a new variant of the novel Nicholson-Bailey model is considered in the sense of the Caputo operator. The model equation is examined for linear stability analysis to guide in ensuring the best choice of parameters when simulating the full dynamical system. Furthermore, this work provides a suitable numerical technique for the approximation of the Caputo fractional derivative with order $\rho \in (0,1]$. To explore the dynamic richness of the model, numerical results are provided for different values of α.

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1. Introduction

In a natural sense, mathematical models often help in the understanding and design of some notable problems in biology, medicine, and life sciences. These mathematical models can appear in many forms namely; the dynamical system, differential equation, and difference equation, among several others [1–5]. It is well-established that the sequence of parasitism and density dependence in the host-life cycle can have a important impact on population dynamics which models the host and parasitoid interactions. Such an effect may lead to significant implications for biological control. For instance, May et al. [6] proposed a three component host-parasitoid model whose formulation was based on the density dependence and timing of parasitism.

In recent years, the concept fractional differential equations have gained significant importance due to their wide-range of applications. Fractional differential equations arise more often in various research areas with engineering applications. Different physical scenarios in the fields of viscoelasticity, diffusion procedures, relaxation vibrations, electro-chemistry, electro-magnetics, among several others are successfully modeled by differential equations [7]. What's more, several problems in biology, finance, physics, chemistry, and applied sciences are mathematically modeled by systems of ordinary and fractional differential equations [7–10]. Based on the successful application of the fractional order derivative to model a number real-life phenomena [11–15], we are motivated to consider a coupled two-component differential system of equations

$$\begin{aligned} {}_0\mathscr{D}_t^{\rho} u(t) &= f(t,u(t),v(t)), \quad t \in (0,T) \\ {}_0\mathscr{D}_t^{\rho} v(t) &= g(t,u(t),v(t)), \quad u(0)=u_0,\ v(0)=v_0, \end{aligned} \tag{1}$$

where ${}_0\mathscr{D}_t^{\rho}$ is the fractional derivative of order $\rho > 0$, $f(\cdot)$ and $g(\cdot)$ are local reactions of u and v, respectively at time t.

Over the years, formulation of numerical methods for the solution of integer-order differential equations has been an active area of study in computational sciences. Fractional calculus, which is a natural extension of the integer-order calculus to non-integer-oder differentiation and integration, has been applied to model a large number of applied and physical phenomena [7,16–19]. There is a critical difference in the behavior of integer-order derivatives and fractional-order cases, that is, fractional-order derivatives are non-local in nature. As a result, new numerical challenges have been posed when deriving the numerical methods to handle this type of differential equation. However, over the years in literature, many authors have suggested different numerical techniques for fractional differential equations. Some of these techniques include: variational iteration, Adomain decomposition method, homotopy perturbation, homotopy analysis, fractional differential transform, spectral collocation methods, see [20, 21] and references therein. Many of the proposed schemes suffer instabilities and lack better accuracy. In this work, a simple and easy to adapt solution method is introduced.

The remainder of this paper is arranged as follows. The numerical method for solving fractional differential equations is introduced in Section 2. In Section 3, the main model as well as its stability analysis is examined to guarantee the correct choice of parameters. Simulation experiments for different instances of fractional order are performed in Section 4 to explore the dynamic behavior of the model. A conclusion is reached with the last section.

2. Numerical approximation of Caputo operator

In a compact form, we consider a general fractional initial value problem

$$\begin{aligned} &{}_0^C\mathscr{D}_t^{\rho} u(t) = f(t,u(t)), \quad t \in (0,T], \\ &u(0) = u_0, \end{aligned} \tag{2}$$

where u_0 and T are positive constants, f is known reaction or local kinetics, and ${}_0^C\mathscr{D}_t^{\rho}u(t)$ is the ρth-order Caputo fractional derivative in $(0,T]$ with $\rho \in (0,1]$ defined as

$$ {}_0^C\mathscr{D}_t^{\rho}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(\xi)}{(t-\xi)^{\rho}}d\rho, \tag{3}$$

with $\Gamma(\cdot)$ denoting the Gamma function. It has been shown that when (2) is solved, it leads to the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\rho)}\int_0^t (t-\xi)^{\rho-1} f(\xi,u(\xi))d\xi, \ \ \rho \in (0,1). \tag{4}$$

Many numerical schemes have been proposed for the solution of (4), among which are the Euler's method, Adams-Bashforth method, and the block by block method, see [7,22,23].

