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TWO-DIMENSIONAL DISTRIBUTED ORDER CABLE EQUATION WITH NON-SINGULAR KERNEL: A NONSTANDARD IMPLICIT COMPACT FINITE DIFFERENCE APPROACH

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Abstract. In this work, a higher-order nonstandard implicit compact finite difference technique is used to study a two-dimensional nonlinear distributed order Cable problem. The distributed fractional order is defined in the Atangana-Baleanu sense. The key advantage of this strategy is the large stability areas it implicitly has. A particular focus is on examining the stability analysis of the proposed scheme through the application of the Jon Von Neumann approach. We show the effectiveness of the numerical scheme using two numerical examples, and we compare our results with the published literature to check the accuracy of the approach we have presented. The technique is a helpful tool for modeling this model, as demonstrated by the results.

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1. Introduction

Distributed-order fractional derivatives indicate fractional integrated over the order of the differentiation within a given range. The concept of distributed order fractional derivative is expanded by Bagley and Torvik in [1]. There are a lot of researchers who have taken this concept and applied it to some fields [2].

Derivatives of fractional order have numerous definitions. The Caputo, Grünwald--Letinkov, and Riemann-Liouville formulas are the most popular [3]. In recent years, other authors have proposed new definitions of the fractional order derivative. Recently, there are many studies to address several problems related to the localization of the kernel [4] and the singularity of the kernel operators [2].

Atangana and Baleanu introduced a novel fractional order derivative [4] in 2016. It is based on the generalised Mittag-Leffler function as a non-local and nonsingular kernel. The recently developed Atangana-Beleanu derivative [4] has been used to mimic a variety of actual world issues in many fields, as can be seen in [5].

The Cable equation is important for modeling brain dynamics and many other electrophysiology-related fields. Many articles, for example [6–8], have looked into this fundamental biological model. All of the above research publications, on the other hand, focused on the one-dimensional fractional Cable equation. Furthermore, only a few academics have examined the 2D fractional Cable equation [5]. However, it has been demonstrated that, due to the absence of long time memory effects, fractional equations may not be suitable for depicting the diffusion processes in multifractal media. This influences diffusion processes need more than fractional time scales, which is important to do in-depth study on the subject of longer memory, that's why numerous scientists have introduced distributed-order fractional partial differential equations [9].

In the compact technique, the ideal trigonal structure is maintained while approximating second-order derivatives [10–14].

In the modeling and simulation of many real problems it is critical that numerical approaches preserve the positivity of the solutions [11]. It is critical to avoid exaggerated negative values for the solution while constructing positivity-preserving methods ([12]). The nonstandard finite difference (NSFD) schemes are developed [13] by Mickens and has been applied in many areas of science including biological and epidemic models [15]. A fundamental physical aspect of NSFD techniques is the preservation of the steady-state convergence of the solution [12].

The motivation and the fundamental accomplishment of this study is the creation of a precise numerical algorithm for estimating the numerical solutions of the Cable equation with a two-dimensional nonlinear distributed order. The distributed order derivative is defined in this article in the Liouville-Caputo meaning of Atangana--Baleanu-Caputo (ABC). The nonstandard implicit Compact Finite Difference Method (NICFDM) is the proposed technique. By comparing the proposed method to the standard implicit compact finite difference method (ICFDM), it will be shown that the proposed method is more accurate.

The following is how the paper is organised: Section 2 contains definitions for fractional and distributed orders as well as the preliminary Nonstandard Finite Difference Method (NSFDM) definitions. In Section 3, we build NICFDM to solve the distributed order fractional Cable equation in two dimensions. The proposed method's stability study is covered in Section 4. Section 5 presents test cases and numerical simulations to verify our findings. Finally, Section 6 provides the conclusions.

2. Preliminaries

2.1. Nonstandard Finite Difference Method

One advantage of NSFDM, which Mickens proposed in (1980) [12–15], is that it may be effectively utilised to examine numerically how ordinary differential equations (ODEs) and partial differential equations (PDEs) behave. The attributes of the exact solution of the original ODEs or PDEs can be preserved by the NSFDM. If at least one of the following criteria is met, the numerical scheme is referred to as NSFDM [15]:

1. The nonlocal approximation is used.
2. The discretization of the derivative is not traditional and uses a nonnegative function.

There are many advantages of the applied technique, such as its accuracy, affectivity, efficiency, and stability, as we clearfied in the results section, and its simplicity to be applied to the fractional models rather than many different methods.

