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A NONSTANDARD FINITE DIFFERENCE METHOD FOR SOLVING THE FRACTIONAL LOGISTIC MODEL

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Abstract. This paper proposes an alternative solution formula for the logistic model, which is derived by substituting the exponential function with the Mittag-Leffler function in the solution of the first-order logistic model. Then, it developed two nonstandard finite difference approaches to solve the fractional logistic model. One method employed Mickens's concepts to construct a nonstandard finite difference scheme, under the assumption that the analytical solution is unknown. The second method relies on the proposed analytical solution of the fractional logistic model. Surprisingly the two nonstandard finite difference algorithms are exactly the same. The convergence of the nonstandard finite difference scheme is proven by establishing its consistency and stability. Furthermore, it has been proven that the proposed numerical method is unconditionally stable. The performance of the method is demonstrated through two numerical examples selected from literature.

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1. Introduction

The fractional logistic model has many real-world applications that arise in many fields, including biology, mathematics and geophysics, population growth, spread of diseases, and diffusion in social networks [1-3]. The fractional logistic model is seen as an extension of the classical logistic model.

The fractional version of the logistic model has many advantages over the classical first-order logistic model. One of the most important advantages is the memory effect (inherited from the kernel of the fractional differential operator), which suggests that the dependent variable's history impacts its present state, resulting in a more accurate model [4]. When considering this advantage, the fractional logistic model would be more suited to modeling complex real-world systems [1].

We consider the fractional order logistic model with Caputo differential operator [5, 6] of the form

$$ {}_0^C D_t^\alpha\, u(t) = \lambda u(t)(1-u(t)), \quad u(0) = u_0, \quad 0 < \alpha \le 1. \tag{1} $$

When $\alpha = 1$, we obtain the classical first-order logistic differential equation. This differential equation is solved by separating variables, using partial fractions, integrating and substituting the initial condition to obtain a solution of the form

$$ u(t) = \frac{u_0 e^{\lambda t}}{(1-u_0) + u_0 e^{\lambda t}} \tag{2} $$

Since the exponential function e^x possesses the property $e^a/e^b = e^{a-b}$, equation (2) can be written in equivalent form as

$$ u(t) = \frac{u_0}{u_0 + (1-u_0)e^{-\lambda t}}. \tag{3} $$

The fractional logistic model (1) has two equilibrium points, $u_{eq}^1 = 0$ and $u_{eq}^2 = 0$, which can be obtained by solving

$$ {}_0^C D_t^\alpha\, u(t) = \lambda u(t)(1-u(t)) = 0. $$

The first equilibrium point $u_{eq}^1 = 0$ is unstable. But the second equilibrium point $u_{eq}^2 = 1$ is stable.

However, the exact solution for the fractional logistic model of order α for $0 < \alpha < 1$ is unknown. Several attempts have been made to obtain numerical solutions for the fractional logistic model.

El-Sayed et al. [7] investigated the stability of the fractional logistic equation and used a one-step Adam-type predictor corrector approach to solve the logistic equation numerically.

Based on the iterative technique provided in [8], Bhalekar and Daftardar-Gejji [9] proposed an iterative method for solving the fractional logistic model, and compared the resulting solution to the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM). They came to the conclusion that the solutions by the NIM were more stable than those obtained by ADM and HPM.

A novel formula for approximating fractional derivatives is developed using the generalized Laguerre polynomials in [10]. These fractional derivatives were expressed using the Caputo sense and were utilized to solve the fractional Logistic differential equation.

In [11], a collocation method utilizing fractional B-splines was developed to address a fractional nonlinear differential equation. The utilization of fractional B-splines enabled the computation of the fractional derivatives of the approximation function in an analytical form.

Arshad et al. [12] presented a novel 2-stage fractional Runge-Kutta method for the fractional logistic growth model using the fractional Taylor series with order $0 < \alpha \leq 1$. For $\alpha = 1$, they found that the solution obtained by their proposed method is convergent to the exact solution. They concluded that the proposed approach may be used to generate higher order fractional Runge-Kutta methods.

Using a series of fractional powers, Area and Neito [13] proposed a depiction of the fractional logistic equation solution. In simpler terms, they investigated the simplest example and demonstrated that the power series is the exact solution.