To approximate (4), we require dividing $(0,T]$ into equal M sub-intervals with mesh points $t_k = kh$ for $k = 0,1,2,\ldots,M$, where $h = T/M$. Therefore, we can write (4) as

$$\begin{aligned} u(t_k) &= u_0 + \frac{1}{\Gamma(\rho)}\int_0^{t_k} (t_k-\xi)^{\rho-1} f(\xi,u(\xi))d\xi \\ &= u_0 + \frac{1}{\Gamma(\rho)}\int_0^{kh} (kh-\xi)^{\rho-1} f(\xi,u(\xi))d\xi \\ &= u_0 + \frac{1}{\Gamma(\rho)}\sum_{s=1}^{k}\int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} f(\xi,u(\xi))d\xi. \end{aligned} \tag{5}$$

To approximate the integral on the right hand side of (5), we assume that the function $f(t,u(t))$ is continuous and differentiable with respect to u and t. By applying Taylor's theorem to $f(\xi,u(\xi))$ at ξ_{sk}, for $s = 1,2,\ldots,k$, one obtains

$$f(\xi,u(\xi)) = f(\xi_{sk},u(\xi_{sk})) + \Phi_{sk}(\xi-\xi_{sk}) + \alpha_{sk}(\xi-\xi_{sk})^2, \tag{6}$$

where α_{sk} denotes a constant which represents the second derivative of f between ξ_{sk} and ξ,

$$\Phi_{sk} = f_{\xi}(\xi_{sk},u(\xi_{sk})) + f_u(\xi_{sk},u(\xi_{sk}))u'_{\xi}(\xi_{sk}).$$

Next, we substitute (6) in (5) for $s = 1, 2, \ldots, k$, so that

$$
\begin{aligned}
\frac{1}{\Gamma(\rho)} & \int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} f(\xi, u(\xi)) d\xi \\
&= \frac{1}{\Gamma(\rho)} \int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} \{ f(\xi_{sk}, u(\xi_{sk})) + \Phi_{sk}(\xi - \xi_{sk}) \} d\xi + \vartheta_{sk} \\
&= \frac{1}{\Gamma(\rho)} f(\xi_{sk}, u(\xi_{sk})) \left\{ \frac{(kh-(s-1)h)^{\rho}}{\rho} - \frac{(kh-sh)^{\rho}}{\rho} \right\} \\
&\quad + \frac{\Phi_{sk}}{\Gamma(\rho)} \int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} (\xi - \xi_{sk}) d\xi + \vartheta_{sk} \\
&= \frac{h^{\rho}}{\Gamma(\rho+1)} f(\xi, u(\xi)) \{ (k-s+1)^{\rho} - (k-s)^{\rho} \} \\
&\quad + \frac{\Phi_{sk}}{\Gamma(\rho)} \int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} (\xi - \xi_{sk}) d\xi + \vartheta_{sk},
\end{aligned}
\tag{7}
$$

where

$$
\vartheta_{sk} = \frac{1}{\Gamma(\rho)} \int_{(s-1)h}^{sh} (kh-\xi)^{\rho-1} \alpha_{sk} (\xi - \xi_{sk})^2 d\xi.
$$

In the present case, the choice of ξ_{sk} is chosen as

$$
\xi_{sk} = \frac{h\{(k-s+1)^{\rho+1} - (k-s)^{\rho+1}\} + (\rho+1)[(k-s+1)^{\rho}(s-1) - (k-s)^{\rho} s]}{(\rho+1)[(k-s+1)^{\rho} - (k-s)^{\rho}]} \tag{8}
$$

so as to ensure that the truncation error in (7) is just ϑ_{sk}.