2.2. Fractional calculus definitions

There are many definitions of fractional derivatives of order $\alpha > 0$ [16] such as Grünwald-Letinkov's definition (GL), Riemann-Liouville's definition (RL), and Caputo's fractional derivative. The RL definition is given as:

$$^{RL}D_t^{\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{(n-\alpha-1)} z(\tau) d\tau, \tag{1}$$

where the value of n cannot be less than α, where n is the first integer, that is, $n-1 < \alpha < n$ and $\Gamma(\cdot)$ is a Gamma function.
The Caputo fractional derivative of $z(t)$ is defined as:

$$^{C}D_t^{\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{(n-\alpha-1)} z^{(n)}(\tau) d\tau. \tag{2}$$

The Atangana-Baleanu fractional derivative in the Caputo sense is defined as [4]:

$$^{ABC}_{a}D_t^{\alpha} z(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t E_{\alpha}\Big(-\alpha \frac{(t-q)^{\alpha}}{(1-\alpha)}\Big) \dot{z}(q) dq, \tag{3}$$

where, $M(\alpha) = 1-\alpha+\dfrac{\alpha}{\Gamma(\alpha)}$ is normalization function.

Definition 1. For $q \neq 0$, $\alpha \in (0,1]$,where $q(\alpha \geq 0)$, and $\int_0^1 q(\alpha) d\alpha = c_0 > 0$, both sides (left and right) of fractional derivatives of distributed orders in the Riemann--Liouville sense are defined, respectively, by [17]:

$$^{RL}_{a}D_t^{q(\alpha)} f(t) = \int_0^1 q(\alpha)\, ^{RL}_{a}D_t^{\alpha} f(t) d\alpha, \quad ^{RL}_{t}D_b^{q(\alpha)} f(t) = \int_0^1 q(\alpha)\, ^{RL}_{t}D_b^{\alpha} f(t) d\alpha. \tag{4}$$

Definition 2. For $q \neq 0$, $\alpha \in (0,1]$, where $q(0 \leq \alpha)$, and $\int_0^1 q(\alpha) d\alpha = c_0 > 0$, both sides (left and right) of fractional derivatives of distributed orders in the Caputo sense are defined, respectively, as follows [18]:

$$ {}_{a}^{C}D_{t}^{q(\alpha)}f(t)=\int_{0}^{1}q(\alpha)\,{}_{a}^{C}D_{t}^{\alpha}f(t)d\alpha,\quad {}_{t}^{C}D_{b}^{q(\alpha)}f(t)=\int_{0}^{1}q(\alpha)\,{}_{t}^{C}D_{b}^{\alpha}f(t)d\alpha. \tag{5} $$

Definition 3. For $q \neq 0$, $\alpha \in (0,1]$, where $q(0 \leq \alpha)$ and $\int_0^1 q(\alpha)d\alpha = c_0 > 0$, both sides (left and right) fractional derivatives of distributed orders in the Atangana--Baleanu-Caputo sense are defined, respectively, by [18]:

$$ {}_{a}^{ABC}D_{t}^{q(\alpha)}f(t)=\int_{0}^{1}q(\alpha)\,{}_{a}^{ABC}D_{t}^{\alpha}f(t)d\alpha,\quad {}_{t}^{ABC}D_{b}^{q(\alpha)}f(t)=\int_{0}^{1}q(\alpha)\,{}_{t}^{ABC}D_{b}^{\alpha}f(t)d\alpha. \tag{6} $$

3. Construction of NICFDM for 2-D Fractional Nonlinear Cable equation

Here, we look at the 2-D fractional Cable equation's initial-boundary value problem, which is typically expressed in the following manner [19]:

$$ \begin{gathered} u_t(x,y,t) = {}_{a}^{ABC}D_t^{1-q(\beta)}\Delta u(x,y,t) - \mu\,{}_{a}^{ABC}D_t^{1-q(\alpha)}u(x,y,t) + f(u,x,y,t), \\ (x,y)\in\Omega, \quad 0<t<T, \quad 0<\alpha,\beta<1, \end{gathered} \tag{7} $$

subject to the initial condition

$$ u(x,y,0) = g_0(x,y), \quad (x,y)\in\Omega, \tag{8} $$

and the Dirichlet boundary conditions

$$ u(0,y,t) = g_1(y,t), \quad u(l,y,t) = g_2(x,y), \quad u(x,0,t) = q_1(x,t), \quad u(x,r,t) = q_2(x,y), $$