West [14] proposed an exact solution using Laplace transform for the fractional logistic equation by substituting the Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ for the exponential function $e^{\lambda t}$ in equation (3), yielding a solution of the form

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)E_\alpha(-\lambda t^\alpha)}, \tag{4}$$

Later on, Area et al. [15] revealed that the solution (4) of the fractional logistic model (1) proposed by West in [14] is only accurate when $\alpha = 1$. The reason for this is that the Mittag-Leffler function does not exhibit the property

$$E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha) \cdot E_\alpha(as^\alpha),$$

as in the exponential function.

Equation (2) suggests an analytical solution for the fractional logistic model (1) of the form:

$$u(t) = \frac{u_0 E_\alpha(\lambda t^\alpha)}{(1 - u_0) + u_0 E_\alpha(\lambda t^\alpha)}. \tag{5}$$

The objective of this paper is to construct a nonstandard finite difference technique (NSFD) for solving the fractional logistic equation (1). Then, compare the solutions produced using the nonstandard finite difference scheme for (5), to those obtained in [8] and [14]. The fractional differential equations will be discretized with the generalized fractional Taylor series expansion, which employs Caputo fractional derivatives.

The main contributions of the paper are that it presents an alternative solution formula for the fractional logistic model, derived from the solution form of the classical logistic model and proposes an exact finite difference scheme for solving it.

The rest of the paper is structured as follows: In Section 2 we construct finite difference schemes for the fractional logistic model. Section 3 discusses the convergence and stability of the proposed nonstandard finite difference method. Section 4 illustrates numerical examples for different values of α and compares them to the analytical solution to demonstrate the accuracy of the numerical solution. Section 5 contains the conclusions.

2. Nonstandard finite difference scheme for the fractional logistic model

In [16], Mickens derived an exact finite difference scheme for solving a first order logistic equation. The denominator function was constructed from the exact solution of the linear part of the differential equation, whereas a nonlocal approximation was used for the nonlinear part. In this section, we construct a nonstandard finite difference scheme for the fractional logistic model (1).

2.1. The nonstandard finite difference scheme

By considering the linear part of the fractional logistic model, we obtain a fractional growth model

$$ {}_0^CD_t^{\alpha} u = \lambda u, \quad u(0) = u_0, \quad 0 < \alpha < 1, \tag{6} $$

whose exact solution is given as

$$ u(t) = u_0 E_{\alpha}(\lambda t^{\alpha}) \tag{7} $$

where $E_{\alpha}(\lambda t^{\alpha})$ is the Mittag-Leffler function, defined by

$$ E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} $$

Following [17], a suitable denominator function ϕ_k that depends on k, h, λ and α can be chosen as

$$ \phi_k(h, \lambda, \alpha) = \frac{\dfrac{E_{\alpha}(\lambda (hk)^{\alpha})}{E_{\alpha}(\lambda (hk-h)^{\alpha})} - 1}{\lambda}, \quad k = 1, \ldots, N. \tag{8} $$

The nonlinear term $u^2(t)$ is approximated at $t = t_k$, using a nonlinear approximation as $u^2(t_k) \approx u_{k-1} \cdot u_k$, yielding a nonstandard finite difference method

$$ \frac{u_k - u_{k-1}}{\phi_k(h, \lambda, \alpha)} \tag{9} $$

where

$$ \phi_k(h, \lambda, \alpha) = \frac{\left(\dfrac{E_{\alpha}(\lambda (hk)^{\alpha})}{E_{\alpha}(\lambda (hk-h)^{\alpha})} - 1 \right)}{\lambda}, $$

Equation (9) can be written as

$$ u_k = \frac{(1 + \lambda \phi_k) u_{k-1}}{1 + \lambda \phi_k u_{k-1}} $$

2.2. Constructing a nonstandard finite difference scheme from (2)

In this section, we construct a nonstandard finite difference for solving the fractional logistic model (1).