3. Mathematical analysis of the main model

Owing to the achievement and successful record of fractional order models over existing classical cases [7,16–18], one is motivated to consider the variant case of the host-parasitoid model in fractional form

$$
\begin{aligned}
&{}^{C}\mathscr{D}_t^{\rho} u(t) = f(u,v) = \frac{\phi u}{1 + \tau u e^{-\varphi v}} e^{-\varphi v} + \psi, \\
&{}^{C}\mathscr{D}_t^{\rho} v(t) = g(u,v) = \beta u (1 - e^{-\varphi v}), \\
&u_0 > 0, v_0 > 0,
\end{aligned}
\tag{9}
$$

where ${}^{C}\mathscr{D}_t^{\rho}$ is the Caputo fractional derivative as defined by (3) above, $u(t)$ denotes the population density of host and time t, while the population of the parasitoid class at time t is given by $v(t)$. The model equation is designed in a way that an individual parasitoid species must look for a host in which to deposit its eggs to maintain continuous reproduction, and it was assumed that the parasitism first occurs before density dependence. The number of young births that a host can produce for parasitoid class is represented by β, φ is the searching efficiency, and the response $e^{-\varphi v}$ [19] stands for the probability that host-species will escape from being parasitized when the parasitoid class is of size V. In the absence of parasitoid species, the host class is modeled by $\phi u/(1+\tau u)$ called Beverton-Holt equation, where ϕ and τ are positive parameters.

In what follows, we examine model (9) for steady states and stability analysis. To achieve this, we set $f(u,t) = g(u,t) = 0\,{}^{C}\mathscr{D}_t^{\rho} u(t) = u^*$ and ${}^{C}\mathscr{D}_t^{\rho} v(t) = v^*$, so that

$$\begin{aligned} u^* &= \frac{\phi u}{1+\tau u e^{-\varphi v}} e^{-\varphi v} + \psi, \\ v^* &= \beta u (1 - e^{-\varphi v}). \end{aligned} \tag{10}$$

It is not difficult to see that model (9) has no steady state at point $(0,0)$ which corresponds to a total washout of species, since $\psi \in (1,\infty)$. This state is not biologically meaningful. Hence we consider the point $(u^*,0)$ which indicates the existence of only the host population. Our interest is in the nontrivial case $(\bar{u},\bar{v})$ which guarantees the existence of both species. This state is demonstrated in the following result.

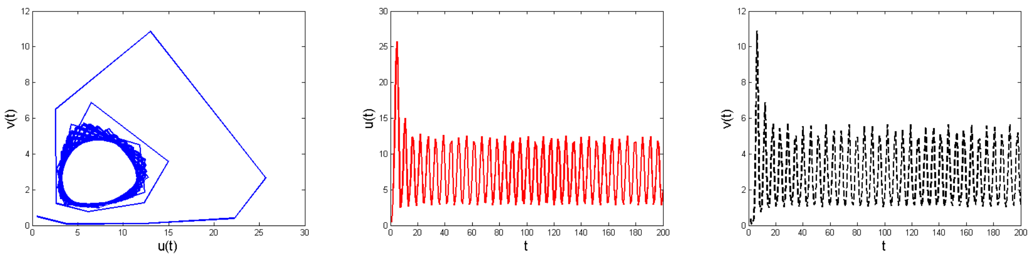

Fig. 1. Solution of model (9) with $\rho = 0.88$

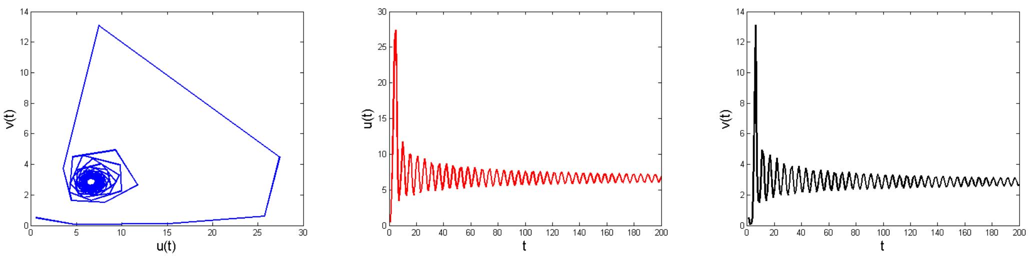

Fig. 2. Numerical solution of model (9) at $\rho = 0.63$ with $\phi = 3.7, \varphi = 0.7, \psi = 3.5, \beta = 0.5, \tau = 0.095$

Theorem 1 The dynamic system (9) has equilibrium state $(\bar{u},\bar{v})$, if it satisfies the following condition

$$\psi < \bar{u} < \frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau}.$$