where,

$$ \Omega = \{(x,y) | 0 \leq x \leq l, \quad 0 \leq y \leq r\}, $$

To find numerical solutions to the nonlinear Cable problem (7), we create NICFDM in this instance. Let's take into account the grid point's numerical value of u, $(t_{n+1}, x_i, y_j) = ((n+1)\triangle t, i\Delta X, j\Delta y)$ is denoted by $u_{i,j}^{n+1}$. The implicit compact finite differences approximations for the proposed model is given as the following [20]:

$$ (u_{xx})_{i,j}^{n+1} := \left(\frac{\frac{-1}{12}u_{i-2,j}^{n+1} + \frac{4}{3}u_{i-1,j}^{n+1} - \frac{5}{2}u_{i,j}^{n+1} + \frac{4}{3}u_{i+1,j}^{n+1} - \frac{1}{12}u_{i+2,j}^{n+1}}{\Delta X^2} \right) + O((\Delta X)^4), $$

$$ (u_{yy})_{i,j}^{n+1} := \left(\frac{\frac{-1}{12}u_{i,j-2}^{n+1} + \frac{4}{3}u_{i,j-1}^{n+1} - \frac{5}{2}u_{i,j}^{n+1} + \frac{4}{3}u_{i,j+1}^{n+1} - \frac{1}{12}u_{i,j+2}^{n+1}}{(\Delta y)^2} \right) + O((\Delta y)^4), $$

$$(u_t)^n_{i,j} = \frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2(\Delta t)} + O(\Delta t)^2. \tag{9}$$

The nonstandard implicit compact finite differences approximations for the proposed model is given as the following

$$(u_{xx})^{n+1}_{i,j} := \left(\frac{\frac{-1}{12}u^{n+1}_{i-2,j} + \frac{4}{3}u^{n+1}_{i-1,j} - \frac{5}{2}u^{n+1}_{i,j} + \frac{4}{3}u^{n+1}_{i+1,j} - \frac{1}{12}u^{n+1}_{i+2,j}}{\phi(\Delta X)^2} \right) + O((\phi(\Delta X))^4),$$

$$(u_{yy})^{n+1}_{i,j} := \left(\frac{\frac{-1}{12}u^{n+1}_{i,j-2} + \frac{4}{3}u^{n+1}_{i,j-1} - \frac{5}{2}u^{n+1}_{i,j} + \frac{4}{3}u^{n+1}_{i,j+1} - \frac{1}{12}u^{n+1}_{i,j+2}}{(\xi(\Delta y))^2} \right) + O(\xi(\Delta y)^4),$$

$$(u_t)^n_{i,j} = \frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2(\theta(\Delta t))} + O(\theta(\Delta t))^2. \tag{10}$$

$${}^{ABC}_{0}D^{q(\alpha)}_t u(t) = \int_0^1 q(\alpha) {}^{ABC}_{0}D^{\alpha}_t u(t) d\alpha, \tag{11}$$

To approximate the integration $\int_0^1 q(\alpha)d\alpha$, we will use the composite Simpson's rule by letting $\triangle\alpha = \frac{1}{2j}$, and $\alpha_i = i \triangle \alpha$. Then

$$\int_0^1 q(\alpha)d\alpha = \triangle\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) - \frac{(\triangle\alpha)^4}{180} q^4(\xi), \;\; \xi \in [0,1], \tag{12}$$

where,

$$\gamma_i = \begin{cases} \frac{1}{3}, & i = 0, 2j, \\ \frac{2}{3}, & i = 2, 4, ..., 2j-4, 2j-2, \\ \frac{4}{3}, & i = 1, 3, ..., 2j-3, 2j-1. \end{cases}$$

We discretize (11) as follows:

$$\begin{aligned} &\triangle \alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) {}^{ABC}_{0}D^{\alpha_i}_t u(t)|_{t=t^k} - \frac{(\triangle\alpha)^4}{180} \omega^4(\alpha;\xi)|_{\alpha=\xi_k} \\ &= \triangle\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) {}^{ABC}_{0}D^{\alpha_i}_t u(t)|_{t=t^k} + O(\triangle\alpha)^4, \text{ where } \xi_k \in [0,1]. \end{aligned} \tag{13}$$

