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THE DIVERGENCE THEOREM FOR VECTOR-VALUED FORMS

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Abstract. The divergence theorem for a vector valued form of any degree $p = 0,1,\dots,n$ is derived on a Riemannian manifold M of dimension n with a nonempty boundary ∂M. In analogy to the classic theorem, it relates the integration over M to the integration over ∂M. In the particular case $p = 0$, when the vector valued form reduces to a vector field, the theorem reduces to the classic divergence theorem.

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1. Introduction

Let M be an oriented compact Riemannian manifold of dimension n with a smooth nonempty boundary ∂M. Let $g = \langle , \rangle$ be the Riemannian metric (scalar product) on M. A particular example of such M is a bounded domain in $\mathbb{R}^n$ with a smooth boundary and with the standard scalar product. Though a general Riemannian manifold looks locally like $\mathbb{R}^n$, the scalar product on M may vary when passing from point to point. More information on manifolds and their geometric structures, especially the Riemannian one, may be found in [1-5].

The classic *divergence theorem* is one of the most important results of modern calculus. It relates the integration over a domain to the integration over the boundary. It has many versions in the dependence on analytic and geometric structures of M. In the case of a Riemannian manifold M with a nonempty boundary ∂M, it can be stated as follows (cf. eg. [1] Theorem 5-8):

Theorem 1 *If X is a (smooth) vector field on M then*

$$\int_M \operatorname{div} X \, \Omega_M = \int_{\partial M} \langle X, N \rangle \, \Omega_{\partial M}. \tag{1}$$

Ω_M is the volume form of M determined uniquely by the Riemannian metric and the orientation, N is the outer vector field normal to the boundary, $\Omega_{\partial M}$ is the volume form of ∂M compatible with Ω_M and $\operatorname{div} X$ is the divergence of X. For more details we refer to Section 3.

The theorem has a wide spectrum of surprising applications. The uniqueness of the theorem follows from its vector character. Vector fields play namely an essential role in mathematical, physical and engineering applications (see [2, 3, 5]). The examples are: the dynamical systems, fluid mechanics, electrostatics, the statics and the dynamics of gases, liquids and solid bodies. If a manifold M represents a mathematical or physical object, a vector field X represents forces acting on the object. In physics or engineering practice, the values of the force can usually be observed and measured only at the boundary. The divergence theorem gives then information on "what is happening" inside. An example of surprising applications is a proof of the known from physics Archimedean principle on hydro-static buoyancy (see exercise 5-36 in [1]). Another surprising application is a simple and elegant proof of the n-dimensional Pythagorean theorem for simplexes [6].

The aim of the paper is to generalize the classic divergence theorem for vector fields to an analogous theorem for vector valued forms of any degree $p = 0, 1, \dots, n$, i.e. to the sections of bundle $\Lambda^p \otimes T$, where Λ^p is the bundle of (scalar) forms of degree p on M (theorems 2 and 3 in Section 6). Vector fields, i.e. the sections of the tangent bundle T, can be regarded as vector valued forms of degree zero. Obviously, in this case ($p = 0$), the main result: Theorem 2, reduces to the classic version: Theorem 1.

The operators of gradient and divergence in the bundle of forms are discussed in Section 5. Interesting examples of vector forms are the gradients of characteristic forms on foliated manifolds. A *characteristic form* $\chi_{\mathscr{F}}$ (see [7] for the exact definition) of a p-dimensional foliation $\mathscr{F}$ on a Riemannian manifold M is the unique p-form on M that arises by gluing together the volume forms of leaves of $\mathscr{F}$. The gradients of such forms encode information on the geometry of foliation (cf. [7], part II). This will be used in Section 7 for constructing examples.

All manifolds and mappings are assumed to be smooth, i.e. of class C^∞. For any bundle E over M, the space of sections of E is denoted by $C^\infty(E)$.

For the notions of manifolds, bundles, vector and tensor fields, forms and also for the tensor and exterior products discussed in this paper, we refer to [2, 4, 5]. We also refer to [8] where the operators of the gradient and the divergence on vector valued forms are discussed in detail.

2. Forms, vector forms, exterior products

Let M be an oriented Riemannian manifold, possibly also with a boundary, $\dim M = n$, with a scalar product (Riemannian metric) $\langle , \rangle_g$ in the tangent bundle T. The metric can naturally be extended to the cotangent bundle T^*. The extension will be denoted by the same symbol.