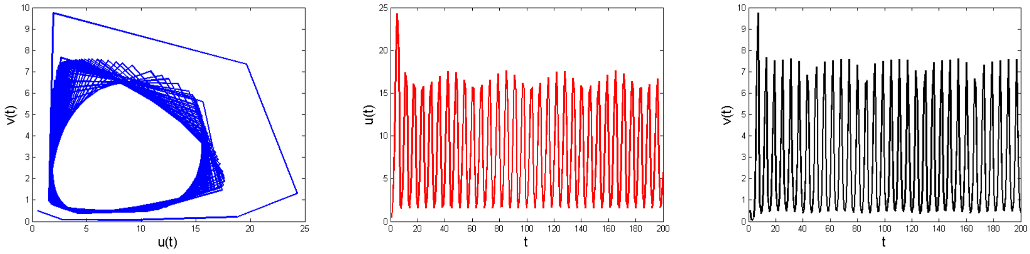

Fig. 3. Numerical solution of model (9) at $\rho = 0.70$ with $\phi = 3.7, \varphi = 0.7, \psi = 1.5, \beta = 0.5, \tau = 0.1$

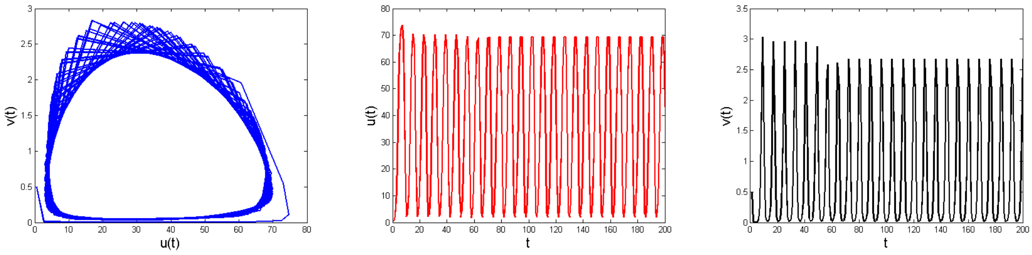

Fig. 4. Numerical solution of model (9) at $\rho = 0.97$ with $\phi = 4.7, \varphi = 1.4, \psi = 1.5, \beta = 0.05, \tau = 0.05$

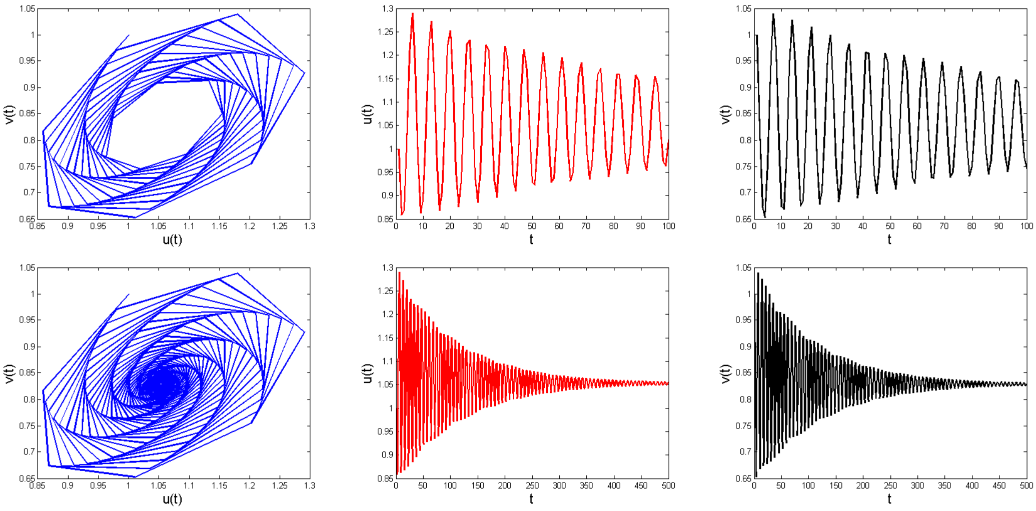

Fig. 5. Numerical solution of model (20) at $\beta = 4.7$ and $\alpha = 1.7$ with $\rho = 0.7$. The upper and lower rows correspond to $t = 100$ and $t = 500$, respectively

PROOF With $\bar{u} \neq 0$ and $\bar{v} \neq 0$ in mind, one gets

$$\bar{u}(1+\tau\bar{u}e^{-\varphi\bar{v}}) = \phi\bar{u}e^{-\varphi\bar{v}} + \psi(1+\tau\bar{u}e^{-\varphi\bar{v}})$$