The corresponding difference equation for (2) is

$$\begin{vmatrix} \dfrac{u_{k-1}}{u_{k-1}-1} & E_\alpha(\lambda(h(k-1))^\alpha) \\ \dfrac{u_k}{u_k-1} & E_\alpha(\lambda(hk)^\alpha) \end{vmatrix} = \frac{u_{k-1}}{u_{k-1}-1}E_\alpha(\lambda(hk)^\alpha) - \frac{u_k}{u_k-1}E_\alpha(\lambda(h(k-1))^\alpha) = 0$$

By applying few algebraic processes, we obtain

$$u_k = u_{k-1}\left(\frac{E_\alpha(\lambda(hk)^\alpha)}{E_\alpha(\lambda(hk-h)^\alpha)}\right) - u_k u_{k-1}\left(\frac{E_\alpha(\lambda(hk)^\alpha)}{E_\alpha(\lambda(hk-h)^\alpha)}\right) + u_k u_{k-1}$$

Subtracting u_{k-1} from both sides, simplifying and multiplying both sides by λ, lead to the nonstandard finite difference scheme

$$\frac{u_k - u_{k-1}}{\dfrac{\left(\dfrac{E_\alpha(\lambda(hk)^\alpha)}{E_\alpha(\lambda(hk-h)^\alpha)} - 1\right)}{\lambda}} = \lambda u_{k-1}(1-u_k) \tag{10}$$

The nonstandard finite difference scheme for the logistic model, as obtained from the solution provided by equation (2) and represented by equation (10), is equivalent to the numerical scheme represented by Equation (9).

Therefore, the best nonstandard finite difference scheme for the fractional logistic differential equation (1) is obtained by creating the denominator function using the linear term and utilizing a nonlocal approximation for the nonlinear term. This approach expands upon the methodology of developing a nonstandard finite difference scheme for the first-order logistic model, as described in [16].

3. Convergence and stability of the nonstandard finite difference scheme

Now we prove the convergence of the nonstandard finite difference scheme (9). From Taylor expansion for fractional derivatives:

$$u(t_k) = u(t_{k-1}) + \frac{h^\alpha}{\Gamma(\alpha+1)}\, {}_0^C D_t^\alpha(u(t_{k-1})) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\, {}_0^C D_t^{2\alpha}(u(\xi)),\ \xi \in (t_{k-1}, t_k)$$

From which,

$${}_0^C D_t^\alpha(u(t_{k-1})) = \frac{u(t_k) - u(t_{k-1})}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} h^\alpha\, {}_0^C D_t^{2\alpha}(u(\xi)) = \frac{u(t_k) - u(t_{k-1})}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + O(h^\alpha)$$

At $t = t_{k-1}$,

$$
\begin{aligned}
{}_0^C D_t^{\alpha}(u(t_{k-1})) &= \lambda u(t_{k-1})(1-u(t_{k-1})) = \lambda u(t_{k-1}) - \lambda u^2(t_{k-1}) \\
&\Rightarrow \frac{u(t_k)-u(t_{k-1})}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}} - \lambda u(t_{k-1}) - \lambda u^2(t_{k-1}) = O(h^{\alpha}) \qquad (11)
\end{aligned}
$$

The local truncation error of (9) at the point $t = t_k$ denoted by LTE_k is:

$$
LTE_k = \frac{u(t_k)-u(t_{k-1})}{\phi_k(h,\lambda,\alpha)} - \lambda u(t_{k-1})(1-u(t_k)) \qquad (12)
$$

By adding and subtracting the term $\frac{u(t_k)-u(t_{k-1})}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}}$, we obtain

$$
LTE_k = \left(\frac{1}{\phi_k} - \frac{1}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}}\right)(u(t_k)-u(t_{k-1})) - \lambda u(t_{k-1}) + \lambda u(t_{k-1})(u(t_{k-1})+O(h))
$$

Now,

$$
\begin{aligned}
|LTE_k| &\leq \left|\left(\frac{1}{\phi_k} - \frac{1}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}}\right)(u(t_k)-u(t_{k-1}))\right| \\
&+ \left|\frac{u(t_k)-u(t_{k-1})}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}} - \lambda u(t_{k-1})(u(t_{k-1})+O(h))\right| \\
&= \frac{\left|\frac{h^{\alpha}}{\Gamma(\alpha+1)} - \phi_k\right|}{\phi_k}\left|\frac{(u(t_k)-u(t_{k-1}))}{\frac{h^{\alpha}}{\Gamma(\alpha+1)}}\right| + |O(h^{\alpha})| + |O(h)| \\
&= \frac{\left|\frac{h^{\alpha}}{\Gamma(\alpha+1)} - \phi_k\right|}{\phi_k}\left|O(h^{\alpha})\right| + |O(h^{\alpha})| + |O(h)|) \qquad (13)
\end{aligned}
$$

From equation (13), we notice that $|LTE_k| \to 0$ as $h \to 0$, which proves the consistency of the numerical scheme (9).