Let $\Lambda^p = \Lambda^p T^*$ be the bundle of (scalar) forms of degree p on M. The exterior product $\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q}$ is defined by

$$(\varphi \wedge \psi)(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = \sum_{\sigma \in \mathrm{sh}(p,q)} \mathrm{sign}\sigma \varphi(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \psi(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$$

for $v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q} \in T$, where $\mathrm{sh}(p,q)$ is the set of all such permutations $\sigma \in S_{p+q}$ that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

The *scalar product* of two simple *p-forms* $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_p$ and $\psi = \psi_1 \wedge \cdots \wedge \psi_p$ is defined as the determinant:

$$\langle \varphi_1 \wedge \cdots \wedge \varphi_p, \psi_1 \wedge \cdots \wedge \psi_p \rangle_{\Lambda^p} = \sum_{\sigma \in S_p} \mathrm{sign}\sigma \langle \varphi_1, \psi_{\sigma_1} \rangle_g \cdots \langle \varphi_p, \psi_{\sigma_p} \rangle_g, \tag{2}$$

where $\varphi_1, \ldots, \varphi_p, \psi_1, \ldots, \psi_p \in \Lambda^1 = T^*$, and then extended to the space of all p-forms be linearity.

Consider also the bundle of *vector p-forms*: $\vec{\Lambda}^p = \Lambda^p \otimes T$ for $p = 1, \ldots, n$ and $\vec{\Lambda}^0 = T$ (zero vector forms are).

The exterior product is extended onto the following pairs of forms:

$$\wedge : \begin{cases} \Lambda^p \times \vec{\Lambda}^q \longrightarrow \vec{\Lambda}^{p+q} \\ \vec{\Lambda}^p \times \Lambda^q \longrightarrow \vec{\Lambda}^{p+q} \\ \vec{\Lambda}^p \times \vec{\Lambda}^q \longrightarrow \Lambda^{p+q} \end{cases}$$

according to the following rules that enable the natural understanding in all the other cases:

$$\varphi \wedge (\psi \otimes Y) = \varphi \wedge \psi \otimes Y \tag{3}$$

$$(\varphi \otimes X) \wedge \psi = \varphi \wedge \psi \otimes X \tag{4}$$

$$(\varphi \otimes X) \wedge (\psi \otimes Y) = \varphi \wedge \psi \cdot \langle X, Y \rangle_g \tag{5}$$

The defined product unifies all the possible actions in the sets of both scalar and vector forms within the only one symbol $\wedge$.

The scalar products in T and Λ^p define the natural scalar product in $\vec{\Lambda}^p$ by

$$\langle \varphi \otimes X, \psi \otimes Y \rangle_{\vec{\Lambda}^p} = \langle \varphi, \psi \rangle_{\Lambda^p} \cdot \langle X, Y \rangle_g.$$

From now on, all the scalar products will be denoted simply by $\langle , \rangle$.

Scalar product of forms of mixed degree like

$$\langle \varphi \otimes X, Y \rangle = \varphi \cdot \langle X, Y \rangle \text{ or } \langle \varphi \otimes X, \psi \rangle = \langle \varphi, \psi \rangle \cdot X \tag{6}$$

will be also accepted.

3. The volume forms of M and ∂M

The Riemannian structure and the orientation define on M a unique form $\Omega_M \in C^\infty(\Lambda^n)$ characterized locally by the condition $\Omega_M(e_1,\ldots,e_n) = 1$ for any local positively oriented orthonormal frame $e_1,\ldots,e_n$ on M. The form is called the *volume form* of M.

By taking the dual frame: $e_1^*,\ldots,e_n^*$, we get that locally $\Omega_M = e_1^* \wedge \ldots \wedge e_n^*$.

Recall (cf. [5]) that the classical *Hodge star*, $\star : \Lambda^p \longrightarrow \Lambda^{n-p}$, is the linear operator defined by

$$\varphi \wedge \star\psi = \langle \varphi, \psi \rangle \Omega_M \tag{7}$$

for $\varphi, \psi \in \Lambda^p$.