The discretization of the Atangana-Baleanu fractional operator in the Liouville-Caputo sense is given by the following equations:

$$ {}^{ABC}_{a}D_t^{1-\alpha}u(x,y,t)|_{x_i,y_j,t_{n+1}} = \frac{M(1-\alpha)}{\alpha}\sum_{k=1}^{n} u_{i,j}^{n+1}\int_{(k-1)\Delta t}^{(k)\Delta t} E_{1-\alpha}(-\frac{(1-\alpha)}{\alpha}(t_n-x)^{1-\alpha})dx, $$

$$ {}^{ABC}_{a}D_t^{1-\beta}u_{xx}(x,y,t)|_{x_i,y_j,t_{n+1}} = \frac{M(1-\beta)}{\beta}\left[\sum_{k=1}^{n}\frac{\frac{-1}{12}u_{i-2,j}^{n+1}+\frac{4}{3}u_{i-1,j}^{n+1}-\frac{5}{2}u_{i,j}^{n+1}+\frac{4}{3}u_{i+1,j}^{n+1}}{\phi(\Delta X)^2}\right. $$
$$ \left. -\sum_{k=1}^{n}\frac{\frac{1}{12}u_{i+2,j}^{n+1}}{\phi(\Delta X)^2}\right]\times\int_{(k-1)\Delta t}^{(k)\Delta t} E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-x)^{1-\beta})dx + O((\phi(\Delta X))^4), $$
$$ {}^{ABC}_{a}D_t^{1-\beta}u_{yy}(x,y,t)|_{x_i,y_j,t_{n+1}} = \frac{M(1-\beta)}{\beta}\left[\sum_{k=1}^{n}\frac{\frac{-1}{12}u_{i,j-2}^{n+1}+\frac{4}{3}u_{i,j-1}^{n+1}-\frac{5}{2}u_{i,j}^{n+1}+\frac{4}{3}u_{i,j+1}^{n+1}}{(\xi(\Delta y))^2}\right. $$
$$ \left. -\sum_{k=1}^{n}\frac{\frac{1}{12}u_{i,j+2}^{n+1}}{(\xi(\Delta y))^2}\right]\times\int_{(k-1)\Delta t}^{(k)\Delta t} E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-x)^{1-\beta}dx + O((\xi(\Delta y))^4), \tag{14} $$

where, $\int_{(k-1)\Delta t}^{k\Delta t} E_{1-\alpha}(-\frac{(1-\alpha)}{\alpha}(t_n-x)^{1-\alpha})dx =$

$$ (t_n-t_{k+1})E_{1-\alpha}(-\frac{(1-\alpha)}{\alpha}(t_n-t_{k+1})^{1-\alpha}) - (t_n-t_k)E_{1-\alpha}(-\frac{(1-\alpha)}{\alpha}(t_n-t_k)^{1-\alpha}), $$

also, $\int_{(k-1)\Delta t}^{(k)\Delta t} E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-x)^{1-\beta})dx =$

$$ (t_n-t_{k+1})E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-t_{k+1})^{1-\beta}) - (t_n-t_k)E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-t_k)^{1-\beta}). $$

Put

$$ \delta_1^{\alpha} = \int_{(k-1)\Delta t}^{k\Delta t} E_{1-\alpha}(-\frac{(1-\alpha)}{\alpha}(t_n-x)^{1-\alpha})dx,\ \delta_2^{\beta} = \int_{(k-1)\Delta t}^{(k)\Delta t} E_{1-\beta}(-\frac{(1-\beta)}{\beta}(t_n-x)^{1-\beta})dx. $$

Substituting (10) and (14) into (7), the resulting equation can be written as follows:

$$ \mu\frac{M(1-\alpha)}{\alpha}\triangle\alpha\sum_{g=0}^{2K}\gamma_g q(\alpha_g)\sum_{k=1}^{n}\delta_1^{\alpha_g}u_{i,j}^{n+1-k} + \frac{u_{i,j}^{n+1}-u_{i,j}^{n-1}}{2\theta(\Delta t)} - \frac{M(1-\beta)}{\beta}\triangle\beta $$
$$ \times\sum_{g=0}^{2K}\gamma_g q(\beta_g)\left[\sum_{k=1}^{n}\delta_2^{\beta_g}\left[\frac{-\frac{1}{12}u_{i-2,j}^{n+1-k}+\frac{4}{3}u_{i-1,j}^{n+1-k}-\frac{5}{2}u_{i,j}^{n+1-k}+\frac{4}{3}u_{i+1,j}^{n+1-k}-\frac{1}{12}u_{i+2,j}^{n+1-k}}{\phi(\Delta X)^2}\right]\right. $$
$$ \left. -\sum_{k=1}^{n}\delta_2^{\beta_g}\left[\frac{-\frac{1}{12}u_{i,j-2}^{n+1-k}+\frac{4}{3}u_{i,j-1}^{n+1-k}-\frac{5}{2}u_{i,j}^{n+1-k}+\frac{4}{3}u_{i,j+1}^{n+1-k}-\frac{1}{12}u_{i,j+2}^{n+1-k}}{\xi(\Delta y)^2}\right]\right] $$
$$ -f(t_{n+1},u_{i,j}^{n+1},x_i,y_j) = R_{1_{i,j}}^{n}. \tag{15} $$

$R^n_{1_{i,j}}$ stands for the truncation error. Scheme (15) creates a nonlinear algebraic system of $(N+1)(M+1)(Q+1)$ equations with both initial and boundary conditions. To solve this problem, Newton's iteration approach will be applied [21].

