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ADVANCING THE DYNAMIC PROPERTIES OF A COLUMN WITH VARIABLE FLEXURAL STIFFNESS IN TERMS OF STRUCTURAL MOUNTING DAMPING, EXTERNAL AND INTERNAL DAMPING

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Abstract. The paper presents an analysis of the influence of damping on the free vibrations of a slender column subjected to a specific load. The analyzed system is characterized by a step-variable geometry modeled by connected prismatic segments. The problem was formulated according to the Bernoulli-Euler theory and solved by the variational method (Hamilton's principle). Boundary and continuity conditions were determined. The paper shows the influence of different types of damping on the free vibrations of the column, and indicates damping as one of the methods for passive controlling and steering of the dynamic properties. The proposed mathematical and numerical model is universal and can be applied to any variable column shape, taking into account any combination of the occurring damping types.

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1. Introduction

Beams and columns are basic mechanical structures used as elements of machines and rod structures (e.g. bridges). Their stability and free vibrations determine the strength of the entire structure, hence the wide interest of scientists in these topics. The latest scientific works concern the analysis of the influence of the variable geometry of such systems on their dynamic properties and critical loads. This is related to the increasingly common need to optimize such systems with regard to strength or mass reduction. In addition to the issue of non-prismaticity, additional parameters are often taken into account in the works, which are intended to best represent real systems with a mathematical model. Therefore, in the scientific literature there are studies on the behavior of such systems under the influence of, for example, damping, support by an elastic base, pre-stressing or the occurrence of cracks.

The influence of materials and different types of damping on the dynamic stability of the Bernoulli-Euler beam was presented in paper [1]. The problem of dynamic stability was solved using the mode summation method and applying an orthogonal condition of eigenfunctions, then describing the system with the Mathieu equation. In paper [2], the closed-formed analytical solutions of the steady-state forced vibration of the multi-cracked Timoshenko beams with damping effects were obtained. The authors, using the solution of the one-cracked situation and the transfer matrix method, presented the Green's functions of beams for various boundary conditions. In conclusions, the effects of some physical parameters (the crack depth and locations) were also discussed based on numerical examples. The results of similar studies on systems with cracks were also presented in the works [3-6]. The last two works concern curved beams. Paper [6] studies not only nonlinear forced vibration of a multi-cracked Euler-Bernoulli curved beam, but also considers damping effects and derives the closed-formed analytical solution of steady-state forced vibration by means of Green's functions. The issue of free and forced vibrations of damped locally-resonant sandwich beams was considered in paper [7]. In the described complex modal analysis approach, the issue of dynamics is solved applying a recently-introduced contour-integral algorithm to an exact dynamic stiffness matrix. Numerical applications proved the exactness of the proposed solutions. The issue of vibration characteristics of rectangular cross-sectioned and straight beams with imperfect supports, focusing on the role of friction damping, was presented in paper [8]. The differential equation describing the system was solved analytically, separating the motion into two distinct regimes, using the Galerkin method. The obtained analytical results were then compared to those from a numerical model, which is built and solved using the finite element method combined with a frequency sweep and time-marching. A method proposed for frequency response analysis of Euler-Bernoulli beams subjected by a constant axial load, and carrying an arbitrary number of translational and rotational dampers with Kelvin-Voigt viscoelastic behaviour was presented in paper [9]. The proposed solution relies on the theory of generalized functions, within a standard 1D formulation of the equation of motion. The authors concluded with the exact closed-form expressions derived for the frequency response of the beam with dampers, subjected to harmonically varying, arbitrarily placed transverse point/polynomial loads. In article [10], the vibration analysis of functionally graded and viscoelastic/fractionally damped beams with Pasternak foundation is presented. The governing equations were derived and solved analytically in the Laplace domain by considering the fractional three-parameter Kelvin-Voigt model. Studies on vibration reduction of the building structure equipped with an intermediate column-lever viscous damper was shown in paper [11].

The presented selected examples from scientific literature show how complicated and complex the problem of free vibrations of slender systems subjected to various types of conservative and non-conservative loads is, constant or variable in time, when other parameters of the systems, such as damping or variable geometry, are additionally taken into account. This paper attempts to analyze the change in the dynamic properties of a column with a stepwise variable cross-section, subjected to a selected case of specific load, under the influence of internal and structural damping.

