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SYMMETRY ANALYSIS, EXACT SOLUTIONS AND CONSERVATION LAWS OF THE NONLINEAR TIME-FRACTIONAL SHARMA-TASSO-OLEVER EQUATION

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Abstract. The Lie symmetry analysis method (LSAM) is applied to obtain all Lie symmetries of the nonlinear time-fractional Sharma-Tasso-Olever equation. The studied fractional partial differential equation (FPDEs) is reduced to some fractional ordinary differential equations (FODEs), of which some exact solutions including the convergent power series solution are obtained. The dynamic behaviors of these exact solutions are presented graphically. In addition, the conservation laws for the obtained symmetries are constructed by Ibragimov's theory.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) are an important tool in the nonlinear modelling of phenomena of the nature. Finding solutions to NLPDEs can help people gain a deeper understanding of the phenomena behind the models. There are some recent works about the NLPDEs and various methods to solve them [1-4]. Among NLPDEs, the following nonlinear Sharma-Tasso-Olever equation is considered [5, 6]:

$$\frac{\partial u}{\partial t} + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \tag{1}$$

which elucidates the dynamics of waves exhibiting infinitesimal amplitudes propagating within a nonlinear dispersive medium and is used in many fields of physics, including relativistic physics, quantum field theory, fusion processes for solitons and fission, quantum relativistic atom theory and nonlinear optics, etc. [7]. Recently, the fractional version of the classical Sharma-Tasso-Olever equation has received great attention and has been studied by different scholars using different methods (see [8-13] and the references therein).

In this paper, we use the LSAM to study the following nonlinear time-fractional Sharma-Tasso-Olever equation:

$$D_t^{\alpha}u(t,x)+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}=0,\ \ 0<t,\ \ 0<\alpha\leqslant 1. \qquad (2)$$

There are many types of definitions for fractional derivative, such as the Riemann--Liouville type, Caputo type, Weyl type, and so on. This paper adopts the most widely used Riemann-Liouville fractional derivative D_t^{α} defined by [14]

$${}_0D_t^{\alpha}f(t,x)=D_t^n\,{}_0I_t^{n-\alpha}f(t,x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t\dfrac{f(s,x)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, & n-1<\alpha<n\\ D_t^n f(t,x), & \alpha=n\in\mathbb{N}\end{cases}$$

with the Gamma function $\Gamma(z)=\displaystyle\int_a^{\infty}e^{-z}t^{z-1}\mathrm{d}t$. We denote the operator ${}_0D_t^{\alpha}$ as D_t^{α} for simplicity throughout this paper.

Fractional differential equations (FDEs), due to the nonlocality of fractional derivative, exhibit genetic effects and long-range dependence, and are widely used in many fields of mathematics, physics, engineering, etc. Therefore, solving FDEs is of great significance. At present, there are only some specialized numerical and analytical solutions available, such as the Adomian decomposition method [15], finite difference method [16], homotopy perturbation method [17], the sub-equation method [18], the variational iteration method [19], invariant subspace method [20], Lie symmetry analysis method [21], and so on. Among them, the LSAM has received increasing attention because it can treat differential equations uniformly regardless of their forms, transforming some solutions of these equations into other forms of solutions [22]. It was introduced to solve FDEs by Gazizov et al. [21] in 2007, and recently used to analyze many important FDEs (see [23-33]).

This paper mainly utilizes the LSAM to find all Lie symmetries for Eq. (2) and uses them to reduce Eq. (2) and gets its exact solutions. For power series solutions, we proved their convergence and showed the dynamic analysis of their truncated graphs. Moreover, we constructed the conserved vector for each symmetry by Ibragimov's theory [34, 35].

2. Lie symmetries of Eq. (2)

Assume the nonlinear time-fractional Sharma-Tasso-Olever equation (2) is invariant under the continuous single-parameter transformation group below:

$$\begin{aligned} &t^* = t + \varepsilon\tau(t,x,u) + o(\varepsilon), \quad x^* = x + \varepsilon\xi(t,x,u) + o(\varepsilon), \\ &u^* = u + \varepsilon\eta(t,x,u) + o(\varepsilon), \quad D^{\alpha}_{t^*}u^* = D^{\alpha}_t u + \varepsilon\eta^{\alpha,t} + o(\varepsilon), \\ &D_{x^*}u^* = D_x u + \varepsilon\eta^x + o(\varepsilon), \quad D^2_{x^*}u^* = D^2_x u + \varepsilon\eta^{xx} + o(\varepsilon), \\ &D^3_{x^*}u^* = D^3_x u + \varepsilon\eta^{xxx} + o(\varepsilon), \end{aligned} \tag{3}$$

where τ, ξ, η are infinitesimals, and $\eta^{\alpha,t}$, η^x, η^{xx}, η^{xxx} are the corresponding prolongations of η. So the transformation group (3) admits the following group generator:

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{4}$$

and its corresponding prolongation:

$$prX = X + \eta^{\alpha,t}\frac{\partial}{\partial u^{\alpha}_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \cdots, \tag{5}$$

where

$$\eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \tag{6}$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \tag{7}$$

$$\eta^{xxx} = D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \tag{8}$$

and

$$\begin{aligned} \eta^{\alpha,t} =& \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + (\eta_u - \alpha D_t(\tau))\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D^n_t(\xi) D^{\alpha-n}_t(u_x) \\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1} D^{n+1}_t(\tau)\right] D^{\alpha-n}_t(u) + \mu, \end{aligned} \tag{9}$$

with

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Remark 1 From the definition of the Riemann-Liouville fractional derivative, the invariance determined by (3) requires that $t = 0$ should be invariant, i.e.,

$$\tau(t,x,u)|_{t=0} = 0. \tag{10}$$

Remark 2 Based on the expression of μ, it vanishes under the following condition:

$$\frac{\partial^2\eta}{\partial u^2} = 0. \tag{11}$$

The assumption that the infinitesimal transformations (3) are admitted by Eq. (2) holds, provided that it satisfies the following invariance criterion:

$$prX\left(D_t^{\alpha}u(t,x)+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}\right)|_{(2)}=0, \tag{12}$$

which is rewritten as

$$\Big(\eta^{\alpha,t}+a\eta^{xxx}+3au\eta^{xx}+(6au_x+3au^2)\eta^x+(6auu_x+3au_{xx})\eta\Big)|_{(2)}=0. \tag{13}$$

Putting $\eta^{\alpha,t}$, η^x, η^{xx} and η^{xxx} into (13) and equating the coefficients of various derivatives of u arrives the following results:

$$\tau=c_1t,\;\; \xi=\frac{\alpha}{3}c_1x+c_2,\;\; \eta=-\frac{\alpha}{3}c_1u, \tag{14}$$

where c_1 and c_2 are arbitrary constants. So we can get the following group generators:

$$X_1=\frac{\partial}{\partial x},\;\; X_2=t\frac{\partial}{\partial t}+\frac{\alpha}{3}x\frac{\partial}{\partial x}-\frac{\alpha}{3}u\frac{\partial}{\partial u}. \tag{15}$$

3. Exact solutions of Eq. (2)

In this section, we perform similarity reductions and obtain exact solutions for Eq. (2) through the obtained group generators (15).

Case 1 X_1

For X_1, the characteristic equation is

$$\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}u}{0}, \tag{16}$$

of which the similarity variables are t and u. So the form of the invariant solution of Eq. (2) is

$$u(t,x)=f(t). \tag{17}$$

Substituting (17) into Eq. (2) yields

$$D_t^{\alpha}f=0. \tag{18}$$

So we can easily obtain the following trivial solution of Eq. (2):

$$u(t,x)=f(t)=\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}. \tag{19}$$

where C_1 is determined by the initial condition, that is, $C_1=D_t^{-(1-\alpha)}f(0)$. Figure 1 shows the dynamic behavior of the trivial solution (19), which demonstrates the asymptotic stability of (19) for some different values of fractional order α.

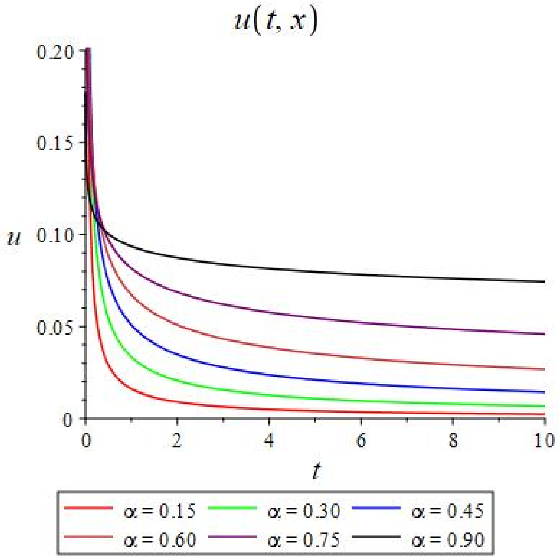


Fig. 1. Graphs of the solution (19) with $C_1 = 0.1$

Case 2 X_2

For X_2, the characteristic equation is

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{\frac{\alpha}{3}x} = \frac{\mathrm{d}u}{-\frac{\alpha}{3}u}, \tag{20}$$

of which the similarity variables are $xt^{-\frac{\alpha}{3}}$ and $ut^{\frac{\alpha}{3}}$. So we obtain the following invariant solutions:

$$u(t,x) = t^{-\frac{\alpha}{3}} f(\omega), \ \ \omega = xt^{-\frac{\alpha}{3}}. \tag{21}$$

