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SERIES SOLUTIONS TO HIGHER-ORDER PARAMETRIC BOUNDARY VALUE PROBLEMS

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Abstract. Solving boundary value problems with parameters is challenging. Based on the homotopy analysis method, explicit formulas for the approximate solutions to a class of higher-order parametric boundary value problems are obtained. These explicit formulas give more insight into the solution structures of the given problems. The effectiveness of this approach is demonstrated by solving two specific parametric boundary value problems.

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1. Introduction

The higher-order boundary value problem is an active research field; see [1–3] and the references therein. Consider the fourth-order parametric boundary value problem:

$$u^{(4)}(x) = (1+\gamma)u''(x) - \gamma u(x) + \frac{1}{2}\gamma x^2 - 1, \tag{1}$$

$$u(0) = 1,\ u'(0) = 1,\ u(1) = \frac{3}{2} + \sinh(1),\ u'(1) = 1 + \cosh(1), \tag{2}$$

and the sixth-order parametric boundary value problem:

$$u^{(6)}(x) = (1+\gamma)\,u^{(4)}(x) - \gamma u''(x) + \gamma x, \tag{3}$$

$$u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 0,$$

$$u(1) = \frac{7}{6} + \sinh(1),\ u'(1) = \frac{1}{2} + \cosh(1),\ u''(1) = 1 + \sinh(1). \tag{4}$$

The general solutions to these problems depend upon the parameter γ, but the specific boundary conditions cause the coefficients of the terms containing γ to be identically zero. Thus, the exact solutions for the full boundary-value problems are

$$u_{exa4}(x) = 1 + \frac{1}{2}x^2 + \sinh(x) \quad \text{and} \tag{5}$$

$$u_{exa6}(x) = 1 + \frac{1}{6}x^3 + \sinh(x) \tag{6}$$

respectively.

Although the parameter γ is not visible in the exact solutions, it strongly influences the solution process, and causes the task of correctly identifying the relevant coefficients to be a difficult challenge.

Problems (1)-(2) and (3)-(4) were investigated by applying the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) [4, 5].

For each of the two problems, the two methods obtained the same series solutions. However, the solutions found were in good agreement with the exact solutions only for small values of γ. For large values of γ, these methods did not give accurate series solutions. For example, when $\gamma = 100$, the relative error of the nth-order approximation to the fourth-order problem (1)-(2) given by the ADM increased exponentially as n increased, as shown in Table 1, in which $\alpha E\beta$ stands for $\alpha \times 10^{\beta}$ [6]. It was justified in [7] that the HPM or the classical ADM, being a series method, fails to converge to the true solution.

Table 1. Relative errors of the ADM solutions when $\gamma = 100$

x	5th order	10th order	15th order	20th order
0.1	0.8	1.3E2	2.1E4	3.3E6
0.2	2.7	4.3E2	6.8E4	1.1E7
0.3	4.6	7.3E2	1.2E5	1.8E7
0.4	5.8	9.1E2	1.4E5	2.3E7
0.5	5.8	9.1E2	1.4E5	2.3E7
0.6	4.8	7.5E2	1.2E5	1.9E7
0.7	3.1	4.9E2	7.8E4	1.2E7
0.8	1.5	2.4E2	3.8E4	5.9E6
0.9	0.3	6.0E1	9.5E4	1.6E6

It is expected that the successful solutions can only be achieved by imbedding a convergence parameter in such classical approaches to adjust and control the convergence region and the convergence rate of the resulting series solution; see [8, 9].

The homotopy analysis method (HAM) [10–13] is a popular analytic approach for seeking series solutions to differential equations and related problems. Based on the HAM, accurate series solutions to the fourth-order problem (1)-(2) and the sixth--order problem (3)-(4) for any large values of γ were obtained [6, 14]. Furthermore, some numerical analysis was presented for the relationship between the parameter γ and the convergence-control parameter c_0 for both problems by means of the rational interpolation technique [15].

Based on the optimal variational iteration method, accurate series solutions to the fourth-order problem (1)-(2) for any large values of γ were obtained [3]. Moreover, some numerical analysis was also presented by means of the h-curve technique.