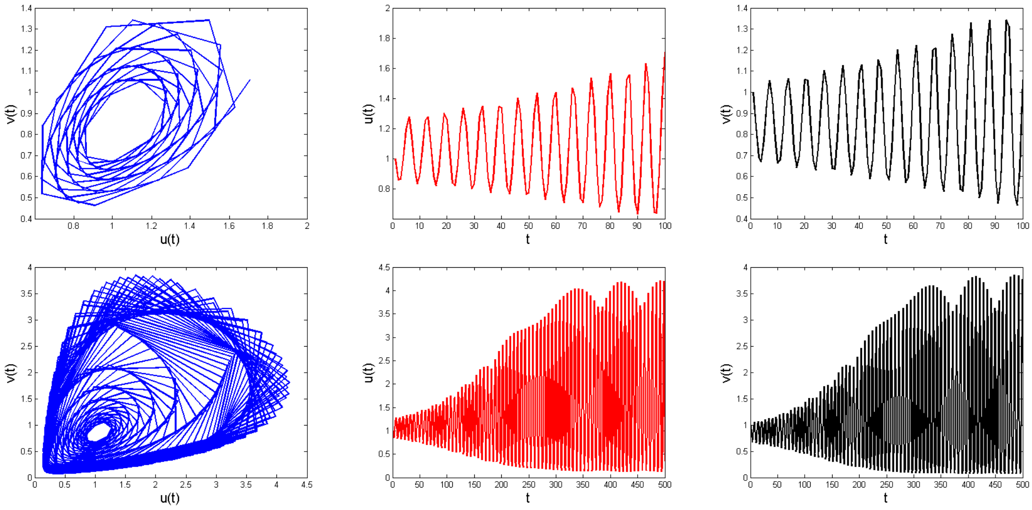

Fig. 6. Numerical solution of model (20) at $\beta = 5.2$ and $\alpha = 1.8$ with $\rho = 0.81$. The upper and lower rows correspond to $t = 100$ and $t = 500$, respectively

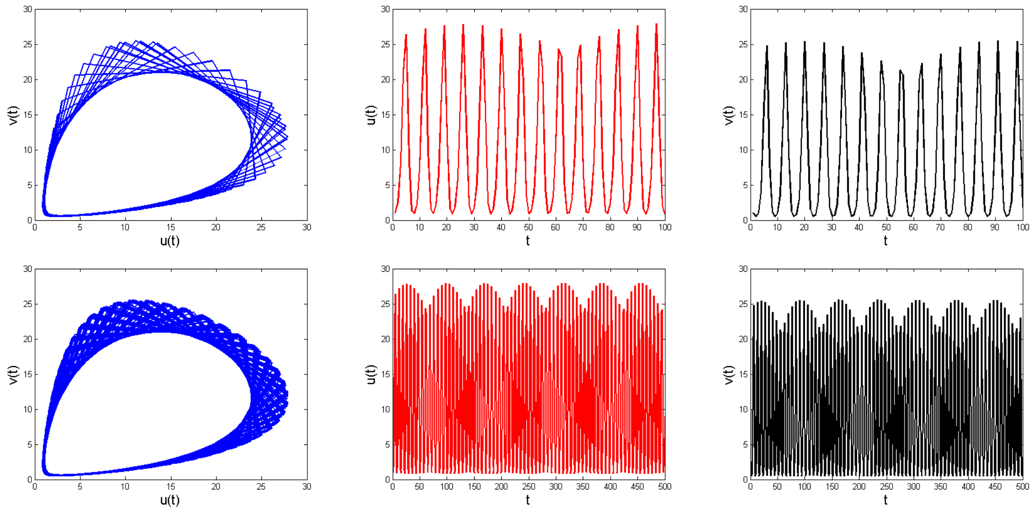

Fig. 7. Numerical solution of model (20) at $\beta = 5.2$ and $\alpha = 0.7$ with $\rho = 0.95$. The upper and lower rows correspond to $t = 100$ and $t = 500$, respectively

which implies that

$$e^{-\varphi\bar{v}} = \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}} \tag{11}$$

and

$$\bar{v} = -\frac{1}{\varphi} \ln \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}. \tag{12}$$