In order to perform numerical studies, the Bernoulli-Euler theory and the variational method were used, based on which a numerical calculation program was written. The proposed approach is universal, thanks to which, with minor changes to the source code, columns of any shape and type of load can be analyzed, with coexisting different types of load. The research results may be of great importance not only for the development of science and a better understanding of the damping phenomenon, but also in the machine or construction industry.

The paper is organized as follows. In Section 2, we present a physical model of the system. Section 3 contains formulas describing mechanical energy and mathematical considerations aimed at defining boundary conditions and differential equations of motion and their solutions. In Section 4, the results of numerical calculations in the dimensionless form are introduced, while Section 5 concludes our paper.

2. Physical model

The analysis was performed on a slender elastic column subjected to a follower force directed towards the positive pole. The external load P is performed by heads with a circular outline (constant radius R of curvature) – loading and receiving the load. The column is connected to the loading system by means of an element, assuming its infinite bending stiffness. The concentrated mass m models the weight of the loading system.

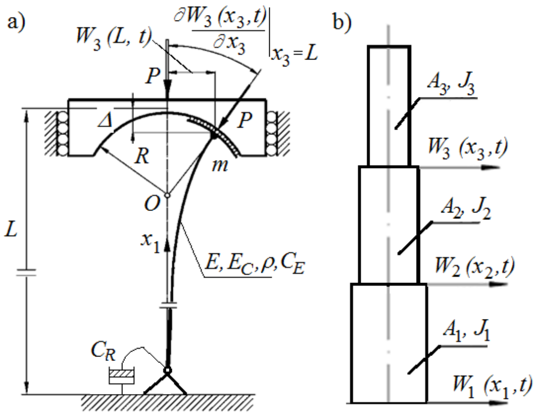

Fig. 1. Physical model of the system: a) column subjected to the follower force directed towards the positive pole, b) model of variable cross-section of the column

The column is characterized by a stepwise variable cross-section – it consists of three prismatic segments of a given geometry. The dimensions of individual segments are changed while maintaining the condition of constant column volume. The system includes structural and internal damping. The structural damping was modeled using a rotational damper C_R in the column mounting (at $x = 0$), and the damping caused by external viscous resistance was taken into account by the C_E coefficient. E_C is the coefficient of viscosity of the material.

3. Mathematical model

The problem of transverse vibrations of a Bernoulli-Euler beam subjected to a follower force directed towards the positive pole, taking into account damping, was solved using Hamilton's variation principle (see [12]) by formulating the boundary value problem:

$$\int_{t_1}^{t_2}(\delta T-\delta V)\,dt+\int_{t_1}^{t_2}\delta W_N\,dt=0, \tag{1}$$

where δT is a variation of kinetic energy, δV is the variation of the potential energy, the sum of the variances of the elastic energy δV_1 and the energy from the external load δV_2, whereas δW_N is the virtual work of non-conservative forces:

$$\delta T=-\sum_{i=1}^{n=3}\rho A_i\int_0^l\frac{\partial^2W_i(x_i,t)}{\partial t^2}\delta W_i(x_i,t)\,dx_i-m\frac{\partial^2W_n(l,t)}{\partial t^2}\delta W_n(l,t)\bigg|_{x_n=l}, \tag{2}$$

$$\begin{gathered}\delta V_1=\sum_{i=1}^{n=3}(EJ)_i\left[\frac{\partial^2W_i(x_i,t)}{\partial x_i^2}\delta\frac{\partial W_i(x_i,t)}{\partial x_i}\bigg|_{x_i=0}^{x_i=l}-\frac{\partial^3W_i(x_i,t)}{\partial x_i^3}\delta W_i(x_i,t)\bigg|_{x_i=0}^{x_i=l}+\right.\\ \left.+\int_0^l\frac{\partial^4W_i(x_i,t)}{\partial x_i^4}\delta W_i(x_i,t)\,dx_i\right],\end{gathered} \tag{3}$$