Different from the contributions above, for each of the two problems, an explicit formula for the resulting series solutions is established in this paper, and the relationship between γ and c_0 is revealed which states that the product of γ and c_0 converges to a constant as $\gamma \to +\infty$. Based on the explicit formula and the relationship between γ and c_0, accurate series solutions are obtained no matter how large the value of γ is, by choosing a proper value of c_0, and some interesting phenomena on the resulting solutions can also be explained. At the end of this paper, an explicit solution formula for the general higher-order linear boundary value problem with one parameter is also established.

2. Solution to the fourth-order problem

As usual, one constructs the zeroth-order deformation equation

$$(1-p)\mathscr{L}[\phi(x;p)-u_0(x)] = p\,c_0\,\mathscr{N}[\phi(x;p)] \quad \text{where} \tag{7}$$

$$\mathscr{L}[\phi(x;p)] = \frac{\partial^4\phi(x;p)}{\partial x^4} \tag{8}$$

$$u_0(x) = x^4 - \left(1+\frac{\mathrm{e}}{2}-\frac{3}{2\mathrm{e}}\right)x^3 - \left(\frac{1}{2}-\mathrm{e}+\frac{2}{\mathrm{e}}\right)x^2 + x + 1 \tag{9}$$

$$\mathscr{N}[\phi(x;p)] = \frac{\partial^4\phi\,(x;p)}{\partial x^4} - (1+\gamma)\frac{\partial^2\phi\,(x;p)}{\partial x^2} + \gamma\phi\,(x;p) - \frac{\gamma}{2}x^2 + 1. \tag{10}$$

The initial approximation $u_0(x)$ is chosen as follows: Let $u_0(x) = x^4 + ax^3 + bx^2 + cx + d$. Then the values of a, b, c, d are determined by the boundary conditions (2). Following the standard procedure of the HAM, one obtains an nth-order approximation to the boundary value problem (1)-(2):

$$U_n(x;c_0,\gamma) = \sum_{m=0}^{n} u_m(x). \tag{11}$$

For this approximation, one has the following:

Theorem 1. Let $\eta = c_0\gamma$. Then $U_n(x;c_0,\gamma)$ can be expressed as

$$U_n(x;c_0,\gamma) = \eta^n p_n(x) + \eta^{n-1}p_{n-1}(x) + \cdots + p_0(x) + c_0\,q(x;c_0,\gamma), \tag{12}$$

where $p_n, p_{n-1}, \ldots, p_0$ and q are polynomials over $\mathbb{R}$, and for every term T in q, provided $q \neq 0$, $\deg(T,c_0) \geq \deg(T,\gamma)$.

Proof 1. We first prove that the general term $u_m(x)$ can be expressed as

$$u_m(x) = \eta^m p_{m,m}(x) + \eta^{m-1}p_{m,m-1}(x) + \cdots + p_{m,0}(x) + c_0\,q_m(x;c_0,\gamma), \tag{13}$$

where $p_{m,m}, p_{m,m-1}, \ldots, p_{m,0}$ and q_m are polynomials over $\mathbb{R}$, and $\deg(T,c_0) \geq \deg(T,\gamma)$ for every term T in q_m, in case $q_m \neq 0$.

In view of the initial guess $u_0(x)$, it is obvious that (13) is true for $m=0$.

Now suppose that (13) is true for $m=k$. We need to prove that (13) is also true for $m=k+1$.

The $(k+1)$th-order deformation equation can be expressed as

$$\begin{aligned} u_{k+1}^{(4)}(x) &= \chi_{k+1}u_k^{(4)}(x) + c_0\mathscr{R}_{k+1}(\vec{u}_k(x)) \\ &= F(x) + c_0G(x) + \eta H(x) \end{aligned}$$

$$\begin{aligned} \text{where}\quad F(x) &= \chi_{k+1}u_k^{(4)}(x) \\ G(x) &= u_k^{(4)}(x) - u_k''(x) + (1-\chi_{k+1}) \\ H(x) &= -u_k''(x) + u_k(x) + \frac{\chi_{k+1}-1}{2}x^2 \end{aligned}$$

$$\text{and}\quad \chi_{k+1} = \begin{cases} 0, & \text{if} \quad k=0 \\ 1, & \text{if} \quad k>0. \end{cases}$$