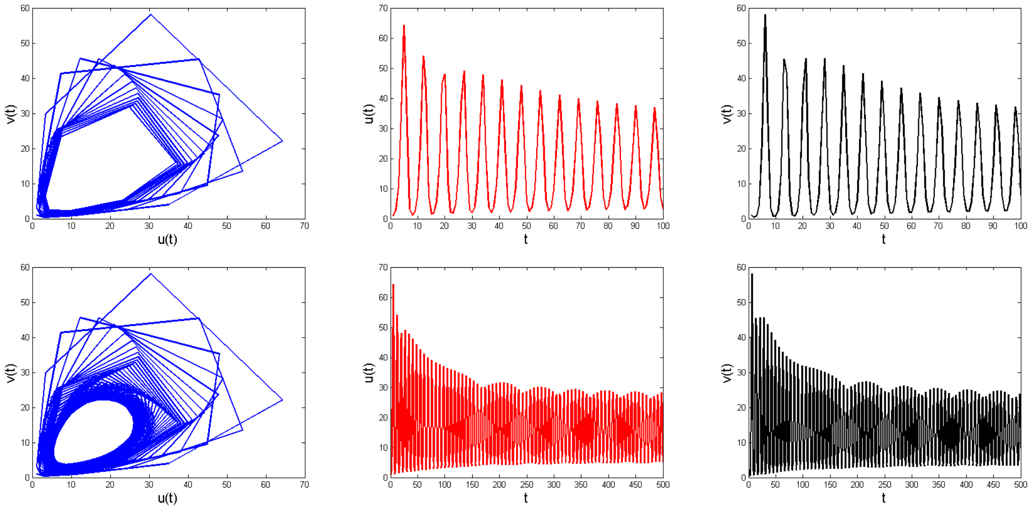

Fig. 8. Numerical solution of model (20) at $\beta = 4.99$ and $\alpha = 0.5$ with $\rho = 1.00$. The upper and lower rows correspond to $t = 100$ and $t = 500$, respectively

If the condition

$$0 < \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}} < 1 \tag{13}$$

holds, then space $\bar{v} > 0$. By combining (11) with the second equation in (10), yields

$$\bar{v} = \beta\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right].$$

By substituting for $\bar{v}$ in (10), gives

$$\bar{u} = \frac{\phi\bar{u}}{1 + \tau\bar{u}\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u}-\psi}{-\tau\bar{u}^2+(\phi+\psi\tau)\bar{u}}\right]\right)}\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right]\right) + \psi$$

which means

$$\begin{aligned}
&\bar{u}\left(1 + \tau\bar{u}\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right]\right)\right) \\
&= \phi\bar{u}\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right]\right) + \psi\left(1 + \tau\bar{u}\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right]\right)\right)
\end{aligned}$$

which implies that

$$\phi = \left(1 - \frac{\psi}{\bar{u}}\right)\left\{\exp\left(-\beta\varphi\bar{u}\left[1 - \frac{\bar{u} - \psi}{-\tau\bar{u}^2 + (\phi + \psi\tau)\bar{u}}\right]\right) + \tau\bar{u}\right\} \tag{14}$$

By simplifying (13), we have

$$\psi < \bar{u} < \frac{\phi + \psi\tau}{\tau} \quad \bar{u} < \frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau}. \tag{15}$$

By comparing $\frac{\phi + \psi\tau}{\tau}$ and $\frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau}$, assuming that

$$\frac{\phi + \psi\tau}{\tau} > \frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau} \tag{16}$$

is true. So if this process holds, we get

$$1 + \phi + \psi\tau > \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau},$$

on squaring both sides we have

$$(1 + \phi + \psi\tau)^2 > (1-\phi-\psi\tau)^2 + 4\psi\tau, \Rightarrow 4\tau > 0.$$

Since $\tau > 0$ is true, we require showing that

$$\psi < \frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau}, \tag{17}$$

from which we have the following result

$$\psi\tau + 1 - \phi < \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}$$

which on squaring both sides implies that

$$(\psi\tau + 1 - \phi)^2 < (1-\phi-\psi\tau)^2 + 4\psi\tau \Rightarrow 0 < 4\psi\tau.$$