Extend the Hodge star operator to the bundle of vector forms, $\star : \vec{\Lambda}^p \longrightarrow \vec{\Lambda}^{n-p}$, by:

$$\star(\varphi \otimes X) = (\star\varphi) \otimes X.$$

Then, in analogy to (7), we have that for $\Phi \in \vec{\Lambda}^p,\ \psi \in \Lambda^p$,

$$\Phi \wedge \star\psi = \langle \Phi, \psi \rangle \Omega_M. \tag{8}$$

To construct the volume form of ∂M, compatible with the volume form of M, we will work in a local coordinate system $x = (y,r)$ on M near ∂M, such that: $y = (y_1,\ldots,y_{n-1})$ is a local coordinate system on ∂M and r is the normal distance to the boundary. Then $\partial M = \{x : r(x) = 0\}$ and $\dfrac{\partial}{\partial r}$ is the inward unit normal vector. The vector fields $\dfrac{\partial}{\partial x_1},\ldots,\dfrac{\partial}{\partial x_{n-1}}, \dfrac{\partial}{\partial x_n} = \dfrac{\partial}{\partial r}$ constitute a local frame in the tangent bundle TM and the 1-forms: $dx_1,\ldots,dx_{n-1}, dx_n = dr$ constitute the dual frame in the cotangent bundle T^*M. Normalize the choice of coordinate by requiring that the curves $x(r) = (y_0, r)$ are unit speed geodesics for r being small enough and for any $y_0 \in \partial M$. Since M is compact, the inward geodesic flow identifies a neighborhood of ∂M in M with the collar $\partial M \times [0,\delta)$ for some $\delta > 0$. The collaring gives a splitting: $TM = T(\partial M) \oplus T\mathbb{R}$ and a dual splitting: $T^*M = T^*(\partial M) \oplus T^*\mathbb{R}$.

Let $N = -\dfrac{\partial}{\partial r}$ in this coordinate system. Then N is a smooth vector field on a neighborhood of ∂M, so, in particular, on ∂M. We will call N the field of *outer vectors normal to the boundary*.

The *volume form* of ∂M is defined by

$$\Omega_{\partial M} = \iota_N \Omega_M. \tag{9}$$

It determines then the orientation of ∂M compatible with the orientation of M. The operation $\iota_X : \Lambda^p \to \Lambda^{p-1}$ of *substitution* of a vector field X to a p-form φ is defined by

$$(\imath_X\varphi)(X_1,\ldots,X_{p-1}) = \varphi(X,X_1,\ldots,X_{p-1}) \text{ for } p>0 \tag{10}$$

and $\imath_X\varphi = 0$ for $p=0$.

The outer normal field N defines near the boundary a unique decomposition of any vector p-form Φ onto its tangent and normal parts:

$$\Phi = \Phi^{tan} + \Phi^{nrm}, \text{ where } \Phi^{nrm} = \langle \Phi, N\rangle \otimes N \text{ and } \Phi^{tan} = \Phi - \Phi^{nrm}. \tag{11}$$

It is obvious that then, for any scalar p-form φ,

$$\langle \Phi^{tan}, \varphi\otimes N\rangle = 0. \tag{12}$$

4. Differential operators

Let ∇ be the Levi-Civita *covariant derivative* in the tangent bundle T, $\nabla : C^\infty(T) \longrightarrow C^\infty(T^*\otimes T)$ i.e. the unique first-order linear differential operator in T such that, for any function f on M, $\nabla(fX) = df\otimes X + f\nabla X$ and with the property of being *metric* and *torsion free* (cf. [2] Sect. 2.7).

The covariant derivative ∇ can be extended naturally to the cotangent bundle T^*, next, by the Leibniz rule, to any tensor bundle on M, in particular to $T^{*p} = T^{*\otimes p}$, $p = 1,2,\ldots$. Finally, it can be restricted to subbundles so, in particular, to the bundles Λ^p, or $\vec{\Lambda}^p$.

With the convention: $(\nabla\varphi)(X,X_1,\ldots,X_k) = (\nabla_X\varphi)(X_1,\ldots,X_k)$, the covariant derivative ∇ in the bundle T^{*p} may be treated as a map: $\nabla : C^\infty(T^{*p}) \longrightarrow C^\infty(T^{*p+1})$ and therefore can be alternated.