Notice that the derivatives above are with respect to x. So, by the inductive hypothesis, one can express $F(x), G(x)$ and $H(x)$ respectively as follows:

$$\begin{aligned} F(x) &= \eta^k f_k(x) + \eta^{k-1} f_{k-1}(x) + \cdots + f_0(x) + c_0 f(x;c_0,\gamma)\ , \\ G(x) &= \eta^k g_k(x) + \eta^{k-1} g_{k-1}(x) + \cdots + g_0(x) + c_0 g(x;c_0,\gamma)\ , \\ H(x) &= \eta^k h_k(x) + \eta^{k-1} h_{k-1}(x) + \cdots + h_0(x) + c_0 h(x;c_0,\gamma)\ . \end{aligned}$$

Therefore, $u_{k+1}^{(4)}(x)$ can be expressed as

$$u_{k+1}^{(4)}(x) = \eta^{k+1}s_{k+1}(x) + \eta^k s_k(x) + \cdots + s_0(x) + c_0 s(x;c_0,\gamma)\ , \tag{14}$$

where $s_{k+1}, s_k, \ldots, s_0$ and s are polynomials over $\mathbb{R}$, and for every term T in s, provided $s \neq 0$, $\deg(T,c_0) \geq \deg(T,\gamma)$.

Solving (14) with the boundary conditions

$$u_{k+1}(0) = u'_{k+1}(0) = u_{k+1}(1) = u'_{k+1}(1) = 0$$

yields

$$u_{k+1}(x) = \eta^{k+1}p_{k+1,k+1}(x) + \eta^k p_{k+1,k}(x) + \cdots + p_{k+1,0}(x) + c_0\, q_{k+1}(x;c_0,\gamma)\ .$$

Therefore, (13) is true for all nonnegative integers.

Finally, in view of the expression (11) of $U_n(x;c_0,\gamma)$, the theorem follows immediately.

Remark 1. It is seen from Theorem 1 that, as $\gamma \to +\infty$, if the product η of c_0 and γ converges to a constant η_0 then $c_0 = \eta/\gamma \to 0$; consequently, $U_n(x;c_0,\gamma)$ converges to

$$\eta_0^n p_n(x) + \eta_0^{n-1} p_{n-1}(x) + \cdots + \eta_0 p_1(x) + p_0(x) \tag{15}$$

which does not depend on γ and c_0.

It will be shown by minimizing the averaged residual error of the approximation (11) that the product η of c_0 and γ indeed converges to a constant as $\gamma \to +\infty$. Therefore, the HAM can always give accurate series solutions to the problem (1)-(2), no matter how large the value of γ is by choosing a proper value of c_0 as demonstrated in the following.

The averaged residual error of the approximation (11) with M sample points was first proposed by Liao [10] and is defined by

$$E(c_0, \gamma, n) = \frac{1}{M} \sum_{j=1}^{M} \left(\mathscr{N}\left[U_n(x_j; c_0, \gamma)\right]\right)^2, \tag{16}$$

where $x_1, x_2, \ldots, x_M \in [0,1]$ are the sample points and $\mathscr{N}$ is the operator defined in (10). In theory, the approximation (11) converges to the exact solution (5) if and only if the averaged residual error $E(c_0, \gamma, n)$ converges to zero. In practice, one wants to minimize $E(c_0, \gamma, n)$ by finding the optimal value of c_0 which can be determined by solving the equation

$$\frac{\partial E(c_0, \gamma, n)}{\partial c_0} = 0.$$

One takes 20 equally-distributed sample points $\{0.05, 0.10, 0.15, \ldots, 1.00\}$ in the interval $[0,1]$ to calculate (16) for $n = 10$. For different values of γ, by minimizing the averaged residual error, one obtains the corresponding optimal values of c_0 as in Table 2. It is seen that, as $\gamma \to +\infty$, the product of γ and c_0 indeed converges to a constant -58.8!

Table 2. Correspondence between γ and c_0 when $n = 10$

γ	10E1	10E3	10E5	10E7	10E9	10E11
c_0	–8.74E-1	–5.90E-2	–5.88E-4	–5.88E-6	–5.88E-8	–5.88E-10

To see how well the approximate solutions corresponding to the (γ, c_0)-pairs in Table 2 are, one calculates the relative errors for the equally-distributed sample points $\{0.1, 0.2, \ldots, 0.9\}$ in the interval $[0,1]$. The results are shown in Table 3.

