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OPTIMIZATION BY THE CONVERGENCE CONTROL PARAMETER IN ITERATIVE METHODS

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Abstract. The classical iterative methods, such as the fixed point iteration, the Adomian decomposition method and the homotopy analysis method are discussed in the present paper. It is proven that adding a convergence control parameter into these makes them powerful and rapidly converging to the true solution, whilst the classical correspondences may fail to or slowly converge to the desired solution. The key is to demonstrate the presence of a continuous interval of the convergence control parameter for the considered problem. This allows convergence of such modified iterative methods with an optimum convergence control parameter obtained from squared residual errors of either the original equation or the derivative of iterative solution with respect to the convergence control parameter.

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1. Introduction

Approximate solutions of nonlinear equations in algebraic or differential form are highly desirable nowadays to gain information about the behavior of solutions. The classical iterative methods are discussed here when a convergence control parameter is plugged into the classical formulation.

There are a significant number of iterative methods available in the literature now. The oldest and the longest lasting method may be the fixed point iteration [1]. Differential equations with deviating arguments were considered by the fixed point iteration technique in [2]. A fixed-point iterative method together with its convergence theorem was given in [3] suitable for system of nonlinear equations. The convergence theorems for fixed point iterative methods based on the admissible function concept were presented in [4]. A family of Newton-type iterative methods for solving nonlinear equations was recently presented in [5]. Self-accelerating parameters are shown to increase the convergence order without extra computational cost.

The Adomian decomposition method is yet another technique of iteratively approximating the solutions through Adomian polynomials made use of the nonlinear terms, refer to [6]. Differential equations were solved in [7] by means of the decomposition method of Adomian. The recent advances in the Adomian method can be grasped from the works [8–10]. The homotopy analysis method is also a flash approximate analytical method nowadays, like the Adomian method. Many good ideas and papers on the homotopy method can be found in the book [11]. Also see the series of publications on the foundations and applications of the HAM method [12–15].

Even though the literature is full of iterative techniques; either in classical or accelerated forms, there is no rigorous mathematical treatment in order to explain their feasibility. The present motivation is to rigorously prove that embedding a convergence control parameter within the classical iterative methods makes them quite strong in view of convergence. Showing the presence of a continuous interval of convergence parameters is the key for the rapid convergence of modified methods, whilst their cousins may be divergent or slowly convergent.

2. Iterative methods

Three iterative methods, namely fixed point, Adomian decomposition and homotopy analysis will be analyzed in what follows.

2.1. Fixed point for algebraic equations

For x real, let $f(x)$ be a differentiable function over a closed interval $I = [a,b] \subset R$ in which at least a solution to

$$f(x) = 0 \tag{1}$$

is supposed to exist. For a better mathematical exposition, let us rewrite (1) in classical fixed point form

$$x = x + f(x). \tag{2}$$

It is well known that the fixed point iteration algorithm associated with (2)

$$x_{n+1} = x_n + f(x_n), \tag{3}$$

with an initial guess, x_0 converges if

$$|1 + f'(x)| < 1, \tag{4}$$

provided that I is mapped onto I by $x + f(x)$. Condition (4) leads to a good approximation interval $-2 < f'(x) < 0$ to be picked up for the initial guess x_0, though not necessary. Also, the fastest convergence rate of (3) is achieved when $1 + f'(x) = 0$, since then, it will converge quadratically, with less restrictive conditions imposed on $f''(x)$.

With h a real parameter, referred to hereafter as the so-called convergence control parameter, it is possible to rewrite (1) in the Krasnoselskij form [2]

$$x = (1-h)x + h(x + f(x)), \tag{5}$$

which can be simplified to

$$x = x + hf(x), \tag{6}$$

whose iterative form with an initial approximation x_0 is

$$x_{n+1} = x_n + hf(x_n). \tag{7}$$

Now, let us assume that $x_n(h)$ is a sequence produced from (7). For a sufficiently large n, changing values of h will generate a convergent set of sequences $x_n(h)$, therefore, over a domain of h, $x_n(h)$ will converge to the same values, implying that $x_n(h)$ will be constant over some $h \in I_h \subset R$, in which case $\frac{dx_n}{dh}$ will diminish for all $h \in I_h$. The author is aware of the fact that this is a rather strong assumption, which may imply convergence after a finite number of steps.