Since ∇ is torsion free, the operator $\mathrm{d} : C^\infty(\Lambda^p) \longrightarrow C^\infty(\Lambda^{p+1})$, $p = 0,1,\ldots,n$, of derivation of scalar forms can be expressed as the alternation of ∇:

$$(\mathrm{d}\varphi)(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^p\nabla\varphi(X_i,X_1,\ldots,\hat{X}_i,\ldots,X_{p+1}). \tag{13}$$

The hat over a term means that the term is omitted. Note that for $p=0$, $\mathrm{d}\varphi = d\varphi$ is just the usual differential of function φ.

The operator d extends to operator $\mathrm{d} : C^\infty(\vec{\Lambda}^p) \longrightarrow C^\infty(\vec{\Lambda}^{p+1})$ of derivation of vector p-forms, as follows:

$$\mathrm{d}(\varphi\otimes X) = \mathrm{d}\varphi\otimes X + (-1)^p\varphi\wedge\nabla X, \text{ for } p = 1,\ldots,n,$$

and

$$\mathrm{d}X = \nabla X, \text{ for } p=0.$$

Here, ∇X is treated as a vector 1-form.

Note that with our rules for the exterior multiplication (cf. (3)-(5)) we have – in each case: $\varphi\in C^\infty(\Lambda^p)$ (or $C^\infty(\vec{\Lambda}^p)$) and $\psi\in C^\infty(\Lambda^q)$ (or $C^\infty(\vec{\Lambda}^q)$) – the same rule saying that d is antiderivation:

$$\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi + (-1)^p \varphi \wedge \mathrm{d}\psi.$$

Now (in analogy to [7], see also [8]), define two operators of order zero that will be used in the construction of the two main operators: the gradient and the divergence.

Definition 1 $\mathrm{j} : \Lambda^p(E) \longrightarrow \vec{\Lambda}^{p-1}(E),\ p = 1,\ldots,n$, is the linear operator defined by: $\mathrm{j}\,\varphi \wedge X = \iota_X\,\varphi$ for $\varphi \in \Lambda^p$, $X \in \vec{\Lambda}^0$ and $\mathrm{j}\,\varphi = 0$ for $\varphi \in \Lambda^0$. □

One can easily see (cf. eg. [7, 8]) that j has the following local expression (Here and afterwards, $e_1,\ldots,e_n \in T$ will be a local orthonormal frame, and $e_1^*,\ldots,e_n^* \in T^*$ is the dual one):

$$\mathrm{j}(e_{i_1}^* \wedge \ldots \wedge e_{i_p}^*) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1}^* \wedge \ldots \wedge e^*{}_{i_{k-1}} \wedge e^*{}_{i_{k+1}} \wedge \ldots \wedge e_{i_p}^* \otimes e_{i_k}.$$

Definition 2 $\mathrm{tr} : \vec{\Lambda}^p \longrightarrow \Lambda^{p-1},\ p = 1,\ldots,n$, is the linear operator defined by: $(\mathrm{tr}\,\Phi)(X_1,\ldots,X_{p-1}) = \mathrm{tr}\,(X \to \Phi(X,X_1,\ldots,X_{p-1}))$ and $\mathrm{tr}\,\Phi = 0$ for $p = 0$. □

Note that with the use of a local orthonormal frame $e_1,\ldots,e_n$ of T the operator tr may be defined equivalently by: $\mathrm{tr}\,\Phi = \sum_{i=1}^{n} \left\langle \iota_{e_i}\Phi, e_i \right\rangle$. In particular, at the boundary with the local frame $e_1,\ldots,e_{n-1}$ tangent to ∂M and $e_n = N$, we have:

$$\mathrm{tr}\,\Phi = \mathrm{tr}\,\Phi^{tan} + \mathrm{tr}\,\Phi^{nrm} = \sum_{i=1}^{n-1} \left\langle \iota_{e_i}\Phi^{tan}, e_i \right\rangle + \left\langle \iota_N \Phi^{nrm}, N \right\rangle. \tag{14}$$

5. The gradient and the divergence

Consider now, in analogy to [7], two first-order linear differential operators grad and div on forms (see also [8]). The first one acts on usual forms and generalizes the classical gradient acting on functions (scalar 0-forms), while the other one acts on vector forms and generalizes the classical divergence acting on vector fields (vector 0-forms).