Theorem 1 The similarity transformation $u(t,x) = t^{-\frac{\alpha}{3}} f(\omega)$ with $\omega = xt^{-\frac{\alpha}{3}}$ reduces Eq. (2) to the following FODE:

$$(\mathscr{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha} f)(\omega) + 3a(f')^2 + 3af^2 f' + 3aff'' + af^{(3)} = 0, \tag{22}$$

of which the Erdélyi-Kober fractional derivative operator is defined as

$$(\mathscr{P}_\delta^{\iota,\kappa}\psi)(\omega) := \prod_{j=0}^{m-1}(\iota + j - \frac{1}{\delta}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathscr{K}_\delta^{\iota+\kappa,m-\kappa}\psi)(\omega), \ m = \begin{cases} [\kappa]+1, & \kappa \notin \mathbb{N}, \\ \kappa, & \kappa \in \mathbb{N}, \end{cases}$$

with

$$(\mathscr{K}_\delta^{\iota,\kappa}\psi)(\omega) := \begin{cases} \dfrac{1}{\Gamma(\kappa)}\displaystyle\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega s^{\frac{1}{\delta}})\mathrm{d}s, & \kappa > 0, \\ \psi(\omega), & \kappa = 0. \end{cases}$$

Proof From (21), we can obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(t^{-\frac{\alpha}{3}} f(\omega)) = \frac{\partial}{\partial t}\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} s^{-\frac{\alpha}{3}} f(xs^{-\frac{\alpha}{3}})\mathrm{d}s\Big].$$

Let $r = \dfrac{t}{s}$, and we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t}\Big[\frac{t^{1-\frac{4\alpha}{3}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha} r^{\frac{4\alpha}{3}-2} f(\omega r^{\frac{\alpha}{3}})\mathrm{d}r\Big] = \frac{\partial}{\partial t}\Big[t^{1-\frac{4\alpha}{3}}(\mathscr{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},1-\alpha} f)(\omega)\Big].$$

Due to the following relation:

$$t\frac{\partial}{\partial t}\psi(\omega) = tx(-\frac{\alpha}{3})t^{-\frac{\alpha}{3}-1}\frac{\mathrm{d}}{\mathrm{d}\omega}\psi(\omega) = -\frac{\alpha}{3}\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\psi(\omega).$$

we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{4\alpha}{3}}\Big[(1-\frac{4\alpha}{3}-\frac{\alpha}{3}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathscr{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},1-\alpha} f)(\omega)\Big] = t^{-\frac{4\alpha}{3}}(\mathscr{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha} f)(\omega).$$

In addition,

$$3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = t^{-\frac{4\alpha}{3}}\big(3a(f')^2 + 3af^2f' + 3aff'' + af^{(3)}\big).$$

This completes the proof. □

Next, we can obtain the power series solutions for (22) by using the power series method. Assuming

$$f(\omega) = \sum_{k=0}^{\infty} a_k\omega^k, \tag{23}$$

we get

$$\begin{aligned} f'(\omega) &= \sum_{k=0}^{\infty}(k+1)a_{k+1}\omega^k, \\ f''(\omega) &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\omega^k, \\ f'''(\omega) &= \sum_{k=0}^{\infty}(k+3)(k+2)(k+1)a_{k+3}\omega^k, \end{aligned} \tag{24}$$

and

$$
\begin{aligned}
&(\mathscr{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}f)(\omega)=(1+\frac{(2n-m)\alpha}{m-2n+1}-\frac{(m-n)\alpha}{m-2n+1}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathscr{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},1-\alpha}f)(\omega)\\
&=(1-\frac{4\alpha}{3}-\frac{\alpha}{3}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\frac{1}{\Gamma(1-\alpha)}\int_1^{\infty}(s-1)^{-\alpha}s^{\frac{4\alpha}{3}-2}\sum_{k=0}^{\infty}a_k\omega^k s^{\frac{k\alpha}{3}}\mathrm{d}s)\\
&=(1-\frac{4\alpha}{3}-\frac{\alpha}{3}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\sum_{k=0}^{\infty}a_k\omega^k\frac{1}{\Gamma(1-\alpha)}\int_1^{\infty}(s-1)^{-\alpha}s^{\frac{k+4}{3}\alpha-2}\mathrm{d}s)\\
&=(1-\frac{4\alpha}{3}-\frac{\alpha}{3}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{k+1}{3}\alpha)}{\Gamma(2-\frac{k+4}{3}\alpha)}a_k\omega^k)=\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{k+1}{3}\alpha)}{\Gamma(1-\frac{k+4}{3}\alpha)}a_k\omega^k.
\end{aligned}
\tag{25}
$$