Theorem 1. Consider the successive iteration formula (7). If there exists $h \in I_h$ such that $x_n(h)$ are constant in the continuous interval I_h, i.e. $x_n(h) \in C^1(I_h)$, then $f(x_n(h)) = 0$, that is, the convergent sequence $x_n(h)$ is a solution to (1).

Proof. Indeed, under the light of the above remarks, $\frac{dx_n}{dh} = 0$ and using this fact, after differentiating (7) with respect to h reads

$$\frac{dx_{n+1}}{dh} = \frac{dx_n}{dh} + f(x_n) + h\frac{df(x_n)}{dx}\frac{dx_n}{dh}, \tag{8}$$

which leads to $f(x_n(h)) = 0$.

This completes the proof $\diamond$

Remark 1. For finite and large n, the graph of $x_n(h)$ versus h will lay out the interval of convergence control parameter I_h.

Corollary 1. Having obtained the approximated $x_n(h)$ at high iteration level n, the best value of the convergence resulting in the quickest rate of convergence of iterative process (7) may be picked up from optimizing control parameter h, to achieve the minimums either

$$\frac{dx_n}{dh} = 0 \tag{9}$$

or

$$f(x_n(h)) = 0. \tag{10}$$

Corollary 2. The initial guess, though not necessary, assures that the convergence can be selected from the analogy of (4)

$$|1 + hf'(x_0)| < 1. \tag{11}$$

In fact, the fastest convergence occurs when $1+hf'(x_0)=0$, and this leads to the iteration (7) to coincide with Newton's iteration.

Remark 2. Double/multiple roots of (1) may be easily gained from the selection of appropriate initial guesses.

Corollary 3. If $x+f(x)=g(x)$ operates as self mapping over a convex subset of a normed space, then the Krasnoselskij iteration (7) will converge over $h\in[0,1]$, whose proof can be inferred from [2].

2.2. Adomian decomposition for algebraic equations

Let $f(x)=g(x)+l$, where $g(x)$ solely depends upon x and l is a constant, and we want to find the solution to (1). From (6), we write

$$x=x+h(g(x)+l), \tag{12}$$

and decomposing x into classical series

$$\sum_{n=0}^{\infty} x_n$$

and $g(x)$ into a series of Adomian polynomials

$$\sum_{n=0}^{\infty} A_n$$

with the definition of Adomian polynomials

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(g\Big(\sum_{i=0}\lambda^i x_i\Big)\right)\Bigg|_{\lambda=0}, \tag{13}$$

we have the recursive relation

$$\begin{aligned} x_0 &= hl,\\ x_{n+1} &= x_n+hA_n. \end{aligned} \tag{14}$$

Theorem 2. If the sum

$$x_M(h)=\sum_{n=0}^{M} x_n(h),$$

approximating the root of (1) and producing it from (14) is nearly constant (for the notion of nearly constant, please refer to Theorem 1) over a continuous interval $h\in I_h\subset R$ for a sufficiently large M, then $f(x_n(h))=g(x_n(h))+l=0$ over I_h.

Proof. Differentiating $x_M(h)$ as a consequence of (14), with respect to h, and neglecting the derivatives with respect to h, it is obtained

$$l + \sum_{n=0} A_n = g(x_n) + l = f(x_n) = 0. \quad (15)$$

This completes the proof ⋄

We should mention that all the previous Corollaries expressed for the fixed point method remain the same for the decomposition method here.