4. Stability and error estimation

4.1. The Atangana-Baleanu fractional operator and NICFDM stability

The following section uses a Jon Von Neumann methodology to discuss the method's stability (15). Then, by linearizing this scheme and letting the nonlinear term $f(t,u,x,y)$ be equal to 0, we will use this technique [22]. Assume that $u^n_{i,j} = \zeta^n e^{miq_1\Delta X + mjq_2\Delta y}$, where $m = \sqrt{-1}$, and q_1, q_2 are the spatial wave numbers (which assumed to be real) [23] hence the requirement $|\zeta(q)| \leq 1$, and the following form can be used to express the scheme (15):

$$\begin{aligned}
&N_{\alpha\alpha} \triangle \alpha \sum_{g=0}^{2K} \gamma_g q(\alpha_g) \sum_{k=1}^{n} \delta_1^{\alpha_g} u_{i,j}^{n+1-k} + (u_{i,j}^{n+1} - u_{i,j}^{n-1}) - \triangle\beta \sum_{g=0}^{2K} \gamma_g q(\beta_g) \\
&\times \Bigg[N_{\alpha\alpha} \sum_{k=1}^{n} \delta_2^{\beta_g} \Bigg[-\frac{1}{12} u_{i-2,j}^{n+1-k} + \frac{4}{3} u_{i-1,j}^{n+1-k} - \frac{5}{2} u_{i,j}^{n+1-k} + \frac{4}{3} u_{i+1,j}^{n+1-k} - \frac{1}{12} u_{i+2,j}^{n+1-k} \Bigg] \\
&+ M_{\alpha\alpha} \sum_{k=1}^{n} \delta_2^{\beta_g} \Bigg[-\frac{1}{12} u_{i,j-2}^{n+1-k} + \frac{4}{3} u_{i,j-1}^{n+1-k} - \frac{5}{2} u_{i,j}^{n+1-k} + \frac{4}{3} u_{i,j+1}^{n+1-k} - \frac{1}{12} u_{i,j+2}^{n+1-k} \Bigg] \Bigg] = 0.
\end{aligned} \tag{16}$$

where,

$$N_{\alpha\alpha} = 2\mu\theta(\Delta t) \frac{M(1-\alpha)}{\alpha}, \ N_{\beta\beta} = 2\theta(\Delta t) \frac{M(1-\beta)}{\beta\phi(\Delta X)^2}, \ M_{\beta\beta} = 2\theta(\Delta t) \frac{M(1-\beta)}{\beta\xi(\Delta y)^2}, \tag{17}$$

by inserting $u^n_{i,j} = \zeta^n e^{miq_1\Delta X + mjq_2\Delta y}$ in (16) and let $\eta = \dfrac{\zeta^{n+1}}{\zeta^n}$, $m = \sqrt{-1}$, and

$$\begin{aligned}
H = {} & N_{\alpha\alpha} \triangle \alpha \sum_{g=0}^{2K} \gamma_g q(\alpha_g) \sum_{k=1}^{n} \delta_1^{\alpha_g} \zeta^{-k} + 1 - N_{\beta\beta} \triangle \beta \sum_{g=0}^{2K} \gamma_g q(\beta_g) \sum_{k=1}^{n} \delta_2^{\beta_g} \\
& \Bigg[-\frac{1}{12} \zeta^{-k} e^{-2mq_1\Delta X} + \frac{4}{3} \zeta^{-k} e^{-mq_1\Delta X} - \frac{5}{2} \zeta^{-k} + \frac{4}{3} \zeta^{-k} e^{mq_1\Delta X} - \frac{1}{12} \zeta^{-k} e^{2mq_1\Delta X} \Bigg] \\
& - M_{\beta\beta} \triangle \beta \sum_{g=0}^{2K} \gamma_g q(\beta_g) \sum_{k=1}^{n} \delta_2^{\beta_g} \Bigg[-\frac{1}{12} \zeta^{-k} e^{-2mq_2\Delta y} + \frac{4}{3} \zeta^{-k} e^{-mq_2\Delta y} - \frac{5}{2} \zeta^{-k} \\
& + \frac{4}{3} \zeta^{-k} e^{mq_2\Delta y} - \frac{1}{12} \zeta^{-k} e^{2mq_2\Delta y} \Bigg].
\end{aligned} \tag{18}$$