Remark 2. It is interesting to notice that the relative error of the approximation for each sample point converges to a fixed number as $\gamma \to +\infty$. The reason is as follows. By Remark 1, as $\gamma \to +\infty$, $U_n(x; c_0, \gamma)$ converges to (15) which does not depend on γ; consequently, the relative error of the approximation for each sample point converges to a fixed number.

It should be pointed out that the limit η_0 of the product η is usually not fixed for different order of approximation (11). For example, when $n = 20$ the limit $\eta_0 = -63.8$ as shown in Table 4, and the corresponding relative errors are calculated in Table 5.

Table 3. Relative errors of the approximation (11) when $n = 10$

x	γ = 10E1	γ = 10E3	γ = 10E5	γ = 10E7	γ = 10E9	γ = 10E11
0.1	1.02E-11	2.08E-4	2.60E-4	2.61E-4	2.61E-4	2.61E-4
0.2	3.26E-11	3.82E-4	2.67E-4	2.66E-4	2.66E-4	2.66E-4
0.3	5.55E-11	5.71E-4	2.89E-4	2.85E-4	2.85E-4	2.85E-4
0.4	6.92E-11	7.04E-4	3.41E-4	3.37E-4	3.37E-4	3.37E-4
0.5	6.92E-11	7.04E-4	3.43E-4	3.39E-4	3.39E-4	3.39E-4
0.6	5.67E-11	5.78E-4	2.80E-4	2.76E-4	2.76E-4	2.76E-4
0.7	3.74E-11	3.85E-4	1.94E-4	1.92E-4	1.92E-4	1.92E-4
0.8	1.80E-11	2.11E-4	1.48E-4	1.46E-4	1.46E-4	1.46E-4
0.9	4.63E-12	9.39E-5	1.17E-4	1.17E-4	1.17E-4	1.17E-4

Table 4. Correspondence between γ and c_0 when $n = 20$

γ	10E1	10E3	10E5	10E7	10E9	10E11
c_0	–9.74E-1	–6.35E-2	–6.38E-4	–6.38E-6	–6.38E-8	–6.38E-10

Table 5. Relative errors of the approximation (11) when $n = 20$

x	γ = 10E1	γ = 10E3	γ = 10E5	γ = 10E7	γ = 10E9	γ = 10E11
0.1	1.46E-14	7.42E-5	9.05E-5	9.10E-5	9.10E-5	9.10E-5
0.2	4.73E-14	1.89E-4	8.32E-5	8.19E-5	8.19E-5	8.19E-5
0.3	8.06E-14	3.15E-4	1.13E-4	1.10E-4	1.10E-4	1.10E-4
0.4	1.01E-13	3.90E-4	1.29E-4	1.25E-4	1.25E-4	1.25E-4
0.5	1.01E-13	3.88E-4	1.24E-4	1.21E-4	1.21E-4	1.21E-4
0.6	8.25E-14	3.20E-4	1.05E-4	1.03E-4	1.03E-4	1.03E-4
0.7	5.43E-14	2.12E-4	7.58E-5	7.40E-5	7.40E-5	7.40E-5
0.8	2.62E-14	1.04E-4	4.58E-5	4.51E-5	4.51E-5	4.51E-5
0.9	6.62E-15	3.36E-5	4.05E-5	4.08E-5	4.08E-5	4.08E-5

One thus concludes that accurate series solutions to the problem (1)-(2) can always be obtained, no matter how large the value of γ is, by choosing a proper value of c_0. The HAM approximation (11) agrees very well with the exact solution (5) as shown in Figure 1.

3. Solution to the sixth-order problem

The solution structure of the sixth-order problem is quite similar to the solution structure of the fourth-order problem above. One first constructs the zeroth-order deformation equation

$$(1-p)\mathscr{L}[\phi(x;p)-u_0(x)] = p\,c_0\,\mathscr{N}[\phi(x;p)] \quad \text{where} \tag{17}$$