2.3. Fixed point for operator equations

Now, let us consider operator problem

$$L(u) + N(u) = f(x), \quad (16)$$

where L is a linear operator, N is the rest of the operator consisting of linear and nonlinear parts, and $u(x)$ is a function to be solved over a domain $\Omega \subset R$ with appropriate initial and boundary conditions. The operators belong to the Banach space, and u is a well-behaved function having a sufficient number of continuous derivatives. The classical Picard iteration is

$$u_{n+1} = -L^{-1}(N(u_n) - f) + g, \quad (17)$$

where g is due to the boundary and initial conditions. The optimum Picard iteration, analogous to the algebraic case (7), can be introduced through

$$u_{n+1} = (1-h)u_n - hL^{-1}(N(u_n) - f) + g, \quad (18)$$

with $u_0(x) = u_0(x)$ as the initial guess. It should be anticipated that $h = 1$ matches the classical Picard iteration (17).

Theorem 3. For the iterative formula (18), if there exists $h \in I_h \subset R$ such that $u_n(x,h)$ is nearly constant for the continuous interval I_h for a sufficiently large n, then these values of h do produce an iterative solution satisfying the nonlinear operator equation (16).

Proof. Taking into account the hypothesis, differentiating (18) with respect to h and omitting the derivatives with respect to h, (18) leads to

$$L(u_n) + N(u_n) = f(x). \quad (19)$$

This completes the proof ⋄

Corollary 4. For finite and large n, the constant h level curves of $u_n(x,h)$ over $x \in \Omega$ will form the region I_h. Since u(x) is a differentiable function, the level curves of $u^{(n)}(x)$ may also be at one's disposal.

Corollary 5. The optimum value of h can be gained from the squared residual error

$$Res(h) = \left(\int_\Omega [L(u_n) + N(u_n) - f(x)]^2 d\Omega \right)^{1/2}. \quad (20)$$

Another alternative to receive the optimum value of h could be

$$Res(h) = \left(\int_{\Omega} \left[\frac{du}{dh} \right]^2 d\Omega \right)^{1/2}. \tag{21}$$

2.4. Adomian decomposition for operator equations

There is no need to repeat the analysis given in section 2.3, and hence the above procedure can be simply adapted to the Adomian decomposition method for the Adomian iterative scheme [8]

$$\begin{aligned} u_0 &= L^{-1}(hf) + g, \\ u_{n+1} &= L^{-1}\left[(1-h)L(u_n) - hN(u_n)\right]. \end{aligned} \tag{22}$$

2.5. Homotopy analysis

Consider the nonlinear equation

$$N(u(x)) = 0, \tag{23}$$

defined over a domain $\Omega \subset R$ with appropriate initial and boundary conditions. Construction of homotopy associated with (23) in the manner

$$(1-p)L(u) - hpN(u) = 0, \tag{24}$$

with $p \in [0,1]$, leads to at $p = 1$ [11]

$$u_M = \sum_{n=0}^{M} u_n(x,h) \tag{25}$$

as the approximate homotopic solution of (23).

Theorem 4. If there exists a continuous interval $I_h \subset R$ such that the homotopic solution u_M in (25) is nearly constant for $h \in I_h$ for a sufficiently large M, then for such values of h, the produced solution satisfies the nonlinear operator equation (23).

Proof. Differentiating (24) with respect to h, and omitting the derivatives of variables with respect to h, (24) gives rise to

$$N(u_M) = 0. \tag{26}$$

This completes the proof $\diamond$

3. Discussions

3.1. Algebraic equations

Example 1. Consider the simple fixed point problem

$$x = 1 - x, \tag{27}$$

or simply the root finding problem $f(x) = 1 - 2x = 0$. The solution is the fixed point $x = 1/2$, but since it is a repelling point, the classical fixed point method (3) fails for any initial guess, except $x_0 = 1/2$. Refer to the paper by Berinde [2].