Then, after some simplifications, we have

$$\eta H = \frac{1}{\zeta}, \text{and} \quad \zeta^2 H = 1, \quad |\zeta| = |\frac{1}{\sqrt{H}}| \leq 1. \tag{19}$$

So the scheme (15) will be stable under the condition: $|\frac{1}{\sqrt{H}}| \leq 1$.

5. Numerical examples

Example 1. [9] Consider the following numerical example to test our theoretical analysis on an elliptical domain

$$\begin{aligned} \frac{\partial u(\mathbf{x},t)}{\partial t} &= \int_0^1 \Gamma(3+\alpha)\,{}_0^R D_t^{1-\alpha} \Delta u(\mathbf{x},t) d\alpha - \int_0^1 \Gamma(3+\beta)\,{}_0^R D_t^{1-\beta} u(\mathbf{x},t) d\beta, \\ &+ f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times I, \\ u(\mathbf{x},t) &= t^3 \sin(2\pi x)\sin(2\pi y), (\mathbf{x},t) \in \partial\Omega \times I, u(\mathbf{x},0) = 0, (x,y) \in \Omega, \end{aligned} \tag{20}$$

where S $\Omega = \left\{(x,y) \mid 4x^2 + y^2 \leq 1\right\}$ is the convex spatial domain and $I = (0,1]$ is the temporal interval. The exact solution of (20) is

$$u(\mathbf{x},t) = t^3 \sin(2\pi x)\sin(2\pi y),$$

then we get the source term as

$$f(\mathbf{x},t) = \left(3t^2 + 6\left(1 + 8\pi^2\right)\frac{t^3 - t^2}{\ln(t)}\right)\sin(2\pi x)\sin(2\pi y).$$

Table 1. The $L_2 - error$ by the proposed numerical method (15) and method presented in [9]

Δt	Error in [9]	Erorr in the proposed method
0.0399	$4.3968E-02$	$2.7872E-09$
0.0299	$9.6083E-03$	$3.5588E-09$
0.0233	$2.2811E-03$	$5.39932E-09$

In Example 1, the approximate solutions have been obtained using the proposed method (15) where $\theta(\Delta t) = 0.001(1 - e^{-\Delta t})$, $\phi(\Delta x) = 0.001(1 - e^{-\Delta x})$, and $\xi(\Delta y = 0.001(1 - e^{-\Delta y}))$ and the error computed by $L_2 - error$. Table 1 shows the $L_2 - error$ which was obtained by the proposed numerical method (15) compared with the one in [9]. It shows that the proposed method has high accuracy and gives results better than the results in [9]. Figure 1 shows the behavior of the numerical solutions of (20) for the following data $N = 20, \alpha = 0.9, \beta = 0.6$. Figure 2 shows the behavior of the numerical solutions of (20) for the following data $N = 30$, $\alpha = 0.4$, $\beta = 0.7$.