Definition 3 The *gradient* is the differential operator, $\mathrm{grad} : C^\infty(\Lambda^p) \longrightarrow C^\infty(\vec{\Lambda}^p)$ $p = 0,1,\ldots,n$, defined by:

$$\mathrm{grad} = \mathrm{j}\,\mathrm{d} + \mathrm{d}\,\mathrm{j}. \tag{15}$$

Note that if, in particular, φ is a 0-form, i.e. a function on M, then

$$\mathrm{grad}\,\varphi = \mathrm{j}\,\mathrm{d}\,\varphi = (\mathrm{d}\,\varphi)^\flat \tag{16}$$

where, for any 1-form (a covector) ω, $\omega^\flat$ is the vector metrically dual to ω in the sense that $\omega(X) = \left\langle \omega^\flat, X \right\rangle$, for any vector X. In this particular case, the gradient from (15) coincides with the usual gradient on functions.

Proposition 1 *(see [7], Theorem 1 or [8], Theorem 1) The operator* grad *is a derivation, i.e. for $\varphi \in \Lambda^p$ and $\psi \in \Lambda^p$:*

$$\text{grad}\,(\varphi \wedge \psi) = \text{grad}\,(\varphi) \wedge \psi + \varphi \wedge \text{grad}\,\psi. \tag{17}$$

Proposition 2 *(see [7], Theorem 2 or [8], Theorem 2)*

$$\star\,\text{grad} = \text{grad}\,\star. \tag{18}$$

Example 1 If, near $x \in M$, $\varphi = f\, e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}$ for a function f and a local orthonormal frame $e^*_1, \ldots, e^*_n$ which is *normal* at x in the sense that $\nabla e^*_i = 0,\ i = 1, \ldots, n$, at x, then

$$\text{grad}\,\varphi = e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes \text{grad} f \quad \text{at} \;\; x. \tag{19}$$

PROOF Fix $x \in M$. By (13) and (15), $\nabla e^*_i = 0$ at x implies that $\text{grad}\, e^*_i = 0$ at x. Now use Proposition 1 and equation (16). ■

Definition 4 The *divergence* is the differential operator, div : $\vec{\Lambda}^p(E) \longrightarrow \Lambda^p(E)$, $p = 0, 1, \ldots, n$, defined by:

$$\text{div} = \text{tr}\,\text{d} + \text{d}\,\text{tr}. \tag{20}$$

Note that if, in particular, X is a vector field, i.e. a vector 0-form, then $\text{tr} X = 0$ and then, by (20),

$$\text{div} X = \text{tr}\,\text{d} X = \sum_{i=1}^{n} \left\langle \nabla_{e_i} X\,,\, e_i \right\rangle.$$

so, $\text{div} X$ coincides with the usual divergence of vector field X.

Proposition 3 *(see [7], Theorem 1 or [8], Theorem 1) For $\varphi \in \Lambda^p$ and $\Psi \in \vec{\Lambda}^p$:*

$$\text{div}\,(\varphi \wedge \Psi) = \text{grad}\,\varphi \wedge \Psi + \varphi \wedge \text{div}\,\Psi. \tag{21}$$

Example 2 Similarly, as in Example 1, we have, in a local orthonormal frame, normal at x, that

$$\text{div}(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes X) = e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \cdot \text{div} X \quad \text{at} \;\; x. \tag{22}$$

6. The divergence theorem for vector forms

Theorem 2 (The divergence theorem for vector forms) *Let M, $\dim M = n$, be a compact oriented Riemannian manifold of dimension n with a nonempty boundary ∂M. Let $p \in \{0,\dots,n\}$.*

For any vector form Φ of degree p on M and a scalar form ψ of the same degree:

$$\int_{\partial M} \langle \Phi, \psi \otimes N \rangle \, \Omega_{\partial M} = \int_M \big\langle \operatorname{div} \Phi, \psi \big\rangle \, \Omega_M + \int_M \langle \Phi, \operatorname{grad} \psi \rangle \, \Omega_M. \tag{23}$$

Under the additional assumption that the form ψ is parallel, i.e. that $\nabla \psi = 0$ or, equivalently, that $\operatorname{grad} \psi = 0$, (23) reduces to

$$\int_{\partial M} \langle \Phi, \psi \otimes N \rangle \, \Omega_{\partial M} = \int_M \big\langle \operatorname{div} \Phi, \psi \big\rangle \, \Omega_M. \tag{24}$$