Instead, if we make use of the improved fixed point iterative process (7) with convergence control parameter h, we may write

$$x_{n+1} = (1-h)x_n + h(1-x_n), \quad x_0 = 0. \tag{28}$$

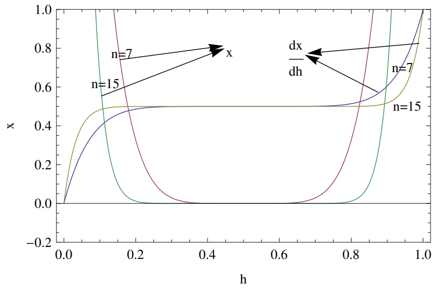

Fig. 1. Control h curves relevant to equation (27) from the fixed point iteration

The constant h level curves generated from (28) are displayed in Figure 1, at the iterative levels 7 and 15, respectively. It is anticipated that the convergence control parameter h can be picked up from the interval $I_h = [0.2, 0.8]$ which makes the improved fixed point iterative scheme (28) convergent. In fact, the condition (4) yields $0 < h < 1$. This figure explains why the classical fixed point iteration should not converge, since $h = 1$ is outside the domain I_h, regardless of what the approximation level n is. Minimizing the iteration at $n = 7$, with the residual $Res(h) = (1 - 2x_7(h))$, or with the derivative $\dfrac{dx_7(h)}{dh}$ results in the same optimum convergence control parameter $h = 0.5$.

To examine the problem (27) with the Adomian decomposition method, because equation (27) is linear in nature, its accelerated Adomian decomposition method from (14) with the initial value $x_0 = h$ coincides with the fixed-point iteration from (28).

Example 2. Let us now solve the transcendental equation

$$f(x) = e^{-x} - \cos x = 0, \tag{29}$$

with the numeric roots $x = 0$ and $x = 1.29269$.

With the initial approximation $x_0 = 0.5$, the improved fixed point iterative process (7) yields the constant h level curves at the iteration level $n = 8$, which is capable of capturing both roots at the same time, as demonstrated in Figure 2.

Figure 2 clearly points to the fact that the convergence control region for the root $x = 0$ is $I_h = [0.6, 1.8]$ and for the other root it is $I_h = [-2.20, -0.8]$, respectively. Actually, minimizing the residuals leads to the optimum convergence control parameters as $h = 1$ and $h = -1.45$, respectively. These lead to the fastest convergence to the real roots.

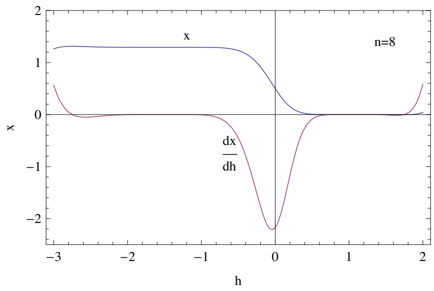

Fig. 2. Control h curves relevant to equation (29) from the fixed point iteration

3.2. Differential equations

Example 3. Consider the nonlinear boundary-value problem

$$u'' = su^2 + \beta^2 u, \quad u'(0) = 0, \quad u(1) = 1, \tag{30}$$

representing the porous fin problem for the heat removal process [8]. There is no exact solution for the current problem.

From (18), one can easily construct the fixed point iteration algorithm

$$u_{n+1} = (1-h)u_n + h\int_1^x \int_0^x (su_n^2 + \beta^2 u_n)dxdx + h, \quad u_0 = 1. \tag{31}$$

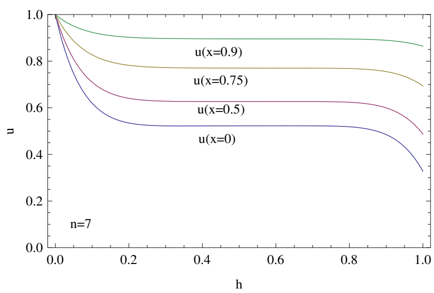

Fig. 3. Control h curves relevant to equation (30) from the fixed point iteration