$$\mathscr{L}[\phi(x;p)] = \frac{\partial^6 \phi(x;p)}{\partial x^6} \tag{18}$$

$$u_0(x) = x^6 - \frac{\left(19+24\,\mathrm{e}-7\,\mathrm{e}^2\right)x^5}{4\,\mathrm{e}} + \frac{\left(23+22\,\mathrm{e}-9\,\mathrm{e}^2\right)x^4}{2\,\mathrm{e}}$$
$$- \frac{\left(87+82\,\mathrm{e}-39\,\mathrm{e}^2\right)x^3}{12\,\mathrm{e}} + x + 1 \tag{19}$$

$$\mathcal{N}[\phi(x;p)] = \frac{\partial^6\phi(x;p)}{\partial x^6} - (1+\gamma)\frac{\partial^4\phi(x;p)}{\partial x^4} + \gamma\frac{\partial^2\phi(x;p)}{\partial x^2} - \gamma x \tag{20}$$

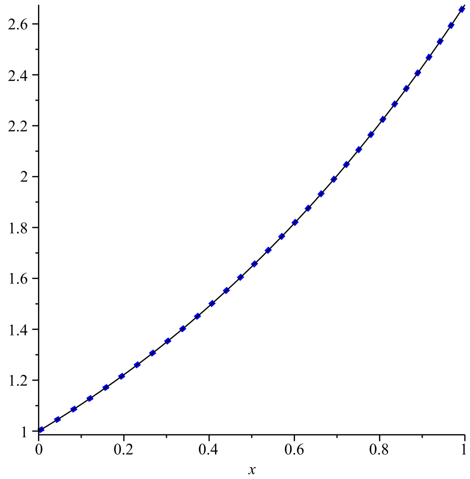

Fig. 1. The 10th-order HAM approximation for $\gamma = 10^9, c_0 = 5.88 \times 10^{-8}$. The solid line: exact solution; the dot line: the HAM approximation

The initial approximation $u_0(x)$ is chosen in a similar way as in Section 2. Again, following the standard procedure of the HAM, one obtains an nth-order approximation to the boundary value problem (3)-(4):

$$V_n(x;c_0,\gamma) = \sum_{m=0}^{n} u_m(x). \tag{21}$$

For this approximation, one has the following:

Theorem 2. Let $\eta = c_0\gamma$. Then $V_n(x;c_0,\gamma)$ can be expressed as

$$V_n(x;c_0,\gamma) = \eta^n f_n(x) + \eta^{n-1} f_{n-1}(x) + \cdots + f_0(x) + c_0\, g(x;c_0,\gamma), \tag{22}$$

where $f_n, f_{n-1}, \ldots, f_0$ and g are polynomials over $\mathbb{R}$, $\deg(T,c_0) \geq \deg(T,\gamma)$ for every term T in g, in case $g \neq 0$.

It is seen that Theorem 1 and Theorem 2 look almost the same, except for the coefficients of the powers of η and c_0. The proof of Theorem 2 is omitted here since it is quite similar to the proof of Theorem 1.

To see if the product η of c_0 and γ converges to a constant as $\gamma \to +\infty$ when minimizing the averaged residual error

$$E(c_0,\gamma,n) = \frac{1}{M}\sum_{j=1}^{M}\left(\mathscr{N}\left[V_n(x_j;c_0,\gamma)\right]\right)^2, \tag{23}$$

one takes 20 equally-distributed sample points $\{0.05, 0.10, 0.15, \ldots, 1.00\}$ in the interval $[0,1]$ to calculate (23) for $n = 10$. For different values of γ, by minimizing the averaged residual error, one obtains the corresponding optimal values of c_0 as in Table 6.

Table 6. Correspondence between γ and c_0 when $n = 10$

γ	10E1	10E3	10E5	10E7	10E9	10E11
c_0	–9.20E-1	–8.98E-2	–1.32E-3	–1.29E-5	–1.29E-7	–1.29E-9

Remark 3. (1) One can see that, as $\gamma \to +\infty$, the product of γ and c_0 indeed converges to a constant $\eta_1 = -129$. It is worth noticing that the constant is different from the constant $\eta_0 = -58.8$ in the fourth-order case above, and the convergence rate to η_1 is slower than the rate to η_0.
(2) In view of Theorem 2, as $\gamma \to +\infty$, $V_n(x;c_0,\gamma)$ thus converges to

$$\eta_1^n f_n(x) + \eta_1^{n-1} f_{n-1}(x) + \cdots + \eta_1 f_1(x) + f_0(x) \tag{24}$$

which does not depend on γ and c_0.
(3) Therefore, one can expect that the relative error of the approximation for each sample point converges to a fixed number as $\gamma \to +\infty$. It is indeed the case as shown in Table 7, but the convergence rate is again slower than the fourth-order case.