To prove the stability of the numerical scheme (9), let $e_k = u(t_k) - u_k$, we prove that $e_k \to 0$ as $k \to \infty$. The numerical scheme (9) can be written as:

$$
u_k = u_{k-1} + \lambda\phi_k u_{k-1} - \lambda\phi_k u_{k-1} u_k \qquad (14)
$$

Substituting $u(t_k)$ instead of u_k in (14), we obtain

$$
u(t_k) = u(t_{k-1}) + \lambda\phi_k u(t_{k-1}) - \lambda\phi_k u(t_{k-1})u(t_k) \qquad (15)
$$

By subtracting (14) from (15) and substituting $e_k = u(t_k) - u_k$, we obtain

$$
\begin{aligned}
e_k &= e_{k-1} + \lambda\phi_k e_{k-1} - \lambda\phi(u(t_k)u(t_{k-1}) - u_k u_{k-1}) \\
&= (1+\lambda\phi_k)e_{k-1} - \lambda\phi_k(u(t_k)u(t_{k-1}) - u(t_k)u_{k-1} + u(t_k)u_{k-1} - u_k u_{k-1})
\end{aligned}
$$

Hence,

$$(1+\lambda\phi_k u_{k-1})e_k = (1+\lambda\phi_k(1-u(t_k)))e_{k-1}$$

Now,

$$e_k = \frac{1+\lambda\phi_k(1-u(t_k))}{1+\lambda\phi_k u_{k-1}} e_{k-1} = \left[\prod_{n=1}^{k} \frac{1+\lambda\phi_n(1-u(t_n))}{1+\lambda\phi_n u_{n-1}}\right] e_0 \tag{16}$$

Considering equation (16), we observe that as the value of $k \to \infty$, $u(t_k) \to 1 = u_{eq}^2$. Consequently, $1-u(t_k) \to 0$, resulting in the expression

$$\left[\prod_{n=1}^{k} \frac{1+\lambda\phi_n(1-u(t_n))}{1+\lambda\phi_n u_{n-1}}\right] \to 0,$$

and therefore, $e_k \to 0$.

This demonstrates that the numerical scheme represented by equation (9) is unconditionally stable.

Since the nonstandard finite difference scheme (9) is both consistent and stable, it is convergent. Hence, we have the following main theorem.

Theorem 1 The nonstandard finite difference scheme described by equation (9) is convergent of order $O(h^\alpha)$ and is unconditionally stable. □

4. Numerical results

Example 1 Consider the fractional logistic equation [7]

$${}_0^C D_t^\alpha u(t) = 0.5u(t)(1-u(t)), \qquad u(0) = 0.1, \qquad \alpha = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.0$$