For the specific parameters $s = \beta = 1$, Figure 3 reveals the constant h level curves corresponding to the 7th order fixed point iteration. The figure implies that for the convergence to take place, the convergence control parameter must be selected from the interval $I_h = [0.2, 0.8]$. The fastest convergence occurs at $h = 0.67$ as a result of minimizing the squared residual error (20) or (21). On the other hand, the classical fixed point iteration may not even converge, since $h = 1$ does not reside within this range. For instance, the optimum h computes the wall temperate $u(0) = 0.522690$ and the fin efficiency $u'(1) = 1.1393791$, versus the classical fixed point iteration, giving $u(0) = -0.03125$ and $u'(1) = 1.21875$ for $h = 1$. We should mention that the exact numerical values are $u(0) = 0.522738093570$ and $u'(1) = 1.13937891581$.

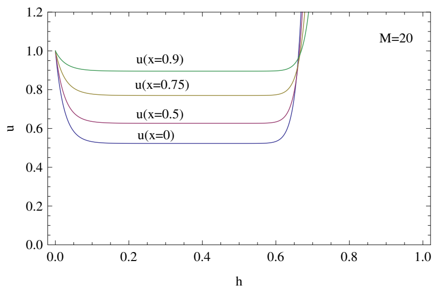

Fig. 4. Control h curves relevant to equation (30) from the Adomian decomposition method

The optimum Adomian decomposition method corresponding to the problem (30) can be better viewed from Figure 4. At this fine approximation level $M = 20$, the boundaries of convergence control parameter interval are very strict nearing to $I_h = [0.1, 0.6]$. It is certain that the classical Adomian decomposition method will diverge when $h = 1$, since no truncation level can attain a close value to the actual solution. On the other hand, the best convergence control parameter is evaluated as $h = 0.47$, yielding an approximate solution correct to 7 significant digits, refer to the outcomes $u(0) = 0.522738297343$ and $u'(1) = 1.13937820236$, and the exact numerical ones aforementioned.

Example 4. Consider the system of nonlinear equations from [16]

$$\begin{aligned} &f'''' + 2Re(ff''' + gg') = 0, \quad f(0) = 0, \quad f'(0) = s_1, \quad f(1) = 0, \quad f'(1) = s_2, \\ &g'' + 2Re(fg' - f'g) = 0, \quad g(0) = 1, \quad g(1) = \Omega. \end{aligned} \tag{32}$$

The nonlinear boundary-value model (32) governs the viscous flow between two rotating disks stretching at the rates s_1 and s_2, and the upper one is also rotating with the angular frequency Ω. Additionally, Re means the Reynolds number.

For the particular parameters $s_1 = s_2 = 0.5$, $Re = 10$ and $\Omega = 0$, as well as with the initial approximations

$$\begin{aligned} f_0(\eta) &= s_1\eta - (2s_1 + s_2)\eta^2 + (s_1 + s_2)\eta^3, \\ g_0(\eta) &= 1 + (\Omega - 1)\eta, \end{aligned} \tag{33}$$

Figure 5 shows the constant h level curves at the fixed point iteration level 6, corresponding to the physical variables $f''(0)$, $f''(1)$, $g'(0)$ and $g'(1)$, respectively. The observation is that the interval $I_h = [0.5, 1.5]$ is the correct interval for the choice of convergence control parameter h. In fact, the minimum squared residual leads to

$h = 0.92$ as the optimum value, producing accurate solutions only up to 2 significant decimal places. This implies that the number of iteration should be increased with more mature iterations.