Remark 1 Note that in the particular case when $p = 0$, so when $\Phi = X$ is a vector field and a test form ψ of degree $p = 0$ (i.e. a function on M) is taken to be identically equal to 1, formula (24) reduces to:

$$\int_{\partial M} \langle X, N \rangle \, \Omega_{\partial M} = \int_M \operatorname{div} X \, \Omega_M,$$

i.e. to formula (1) of Theorem 1. That way our divergence theorem reduces to the classical divergence theorem for vector fields. In the particular case $n = 3$, the theorem is also known as the *Gauss theorem* and relates the *flow* of the vector field X through the closed surface ∂M to the global *charge* of the field within the domain M closed by the surface (see [5], Sect. 5.5.1). □

PROOF (OF THEOREM 2) Take any $\Phi \in \vec{\Lambda}^p$ and $\psi \in \Lambda^p$.

By Proposition 3 we have

$$\operatorname{div}(\Phi \wedge \star\psi) = \operatorname{div}\Phi \wedge \star\psi + \Phi \wedge \star \operatorname{grad}\psi,$$

so, by (8),

$$\operatorname{div}(\Phi \wedge \star\psi) = \big\langle \operatorname{div}\Phi, \psi \big\rangle \Omega_M + \big\langle \Phi, \operatorname{grad}\psi \big\rangle \Omega_M.$$

Integrating over M, we get

$$\int_M \operatorname{div}(\Phi \wedge \star\psi) = \int_M \big\langle \operatorname{div} \Phi, \psi \big\rangle \, \Omega_M + \int_M \langle \Phi, \operatorname{grad} \psi \rangle \, \Omega_M. \tag{25}$$

By (20), the integrand, on the left hand side of the last inequality, is equal to

$$\operatorname{div}(\Phi \wedge \star\psi) = \operatorname{tr} \mathrm{d}(\Phi \wedge \star\psi) + \mathrm{d} \operatorname{tr}(\Phi \wedge \star\psi).$$

The form $\Phi \wedge \star\psi$ is of maximal degree n, so, its differential vanishes, and the last relation reduces to

$$\operatorname{div}(\Phi \wedge \star\psi) = \mathrm{d}\,\operatorname{tr}(\Phi \wedge \star\psi).$$

Integrating over M, and using the Stokes theorem, we get

$$\int_M \operatorname{div}(\Phi \wedge \star\psi) = \int_{\partial M} \operatorname{tr}(\Phi \wedge \star\psi).$$

Substituting this to (25), we get

$$\int_{\partial M} \operatorname{tr}(\Phi \wedge \star\psi) = \int_M \big\langle \operatorname{div}\Phi, \psi \big\rangle\, \Omega_M + \int_M \big\langle \Phi, \operatorname{grad}\psi \big\rangle\, \Omega_M. \tag{26}$$

Now, evaluate the integral over ∂M. By (11),

$$\int_{\partial M} \operatorname{tr}(\Phi \wedge \star\psi) = \int_{\partial M} \operatorname{tr}(\Phi^{tan} \wedge \star\psi) + \int_{\partial M} \operatorname{tr}(\Phi^{nrm} \wedge \star\psi). \tag{27}$$

Since the integration is taken over ∂M then, by (12) and by (8), (6) and the second part of (14), the integrals in the right hand side of (27) are equal to

$$\int_{\partial M} \operatorname{tr}(\Phi^{tan} \wedge \star\psi) = 0$$

and

$$\int_{\partial M} \operatorname{tr}(\Phi^{nrm} \wedge \star\psi) = \int_{\partial M} \big\langle \Phi^{nrm}, \psi \otimes N \big\rangle\, \iota_N \Omega_M = \int_{\partial M} \big\langle \Phi, \psi \otimes N \big\rangle\, \iota_N \Omega_M,$$

respectively. By (9) we have then

$$\int_{\partial M} \operatorname{tr}(\Phi \wedge \star\psi) = \int_{\partial M} \big\langle \Phi, \psi \otimes N \big\rangle\, \Omega_{\partial M}. \tag{28}$$

By (25) and (28) the assertion (23) follows. ∎

For $M = \mathbb{R}^n$ and the Euclidean structure in $\mathbb{R}^n$ the formula (23) simplifies notedly.