The proposed analytical solution

$$u(t) = \frac{0.1E_\alpha(0.5t^\alpha)}{0.1E_\alpha(0.5t^\alpha)+0.9}$$

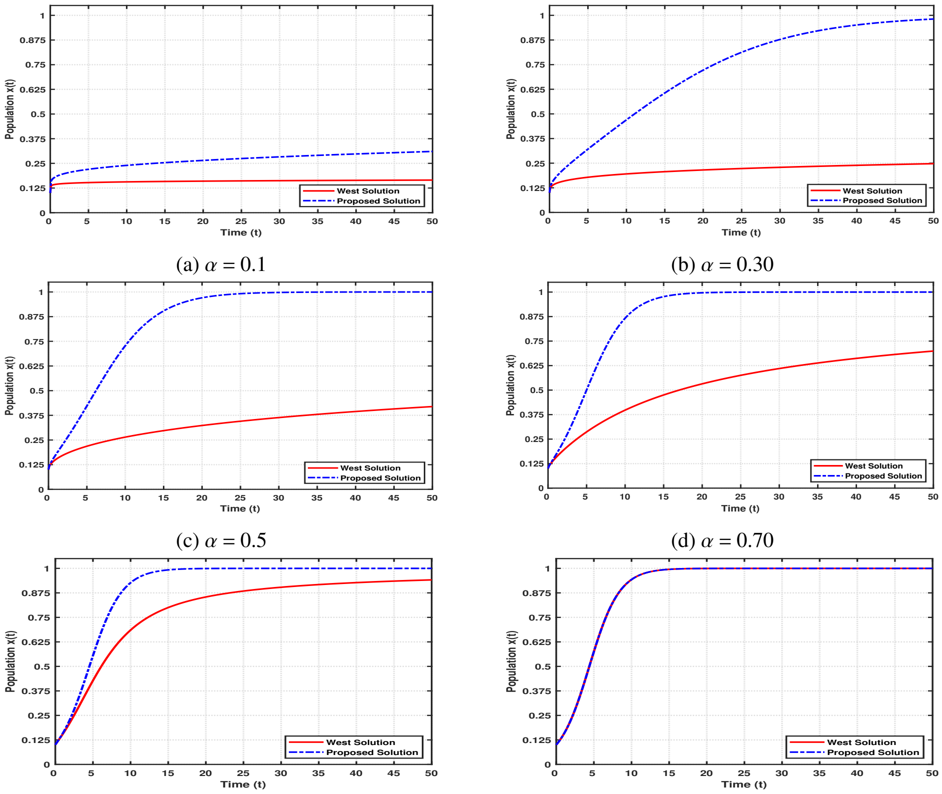
and West's solution

$$u(t) = \frac{0.1}{0.1+0.9E_\alpha(0.5t^\alpha)}$$

Figures 1a-1f depict a comparison between the solution obtained by (14) and West's solution for different values of α.

The nonstandard finite difference scheme is of the form:

$$\frac{u_k-u_{k-1}}{\phi_k(h)} = 0.5u_{k-1}(1-u_k), \text{ where } \phi_k(h) = \frac{\left(\dfrac{E_{\frac{1}{2}}(0.5(hk)^{\frac{1}{2}})}{E_{\frac{1}{2}}(0.5(hk-h)^{\frac{1}{2}})} - 1\right)}{0.5}$$



(a) $\alpha = 0.1$ (b) $\alpha = 0.30$

(c) $\alpha = 0.5$ (d) $\alpha = 0.70$

(e) $\alpha = 0.90$ (f) $\alpha = 1.00$

Fig. 1. Solutions of the fractional logistic model using the proposed and West solutions for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$

That is

$$u_k = \frac{(1 + 0.5\phi_k)u_{k-1}}{1 + 0.5\phi_k u_{k-1}}$$

The absolute errors of the nonstandard finite difference method for Example 1 are demonstrated in Table 1 using various step sizes $h = 1.0$, 1.0, 2.0, 5,0, 10.0, 20.0, 25.0, 50.0, and values of α $(0.1, 0.3, 0.5, 0.7, 0.9$ and $1.0)$.

Table 1. Computed infinity norm errors corresponding to different values of α, for Example 1

h	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
1.0	2.36e-16	1.11e-16	8.88e-16	6.66e-16	4.44e-16	4.44e-16
2.0	9.71e-17	2.50e-16	3.33e-16	3.33e-16	3.33e-16	3.33e-16
5.0	4.16e-17	1.11e-16	2.22e-16	1.11e-16	2.22e-16	2.22e-16
10.0	9.71e-17	2.78e-17	2.22e-16	2.22e-16	2.22e-16	2.22e-16
20.0	1.39e-17	2.78e-17	5.55e-17	1.11e-16	2.22e-16	1.11e-16
25.0	1.39e-17	2.78e-17	8.33e-17	1.11e-16	1.11e-16	2.22e-16
50.0	0.00	2.78e-17	1.11e-16	0.00	1.11e-16	2.22e-16

Table 1 demonstrates that most of the obtained absolute errors computed by the numerical scheme (9) for different step sizes lie within the machine precision, including large step sizes. This indicates that the nonstandard finite difference scheme is nearly exact and is highly stable.

Example 2 Consider the following fractional logistic equation [7]

$$ {}_{0}^{C}D_t^{\alpha} x(t) = 0.5x(t)(1 - x(t)),\ t \in [0, 30],\ x(0) = 0.85 $$

where $\alpha \in \{0.155, 0.499, 0.805, 0.955, 1.0\}$.