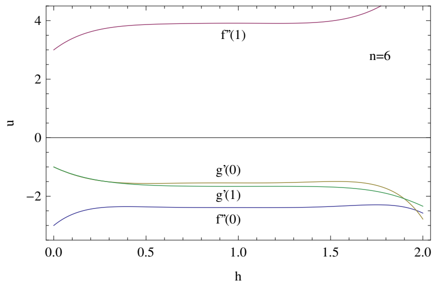

Fig. 5. Control h curves relevant to equation (32) from the fixed point iteration

For the same physical parameters and the leading term (33), the constant h level curves are depicted at the iteration level $M = 20$ for the Adomian decomposition scheme in Figure 6. The interval of convergence control parameter h appears to be $I_h = [0.2, 1.5]$. The optimum convergence occurs at $h = 1.018$, leading to the approximate values $f''(0) = -2.38938054$ and $-g'(0) = 1.54480401$, correct to 6 decimal places, since the exact numerical outcomes are $f''(0) = -2.38938053$ and $-g'(0) = 1.54480443$, see [16].

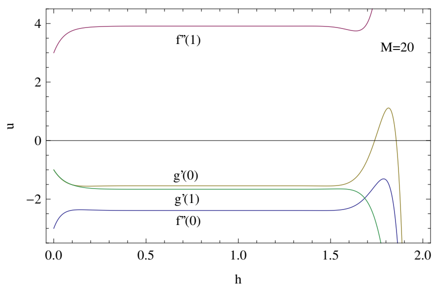

Fig. 6. Control h curves relevant to equation (32) from the Adomian decomposition method

Example 5. This final example is the iterative differential equation or the differential equation with a deviating argument

$$u' = \frac{1}{10}u(u(\eta))^2, \quad u(0) = 1, \quad -1 \le \eta \le 1, \tag{34}$$

taken from the reference [2], see also [17]. It is proven mathematically in [2] that the fixed point iterative method

$$u_{n+1}(\eta) = (1-h)u_n(\eta) + h + \frac{h}{10}\int_0^{\eta} u_n(u_n(\eta))^2 d\eta, \tag{35}$$

approximates well the solution of (34) over $\eta \in [-1,1]$ for $h \in I_h = (0,1)$. No optimum value yielding the best rate of convergence was mentioned in [2].

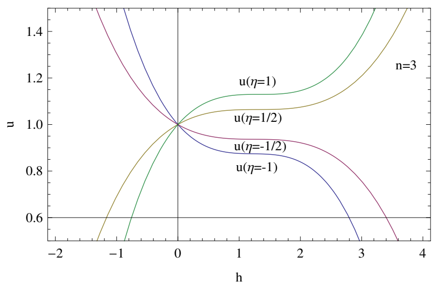

Fig. 7. Control h curves relevant to equation (34)

On the other hand, Figure 7 proves that the convergence region of the fixed point iterative scheme (35) is in fact extended up to a value of h almost 2. The optimum h is worked out as $h = 1.2086$ from the squared residual, which yields accurate solutions correct up to the 5 digits, since

$$\left(\int_{-1}^{1}\left[u'(t)-\frac{1}{10}u(u(t))^2\right]^2 dt\right)^{1/2} \tag{36}$$

turns out to be 0.0000596 at the optimum h. At this order, the approximate solution to (34) is found to be

$$u(\eta) = 1+\eta\,(0.1275336004+\eta\,(0.0018432339+0.0000356532\eta)) \tag{37}$$

4. Conclusions

There are various ways of turning an equation $f(x) = 0$ into a fixed point equation. The present work considers the parameterized fixed point equation $x = x + hf(x)$ as well as a generalization based on the so called Adomian decomposition. The study aims to draw relations between the convergence of the fixed point iteration and (nearly) constant fixed point iterates with respect to h. The further objective is to provide a feasible explanation towards the success of the classical iterative methods of fixed point iteration, the Adomian decomposition method and the homotopy analysis method when a convergence control parameter is embedded.

It is mathematically shown that the classical iterative algorithms may diverge or slowly converge to the physical solution, but the existence of a continuous interval of the convergence control parameter makes them favourable since a rapid convergence is achieved. Even a best convergence control parameter of such modified iterative methods can be derived from squared residual errors of either the original equation or the derivative of iterative solution with respect to the convergence control parameter. It is rigorously explained how to deal with the interval of convergence control parameters for the considered problems.

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