Table 7. Relative errors of the approximation (21) when $n = 10$

x	γ = 10E1	γ = 10E3	γ = 10E5	γ = 10E7	γ = 10E9	γ = 10E11
0.1	2.97E-16	1.34E-5	3.12E-5	2.90E-5	2.90E-5	2.90E-5
0.2	6.10E-16	2.47E-5	8.54E-5	6.83E-5	6.81E-5	6.81E-5
0.3	8.74E-16	2.73E-5	1.53E-4	1.10E-4	1.10E-4	1.10E-4
0.4	1.08E-15	2.89E-5	2.08E-4	1.44E-4	1.44E-4	1.44E-4
0.5	1.10E-15	2.82E-5	2.17E-4	1.49E-4	1.48E-4	1.48E-4
0.6	9.18E-16	2.45E-5	1.77E-4	1.22E-4	1.22E-4	1.22E-4
0.7	6.30E-16	1.97E-5	1.10E-4	7.96E-5	7.93E-4	7.93E-5
0.8	3.71E-16	1.50E-5	5.20E-5	4.16E-5	4.15E-5	4.15E-5
0.9	1.52E-16	6.86E-6	1.60E-5	1.49E-5	1.49E-5	1.49E-5

4. Explicit solution formulas for the general case

The techniques used in Sections 2 and 3 can be applied to general parametric linear boundary value problems. Consider an Nth-order linear two-point boundary value problem with one parameter γ

$$L_1[u(x)] + f_1(x) + \gamma(L_2[u(x)] + f_2(x)) = 0, \tag{25}$$

$$u(a) = A_0,\ u'(a) = A_1,\ \ldots,\ u^{(k)}(a) = A_k, \tag{26}$$

$$u(b) = B_0,\ u'(b) = B_1,\ \ldots,\ u^{(N-k-2)}(b) = B_{N-k-2}, \tag{27}$$

where L_1 and L_2 are Nth-order linear differential operators with polynomial coefficients, and f_1 and f_2 are polynomials over $\mathbb{R}$.

As usual, one constructs the zeroth-order deformation equation

$$(1-p)\mathscr{L}[\phi(x;p) - u_0(x)] = p\,c_0\,\mathscr{N}[\phi(x;p)] \quad \text{where} \tag{28}$$

$$\mathscr{L}[\phi(x;p)] = \frac{\partial^N \phi(x;p)}{\partial x^N} \tag{29}$$

$$\mathscr{N}[\phi(x;p)] = L_1[\phi(x;p)] + f_1(x) + \gamma(L_2[\phi(x;p)] + f_2(x)) \tag{30}$$

and $u_0(x)$ is an initial guess satisfying the boundary conditions (26)-(27).

Again, following the standard procedure of the HAM, one obtains an nth-order approximation to the boundary value problem (25)-(27):

$$W_n(x;c_0,\gamma) = \sum_{m=0}^{n} u_m(x). \tag{31}$$

For this approximation, one has the following:

Theorem 3. Let $\eta = c_0\gamma$. Then $W_n(x;c_0,\gamma)$ can be expressed as

$$W_n(x;c_0,\gamma) = \eta^n r_n(x) + \eta^{n-1} r_{n-1}(x) + \cdots + r_0(x) + c_0\,d(x;c_0,\gamma), \tag{32}$$

where $r_n, r_{n-1}, \ldots, r_0$ and d are polynomials over $\mathbb{R}$, and for every term T in d, provided $d \neq 0$, $\deg(T,c_0) \geq \deg(T,\gamma)$.

We omit the proof of Theorem 3 since it is quite similar to the proofs of Theorems 1 and 2. Based on Theorem 3 and the methods used in Sections 2 and 3, one can solve any problems of type (25)-(27).

5. Conclusion

By establishing an explicit formula for the resulting series solution given by the homotopy analysis method, one can gain more insight into the solution structure of the given parametric linear boundary value problem. However, to obtain an explicit solution formula for a parametric nonlinear boundary value problem, substantial work has to be done.

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