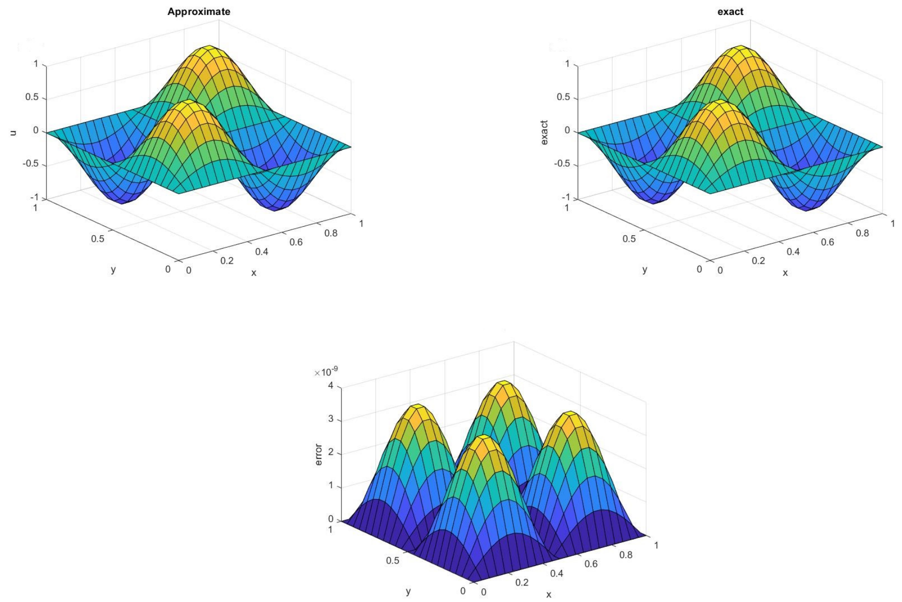

Fig. 1. The comparison between the exact solution, numerical solution and the resulted error at $h = \dfrac{1}{20}$

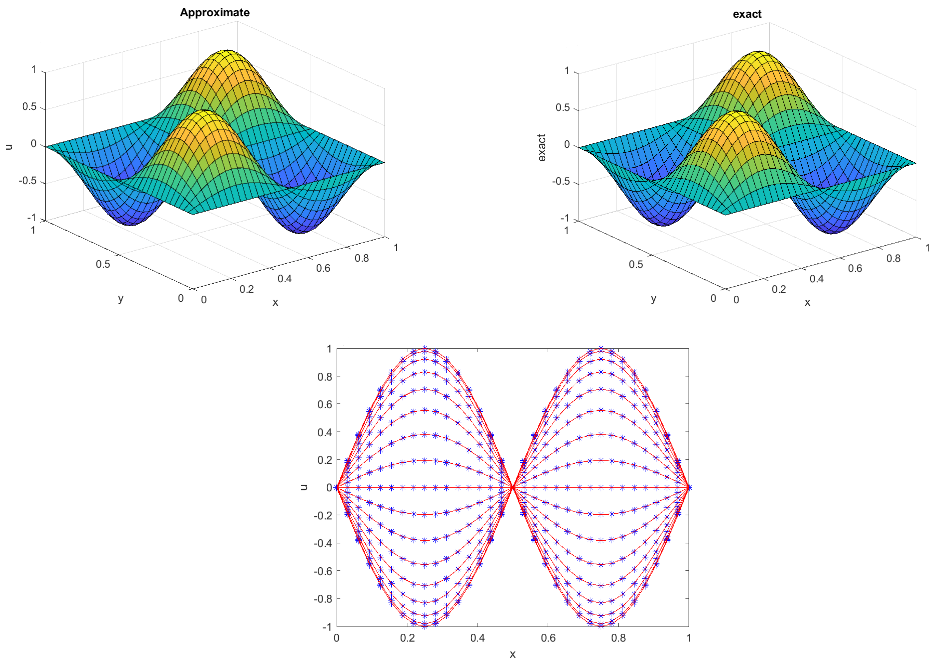

Fig. 2. The comparison between the exact solution, numerical solution and the resulted error at $h = \dfrac{1}{30}$

Example 2. [9] In this example, consider the nonlinear model on the irregular domain

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \int_0^1 K_\alpha {}_0^R D_t^{1-\alpha} \Delta u(\mathbf{x},t) d\alpha - \int_0^1 K_\beta {}_0^R D_t^{1-\beta} u(\mathbf{x},t) d\beta + f(\mathbf{x},t) + g(u),$$

where $K_\alpha = \Gamma(3+\alpha)$, $K_\beta = \Gamma(3+\beta)$ and $u(\mathbf{x},t)$, is the same as Example 1 with nonlinear term $g(u) = u - u^3$.