$$\begin{gathered}\delta V_2=\sum_{i=1}^{3}EA_i\left[\left(\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right)\delta U_i(x_i,t)\bigg|_{x_i=0}^{x_i=l}+\right.\\ \left.+\left(\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right)\frac{\partial W_i(x_i,t)}{\partial x_i}\delta W_i(x_i,t)\bigg|_{x_i=0}^{x_i=l}\right]+\\ +\sum_{i=1}^{3}EA_i\int_0^l\frac{\partial}{\partial x_i}\left(\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right)\delta U_i(x_i,t)\,dx_i+\\ +\sum_{i=1}^{3}EA_i\int_0^l\frac{\partial}{\partial x_i}\left[\left(\frac{\partial U_i(x_i,t)}{\partial x_i}+\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right)\frac{\partial W_i(x_i,t)}{\partial x_i}\right]\delta W_i(x_i,t)\,dx_i+\\ -P\,\delta U_3(x_3,t)\Big|^{x=l}-PR\frac{\partial W_3(x_3,t)}{\partial x_3}\delta\frac{\partial W_3(x_3,t)}{\partial x_3}\bigg|^{x_3=l},\end{gathered} \tag{4}$$

$$\delta W_N = \sum_{i=1}^{n=3} \left(E_C J\right)_i \left[\frac{\partial^3 W_i\left(x_i,t\right)}{\partial x_i^{\,2} \partial t} \delta \frac{\partial W_i\left(x_i,t\right)}{\partial x_i} \Bigg|_{x_i=0}^{x_i=l} - \frac{\partial^4 W_i\left(x_i,t\right)}{\partial x_i^{\,3} \partial t} \delta W_i\left(x_i,t\right) \Bigg|_{x_i=0}^{x_i=l} + \right.$$

$$\left. + \int_0^l \frac{\partial^5 W_i\left(x_i,t\right)}{\partial x_i^{\,4} \partial t} \delta W_i\left(x_i,t\right) dx_i \right] + \sum_{i=1}^{n=3} \int_0^l C_E \frac{\partial W_i\left(x_i,t\right)}{\partial t} \delta W_i\left(x_i,t\right) dx_i + \tag{5}$$

$$+ C_R \frac{\partial^2 W_1\left(0,t\right)}{\partial x_1 \partial t} \delta \frac{\partial W_1\left(0,t\right)}{\partial x_1}.$$

Substituting equations (2)-(5) into the equation describing Hamilton’s principle (1) gives the differential equations of motion as follows:

$$\left(EJ\right)_i \frac{\partial^4 W_i\left(x_i,t\right)}{\partial x_i^{\,4}} + \left(E_C J\right)_i \frac{\partial^5 W_i\left(x_i,t\right)}{\partial x_i^{\,4} \partial t} + P \frac{\partial^2 W_i\left(x_i,t\right)}{\partial x_i^{\,2}} +$$

$$+ \left(\rho A\right)_i \frac{\partial^2 W_i\left(x_i,t\right)}{\partial t^2} + C_E \frac{\partial W_i\left(x_i,t\right)}{\partial t} = 0, \quad i = 1, 2, 3. \tag{6}$$

Boundary conditions for the presented system:

$$W_1\left(0,t\right) = 0, \tag{7}$$

$$\left(E + i\omega E_C\right) J_1 \frac{\partial^2 W_1\left(0,t\right)}{\partial x_1^{\,2}} - i\omega C_R \frac{\partial W_1\left(0,t\right)}{\partial x_1} = 0, \tag{8}$$

$$W_n\left(l,t\right) = \left(R - l_0\right) \frac{\partial W_n\left(x_n,t\right)}{\partial x_n} \Bigg|_{x_n=l}, \tag{9}$$

$$\left(E + i\omega E_C\right) J_3 \frac{\partial^3 W_3\left(x_3,t\right)}{\partial x_3^{\,3}} \Bigg|_{x_3=l} - \frac{\left(E + i\omega E_C\right) J_3}{\left(R - l_0\right)} \frac{\partial^2 W_3\left(x_3,t\right)}{\partial x_3^{\,2}} \Bigg|_{x_3=l} +$$

$$- m \frac{\partial^2 W_3\left(l,t\right)}{\partial t^2} = 0. \tag{10}$$

The solution to the differential equation of motion is the equation:

$$w_i\left(x_i\right) = A_i \cosh\left(\alpha_i x_i\right) + B_i \sinh\left(\alpha_i x_i\right) + C_i \cos\left(\beta_i x_i\right) + D_i \sin\left(\beta_i x_i\right), \tag{11}$$

where:

$$\alpha_i = \sqrt{-\frac{k_i^2}{2} + \sqrt{\frac{k_i^4}{4} + \Omega_i^2}}\,, \tag{12}$$

$$\beta_i = \sqrt{\frac{k_i^2}{2} + \sqrt{\frac{k_i^4}{4} + \Omega_i^2}}\,, \tag{13}$$

$$k_i^{\ 2} = \frac{Pl^2}{\left(E + i\omega E_C\right) J_i}, \tag{14}$$

$$\Omega_i^{\ 2} = \frac{\left(\rho A\right)_i \omega^2 l^4 - i\omega C_E}{\left(E + i\omega E_C\right) J_i}. \tag{15}$$