Since $\tau > 0$ and $\psi > 0$, we can deduce that the assumptions are satisfied. Consequently, (17) holds. Then we have

$$\psi\bar{u} < \frac{-(1-\phi-\psi\tau) + \sqrt{(1-\phi-\psi\tau)^2 + 4\psi\tau}}{2\tau}, \tag{18}$$

and equations (15), (16) and (17), complete the proof. ∎

Next, we examine the stability of system (9) at point $\bar{E} = (\bar{u}, \bar{v})$. The corresponding Jacobian or community matrix at $\bar{E}$ is given as

$$A = \begin{pmatrix} \frac{\phi \exp(-\varphi\bar{v})}{(1 + \tau\bar{u}\exp(-\varphi\bar{v}))^2} & \frac{\varphi\phi\bar{u}\exp(-\varphi\bar{v})}{(1 + \tau\bar{u}\exp(-\varphi\bar{v}))^2} \\ \beta(1 - \exp(-\varphi\bar{v})) & \beta\varphi\bar{u}\exp(-\varphi\bar{v}) \end{pmatrix}_{(\bar{u},\bar{v})} \tag{19}$$

where the determinant and trace of matrix A are respectively given as

$$\det(A) = \frac{\beta\varphi\phi\bar{u}\exp(-\varphi\bar{v})}{(1+\tau\bar{u}\exp(-\varphi\bar{v}))^2}$$

and

$$\mathrm{tr}(A) = \frac{\phi\exp(-\varphi\bar{v})}{(1+\tau\bar{u}\exp(-\varphi\bar{v}))^2} + \beta\varphi\bar{u}\exp(-\varphi\bar{v}).$$

The point $\bar{E} = (\bar{u},\bar{v})$ is said to be local stable if

$$\det(A) < 2+\mathrm{tr}(A) < 1,$$

and by virtue of the above, the local stability condition for interior state $\bar{E} = (\bar{u},\bar{v})$ becomes

$$\frac{\beta\varphi\phi\bar{u}\exp(-\varphi\bar{v})}{(1+\tau\bar{u}\exp(-\varphi\bar{v}))^2} < 1 \ \text{ and } \ \frac{\phi\exp(-\varphi\bar{v})}{(1+\tau\bar{u}\exp(-\varphi\bar{v}))^2} + \beta\varphi\bar{u}\exp(-\varphi\bar{v}) < 1.$$

4. Numerical simulation and results

In this section, we explore the dynamic richness of model (9) for parameters $\phi = 3.7, \varphi = 0.7, \psi = 2.5, \beta = 0.5, \tau = 0.1$, for different instances of fractional order $\rho \in (0,1)$. We utilized the initial condition $(u_0,v_0) = (0.5,0.5)$ as shown in Figures 1-4. Clearly, one can deduce that regardless of the value of ρ chosen, both the host and parasitoid species oscillate in phase. We also observed that both species coexist and are permanent.

The numerical technique suggested in this paper is also extended to solve the fractional Nicholson-Bailey host-parasitoid model in the Caputo sense [24, 25]

$${}^C\mathscr{D}_t^{\rho}u(t) = f(u,v) = \beta u\exp(-\alpha\sqrt{v}),\ {}^C\mathscr{D}_t^{\rho}u(t) = g(u,v) = u(1-\exp(-\alpha\sqrt{v})), \tag{20}$$

where β represents the number of offspring of an unparasitized host surviving to the next year, $\exp(-\alpha\sqrt{v})$ denotes the probability that a host escapes parasitism, α is the proportionality parameter. Model (20) was first developed and named after Alexander John Nicholson and Victor Albert Bailey in the mid 1930s to describe the population dynamics of a coupled host-parasitoid system [24].

Next, we the numerical experiment of model (20) for different instances of ρ and parameters β and α to obtain the dynamic behaviors in Figures 5 to 8. Different dynamical behaviors (both chaotic and spatiotemporal oscillations) were obtained. See Figures 5-8 with their captions for details. We can also observe the effect of simulation time on the dynamics.

5. Conclusion

Mathematical analysis and numerical simulation of the host-parasitoid model in the sense of the Caputo operator have been investigated. The numerical method designed for the approximation of the fractional derivative is easy to implement and inexpensive to compute. The linear stability of the main model is examined in an attempt to capture the correct choice of parameters during simulation experiments. Numerical results obtained for different parameter values show that both host and parasitoid populations oscillate in a phase. The mathematical idea suggested in this work can be extended to time-fractional reaction-diffusion problems in engineering and applied sciences.

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