The proposed analytical solution is of the form:

$$ x(t) = \frac{x_0 E_\alpha(0.5t^\alpha)}{x_0 E_\alpha(0.5t^\alpha) + (1 - x_0)} $$

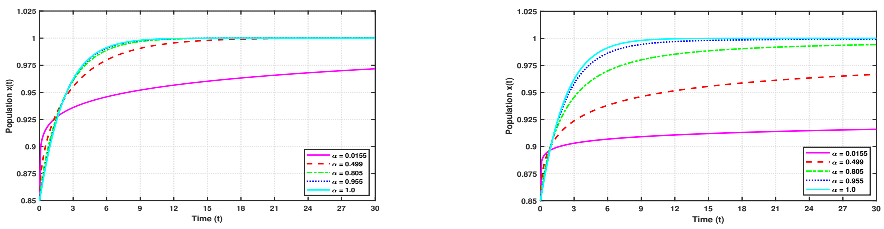
The nonstandard finite difference scheme is given by:

$$ \frac{x_k - x_{k-1}}{\phi_k} = 0.5x_{k-1}(1 - x_k) \text{ where } \phi_k = \frac{\dfrac{E_\alpha(0.5(hk)^\alpha)}{E_\alpha(0.5(hk-h)^\alpha)} - 1}{0.5} $$

That is

$$ x_k = \frac{(1 + 0.5\phi_k)x_{k-1}}{1 + 0.5\phi_k x_{k-1}} $$

Figures 2a and 2b illustrate the solutions of the fractional logistic model obtained by the proposed and West methods for $\alpha \in \{0.155, 0.499, 0.805, 0.955, 1.0\}$ in the interval $[0, 20]$. They demonstrate that the solution achieved by the proposed nonstandard finite difference scheme converges to the carrying capacity of the logistic model more rapidly than West's solution.



(a) Proposed method (b) West method

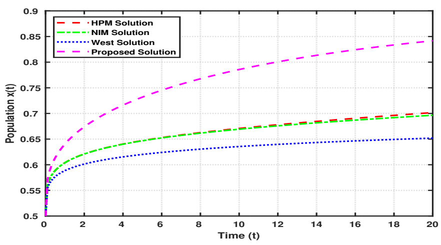
Fig. 2. Solutions of the fractional logistic model using the proposed and West solutions for $\alpha = 0.155, 0.499, 0.805, 0.955, 1.0$

Next, we solve Example 2 for different values of h, especially 0.5, 1.0, 1.5, 2.0, 3.0, 5.0, 6.0, 10.0 and 15.0, using the numerical method (14). The absolute errors generated by these computations are subsequently calculated and summarized in Table 2.

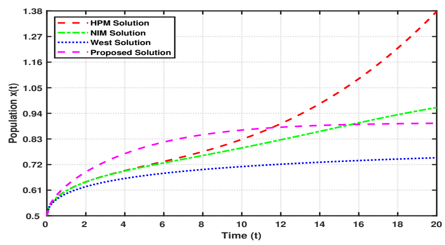
Table 2. Computed infinity norm errors corresponding to different values of α, for Example 2

h	$\alpha = 0.155$	$\alpha = 0.499$	$\alpha = 0.805$	$\alpha = 0.955$	$\alpha = 1.0$
0.50	5.551e-16	4.44e-16	3.33e-16	3.33e-16	4.44e-16
1.00	3.33e-16	2.22e-16	2.22e-16	3.33e-16	2.22e-16
1.50	5.55e-16	1.11e-16	2.22e-16	2.22e-16	3.33e-16
2.00	3.33e-16	2.22e-16	2.22e-16	2.22e-16	2.22e-16
3.00	2.22e-16	3.33e-16	1.11e-16	2.22e-16	2.22e-16
5.00	2.22e-16	1.11e-16	2.22e-16	1.11e-16	2.22e-16
6.00	1.11e-16	1.11e-16	1.11e-16	1.11e-16	2.22e-16
10.00	2.22e-16	2.22e-16	1.11e-16	1.11e-16	0.00
15.00	1.11e-16	2.22e-16	2.22e-16	1.11e-16	1.11e-16

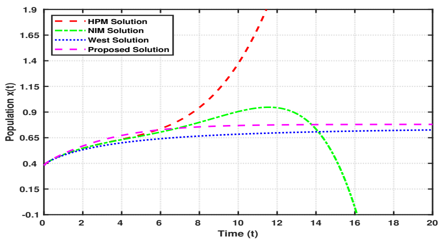
From Table 2, most of the obtained absolute errors obtained by the numerical scheme (9) lie within the machine precision even for large step sizes. This indicates that the nonstandard finite difference scheme is almost exact and is highly stable.