In the canonical coordinate system of coordinates in $\mathbb{R}^n$: $x = (x_1, \dots, x_n)$, vector fields $e_1 = \dfrac{\partial}{\partial x^1}, \dots, e_n = \dfrac{\partial}{\partial x^n}$ and one forms $e_1^* = dx_1, \dots, e_1^* = dx_n$ constitute global orthonormal frames for the tangent and the cotangent bundles over $\mathbb{R}^n$, respectively. The bases are dual to each other, i.e.

$$dx_1\Big(\frac{\partial}{\partial x_j}\Big) = \delta_{ij}. \tag{29}$$

For any fixed p, any vector form $\Phi \in \vec{\Lambda}^p$ can be written in these bases as

$$\Phi = \sum_{i=1}^{n} \sum_{i_1 < \dots < i_p} \Phi_i^{i_1 \dots 1_p}\, dx_{i_1} \wedge \dots \wedge dx_{i_p} \otimes \frac{\partial}{\partial x_i}. \tag{30}$$

where $\Phi_i^{i_1 \dots 1_k}$ are functions – the coefficients of Φ.

Fix an increasing sequence of indices $1 \leq i_1 < \cdots < i_k \leq n$ and take

$$\psi = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{31}$$

as a test form. Obviously (cf. (19)), $\operatorname{grad} \psi = 0$. We obtain then the following version of the divergence theorem for vector forms of degree p in $\mathbb{R}^n$.

Theorem 3 *Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary ∂D. Fix $p \in \{1, \ldots, n\}$. Then, for any vector p-form $\Phi \in \vec{\Lambda}^p(D \cup \partial D)$ of shape (30) and any sequence of indices $1 \leq i_1 < \cdots < i_p \leq n$,*

$$\int_{\partial D} \sum_{i=1}^{n} \Phi_i^{i_1 \ldots i_p} N_i \, \Omega_{\partial D} = \int_D \sum_{i=1}^{n} \frac{\partial \Phi_i^{i_1 \ldots i_p}}{\partial x_i} \, \Omega_D \tag{32}$$

were $N = \sum_{i=1}^{n} N_i \dfrac{\partial}{\partial x_i}$ is the field of outer vectors normal to the boundary. □

PROOF Take – in (24) – Ψ and ψ of shape (30) and (31), respectively and apply (22). ■

7. Examples

Example 3 Let M = $\mathbb{R}^n \setminus \{0\}$ be foliated by two mutually orthogonal foliations: $\mathscr{F}_{sph}$ and $\mathscr{F}_{rad}$. The first with leaves being concentric $(n-1)$-dimensional spheres S_r around the origin with the radius $r \in (0, \infty)$ and the other with leaves being 1-dimensional radii R_p, i.e. half-straight lines out of the origin through $p \in S_1$ (= the unit sphere). Note that for the particular case $n = 3$, such foliations represent in physics the equipotential surfaces and the line of forces, respectively, for the electric field in $\mathbb{R}^3$ generated by a single charge at the origin. In the general case, the characteristic forms of the considered foliations are:

$$\chi_{\mathscr{F}_{sph}}(x_1, \ldots, x_n) = \frac{1}{r} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \tag{33}$$

and

$$\chi_{\mathscr{F}_{rad}}(x_1, \ldots, x_n) = \frac{1}{r} \sum_{i=1}^{n} x_i \, dx_i.$$

It is clear that each of the characteristic forms restricted to any leaf of its foliation is the volume form of this leaf. One can also calculate that, by (19),

$$\operatorname{grad} \chi_{\mathscr{F}_{sph}} = \frac{1}{r} \big(\mathrm{j}\, \Omega_M \big)^{tan} \tag{34}$$

and

$$\operatorname{grad}\chi_{\mathscr{F}_{rad}} = \frac{1}{r}\big(\mathrm{id}_{\mathrm{T(M)}}\big)^{\mathrm{nrm}},$$

where $\mathrm{id}_{\mathrm{T}}(\mathrm{M})$ is the identity automorphism of the tangent bundle of M treated here as a vector 1-form.