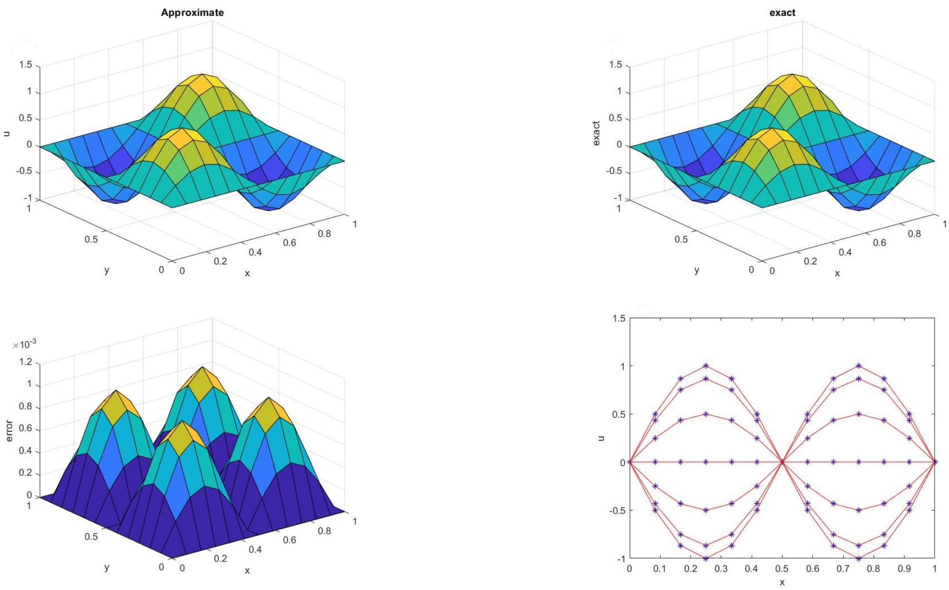

Fig. 3. The comparison between the exact solution, numerical solution and the resulted error at $h = \frac{1}{10}$

Table 2. The maximum error obtained by our proposed numerical method compared with the method presented in [9]

Δt	Error in [9]	Erorr in the proposed method
0.0699	$2.7593E-02$	0.0010
0.0249	$6.7794E-03$	$9.9038E-04$

In Example 2, the approximate solutions have been obtained using the proposed method (15), where $\theta(\Delta t) = 0.001(1-e^{-\Delta t})$, $\phi(\Delta x) = 0.001(1-e^{-\Delta x})$, and $\xi(\Delta y = 0.001(1-e^{-\Delta y}))$ and the error computed by $L_2 - error$. Table 2 shows the $L_2 - error$ which is obtained by the proposed numerical method (15) compared with the one in [9]. It shows that the proposed method has high accuracy and gives results better than the results in [9]. Figure 3 shows the behavior of the numerical solutions of (5) and compares them with the exact solution for the following data $\alpha = 0.1$, $\beta = 0.3$. The same is true for Figure 4 but with different data $\alpha = 0.7, \beta = 0.4$.

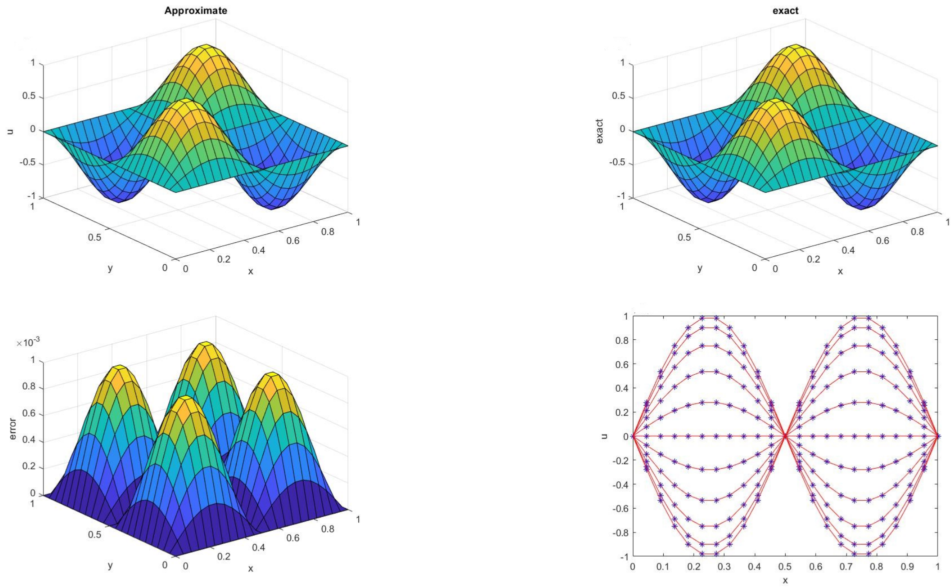

Fig. 4. The comparison between the exact solution, numerical solution and the resulted error at $h = \dfrac{1}{20}$

6. Conclusions

Using a higher accurate non-standard implicit compact finite difference method, numerically approximate solutions of the two-dimensional nonlinear distributed order Cable equation are introduced. The distributed fractional order is defined according to Atangana-Baleanu. The key advantage of this strategy is the large stability areas it implicitly has. The stability analysis of the suggested approaches is the main theoretical part of this study. To do this, we have resorted to a form of John von Neumann stability analysis. Numerical results are given to illustrate the accuracy of the proposed approach. The entire computation in this paper was done using the MATLAB computer language.

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