Substituting the solution of the differential equation (11) into the boundary conditions (7)-(10), supplemented with appropriate continuity conditions, allows one to obtain a system of homogeneous equations. Representing the obtained system in the matrix form and equating the determinant of the matrix to zero allows one to obtain a transcendental equation for the value of the vibration frequency.

4. Results of numerical calculations

In order to compare the results with the reference system (without any damping), dimensionless damping coefficients are introduced in the form of:

– dimensionless internal damping coefficient:

$$H = \frac{E_C}{E\sqrt{l^4 \frac{(\rho A)_{por}}{(EJ)_{por}}}} \tag{16}$$

– dimensionless external damping coefficient:

$$N = \frac{C_E L^2}{\sqrt{(\rho A)_{por}(EJ)_{por}}} \tag{17}$$

– dimensionless viscous structural damping coefficient:

$$M = \frac{C_R}{L\sqrt{(\rho A)_{por}(EJ)_{por}}} \tag{18}$$

– dimensionless parameter of change in geometry of particular segments:

$$Z = \frac{(EJ)_3 - (EJ)_1}{(EJ)_{por}} \tag{19}$$

The values of the coefficients relating to the individual types of damping were selected based on research available in the scientific literature. The parameters with the subscript 'por' in formulas (16)-(19) refer to the analogous values of the reference system (with constant cross-section and the same volume).

The summary of selected results regarding the first three natural frequencies for all analysed cases is presented in Tables 1 and 2. The quoted values can be real or complex numbers. In the second case, the real part is responsible for the value of the vibration frequency, while the imaginary part characterises the degree of decay of the vibration amplitude. Case 1 concerns a system without any damping and is a reference point. In case 2, only the structural damping from the rotary damper C_R, given by the parameter M, is taken into account. As it results from the presented values, this damping has a negligible effect on the values of the vibration frequency. Case 3 refers to external damping. The value of the imaginary part is constant for all three vibration frequencies, but it is worth noting the change in the first vibration frequency in particular.

Table 1. Column vibration frequency values for selected damping cases (P = 100 N, Z = –0.5, R = 0.2L)

Case	1	2	3
H	0	0	0
N	0	0	17.6
M	0	0.04	0
ω_1	120.3	120.3 + 0.322i	18.594 + 118.83i
ω_2	500.8	500.8 + 1.289i	486.5 + 118.836i
ω_3	1134.8	1134.9 + 2.9i	1128.6 + 11.836i

Table 2. Column vibration frequency values for selected damping cases (P = 100 N, Z = –0.5, R = 0.2L)

Case	1	4	5
H	0	0.001	0
N	0	0	4
M	0	0	0.02
ω_1	120.3	120.276 + 0.027i	117.85 + 1.173i
ω_2	500.8	499.743 + 0.54i	498.6 + 1.354i
ω_3	1134.8	1130.74 + 1.219i	1137 + 1.698i

The results presented in Table 2 refer to case 4, in which the internal damping was analyzed, and case 5, in which the external and structural damping were taken into account simultaneously. In the case of internal damping, the vibration frequencies differ little from the frequencies of the comparative system. However, the degree of vibration amplitude decay is significantly smaller. In the case of combining two dampings, a slight decrease in the basic vibration frequencies was observed, while the degree of amplitude decay was greater than in the case of structural or internal damping, but smaller than in the case of external damping.

5. Conclusions

The subject of the work was the free vibrations of a column with a variable cross-section subjected to a specific load in the aspect of changing dynamic properties under the influence of various types of damping. The problem was formulated based on the Bernoulli-Euler theory and solved using the variational calculus (Hamilton's principle). Based on the presented results, it was shown that damping can have a significant effect on the values of the system's vibration frequency, and thus, can be used for passive control and steering of the dynamic properties of systems. The work presents a mathematical model enabling an in-depth analysis of the dependence of the vibration frequency and the analyzed damping, depending on the adopted geometric coefficients of the column, the loading system and the damping itself (rotational damper constant, rheological properties of the medium or viscous damping).

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