By Theorem 7 in [7], foliation $\mathscr{F}_{sph}$ is *parallel* and foliation $\mathscr{F}_{rad}$ is *geodesic*, what is rather obvious in this case. For the notions of parallel and geodesic foliations, see [7]. □

Example 4 Consider the spherical Ring $R_{r_1r_2} \subset \mathbb{R}^n$ being the domain contained between two spheres S_{r_1} and S_{r_2}, $r_1 < r_2$. Apply the divergence theorem to the vector $(n-1)$-form $\operatorname{grad}\Phi = \chi_{\mathscr{F}_{sph}}$ where $\Phi = \chi_{\mathscr{F}_{sph}}$ is given by (33). Then

$$\int_{R_{r_1r_2}} \big\langle \operatorname{div}\Phi, \psi\big\rangle\, \Omega_{R_{r_1r_2}} = 0 \tag{35}$$

for any test $(n-1)$-form ψ in $R_{r_1r_2}$ with $\operatorname{grad}\psi = 0$, so, in particular, for each form $\psi = dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, $i = 1, \ldots, n$.

Indeed, by formula (24) in Theorem 2, the integral in (35) is equal to the integral over the boundary with the integrand: $\big\langle \Phi, \psi \otimes N \big\rangle$ which, by (34), is equal to $\big\langle \Phi^{tan}, \psi \otimes N \big\rangle$, and this is equal to 0 by (12). □

8. Conclusions and possible applications

The importance of the divergence type theorems – especially when possible applications are considered – comes from the fact that they deal with vector fields. Vector fields namely represent forces acting on a physical body. In engineering practice, the values of such forces can usually be measured only at the boundary. The divergence theorem gives then some information on what is going inside the body.

A possible inhomogeneity of the material of the body or its composition from different layers are the cases that can easily be realized within the Riemannian structure on M. The inhomogeneity can namely be carried out throughout a correction of the Riemannian metric. Layers can be represented by a foliation of M.

Interesting examples of vector forms are the gradients of characteristic forms of foliations considered in Section 7. It is known that the gradients of these forms encode some information on the geometry of foliation (see [7], part II, Theorems 7 and 8). In analogy to Example 4, one can also consider applications of the divergence theorem to foliated manifolds in more general and advanced cases.

Another important example are automorphisms of the tangent bundle that can be treated as the vector forms of degree one, i.e. as the sections of the bundle $T^* \otimes T$.

Finally note, that the vector character of the divergence theorem allows setting up systems of nontrivial boundary conditions when solving boundary value problems for differential operators on M. The natural decomposition of a vector form near the boundary ∂M onto its tangent and normal parts, described in Section 3, enables defining a variety of such conditions. For example, in the theory of elastic body, four nontrivial but natural boundary conditions: Dirichlet, Absolute, Relative and Neumann are considered and investigated (cf. [9, 10]).

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References

[1] Spivak, M. (2018). *Calculus on Manifolds, a Modern Approach to Classical Theorems of Advanced Calculus*. CRC Press.

[2] Abraham, R., & Marsden, J. (1988). *Foundation of Mechanics*. Addison-Wesley Publishing Company, lnc.

[3] Nakahara, M. (2003). *Geometry, Topology and Physics*. Institute of Physics Publishing.

[4] De, D. (2022). *Introduction to Differential Geometry with Tensor Applications*. Modern Mathematics in Computer Science Wiley-Scrivener.

[5] Nguyen-Schaffer, H., & Schmidt, J.-P. (2017). *Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers*. Mathematical Engineering, Springer.

[6] Eifler, L., & Rhee, N. (2008). The n-dimensional Pythagorean theorem via the divergence theorem. *Amer. Math. Monthly*, *115*, 5, 456-457.

[7] Rummler, H. (1989). Differential forms, Weitzenböck formulae and foliations. *Publications Mathematiques*, *33*, 543-554.

[8] Klekot, A, & Pierzchalski, A. (2023). The garadients and the divergence for vector valued forms. *J. Appl. Math. Comput. Mech.*, *22*, 1, 27-41.

[9] Kozłowski, W., & Pierzchalski, A. (2008). Natural boundary value problems for weighted form Laplacians. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, *7*(5), 343-367.

[10] Pierzchalski A. (2017). Gradients: the ellipticity and the elliptic boundary conditions – a jigsaw puzzle. *Folia Mathematica*, *19*, 65-83.