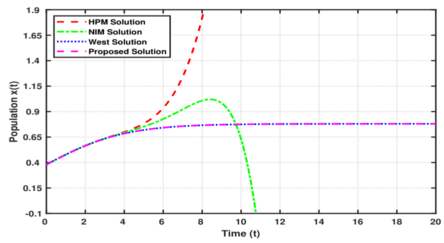
(a) $\alpha = 0.25$



(b) $\alpha = 0.50$



(c) $\alpha = 0.75$



(d) $\alpha = 1.00$

Fig. 3. Comparisons between the solution of the fractional logistic model using the HPM, NIM, West and the proposed method for $\alpha = 0.25, 0.5, 0.75, 1.0$

In order to compare the solutions produced by the proposed approach with West's method, the homotopy perturbation method (HPM), and the new iterative method (NIM), the solutions of Example 2 obtained by these methods are illustrated in Figure 3. Figure 3 shows that each of the four methods yields bounded solutions for Example 2 when considering a short time period and specific values of α. However, the HPM and NIM lack horizontal asymptotes, resulting in solutions that exceed the carrying capacity of the model for longer periods of time or high values of α. However, it is apparent that both the two solution forms of West, and the suggested solution are bounded, with the proposed form reaching the model's carrying capacity more rapidly than West's solution. The two solutions of West and the proposed are in complete accord for $\alpha = 1.0$.

5. Conclusions

This work investigated the solution of the fractional logistic model. In accordance with West's concept as presented in [14], a secondary solution formula for the logistic model is provided. This formula replaces the exponential function with the Mittag--Leffler function in the solution of the first-order logistic model. Subsequently, it derived two nonstandard finite difference methods to solve the fractional logistic model. One approach utilized Mickens's principles to develop a nonstandard finite difference scheme, assuming that the analytical solution is unknown. The second approach is based on the proposed analytical solution of the fractional logistic model. Interestingly, the two nonstandard finite difference methods are identical.

The convergence of the nonstandard finite difference scheme is established by demonstrating its consistency and stability. It has been proved that the proposed numerical scheme is unconditionally stable.

This paper has three major contributions. The first is that it presents an alternate solution formula for the fractional logistic model, derived from the solution form of the classical logistic model. The second is that it provides an exact finite difference scheme for solving the fractional logistic model. The proposed numerical method exhibits unconditional stability and is capable of functioning with significantly large step sizes while maintaining stability and dynamical consistency.

The performance of the proposed nonstandard finite difference scheme is illustrated using two numerical examples chosen from the literature.

Figures 1a-1f demonstrate that when the fractional order α is small, both West's and the proposed solutions of the fractional logistic model exhibit a very slow approach towards reaching their carrying capacity. By raising the value of α, both solutions converge to the carrying capacity more quickly. The proposed nonstandard finite difference scheme converges to the carrying capacity faster than West's solution. Similar behaviours are noticed in Figures 2a-2b and 3a-3d.

We can see from Figures 3a-3d that the proposed nonstandard finite difference method and West's solution work better than the HPM method, and the NIM methods. The HPM and NIM solutions are not bounded by the carrying capacity of the fractional logistic model, but the NSFDM and West's solutions are.

Tables 1 and 2 demonstrate that the solutions obtained from the proposed nonstandard finite difference method (9) closely approximate the solution (2). In Table 1, step sizes are taken in the range from $h = 1$ to $h = 50$. In Table 2, step sizes are taken in the range from $h = 0.5$ to $h = 15$. The majority of the absolute errors obtained in the two tables fall within the limits of machine precision, indicating that the proposed NSFDM is almost exact.

Finally, the results obtained in the two examples agree with the theoretical results stated in Section 5.

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