Substituting (23)-(25) into (22), we obtain the following equations:

$$
\begin{aligned}
&\frac{\Gamma(1-\frac{k+1}{3}\alpha)}{\Gamma(1-\frac{k+4}{3}\alpha)}a_k+3a\sum_{i+j=k}(i+1)(j+1)a_{i+1}a_{j+1}+3a\sum_{i+j+m=k}(m+1)a_ia_ja_{m+1}\\
&\qquad+3a\sum_{i+j=k}(j+2)(j+1)a_ia_{j+2}+a(k+3)(k+2)(k+1)a_{k+3}=0.
\end{aligned}
\tag{26}
$$

So we get the following explicit expressions:

$$
\begin{aligned}
a_{k+3}=&\frac{-1}{(k+3)(k+2)(k+1)}\Big[\frac{\Gamma(1-\frac{k+1}{3}\alpha)}{a\Gamma(1-\frac{k+4}{3}\alpha)}a_k+3\sum_{i+j=k}(i+1)(j+1)a_{i+1}a_{j+1}\\
&+3\sum_{i+j+m=k}(m+1)a_ia_ja_{m+1}+3\sum_{i+j=k}(j+2)(j+1)a_ia_{j+2}\Big],\ \ k\geqslant 0,
\end{aligned}
\tag{27}
$$

with $a_0=f(0)$, $a_1=f'(0)$, $a_2=f''(0)$.

Therefore, we obtain the power series solution as follows:

$$
\begin{aligned}
u(t,x)=&\,a_0t^{-\frac{\alpha}{3}}+a_1xt^{-\frac{2\alpha}{3}}+a_2x^2t^{-\alpha}+\sum_{k=0}^{\infty}\frac{-x^{k+3}t^{-\frac{(k+4)\alpha}{3}}}{(k+3)(k+2)(k+1)}\\
&\times\Big[\frac{\Gamma(1-\frac{k+1}{3}\alpha)}{a\Gamma(1-\frac{k+4}{3}\alpha)}a_k+3\sum_{i+j=k}(i+1)(j+1)a_{i+1}a_{j+1}\\
&+3\sum_{i+j+m=k}(m+1)a_ia_ja_{m+1}+3\sum_{i+j=k}(j+2)(j+1)a_ia_{j+2}\Big].
\end{aligned}
\tag{28}
$$

Theorem 2 For a neighborhood of $(0,|a_0|)$, (28) is convergent.

Proof From Eq. (27), we can obtain

$$|a_{k+3}| \leq \frac{1}{(k+3)(k+2)(k+1)}\Big[\frac{|\Gamma(1-\frac{k+1}{3}\alpha)|}{|a\Gamma(1-\frac{k+4}{3}\alpha)|}|a_k| + 3\sum_{i+j=k}(i+1)(j+1)|a_{i+1}||a_{j+1}| + 3\sum_{i+j+m=k}(m+1)|a_i||a_j||a_{m+1}| + 3\sum_{i+j=k}(j+2)(j+1)|a_i||a_{j+2}|\Big]. \tag{29}$$

From the Gamma function, the property $\dfrac{|\Gamma(1-\frac{k+1}{3}\alpha)|}{|\Gamma(1-\frac{k+4}{3}\alpha)|} \leq 1$ holds for arbitrary k. So (29) is written as

$$|a_{k+3}| \leq M\big(|a_k| + \sum_{i+j=k}|a_{i+1}||a_{j+1}| + \sum_{i+j+m=k}|a_i||a_j||a_{m+1}| + \sum_{i+j=k}|a_i||a_{j+2}|\big), \tag{30}$$

where $M = \max\{\frac{1}{|a|(k+3)(k+2)(k+1)}, \frac{3(k+1)}{(k+3)(k+2)}, \frac{3}{(k+3)(k+2)}, \frac{3}{(k+3)}\}$.

Another power series is defined as

$$B(\omega) = \sum_{k=0}^{\infty} b_k\omega^k, \tag{31}$$

where $b_0 = |a_0|$, $b_1 = |a_1|$, $b_2 = |a_2|$ and

$$b_{k+3} = M(b_k + \sum_{i+j=k} b_{i+1}b_{j+1} + \sum_{i+j+m=k} b_ib_jb_{m+1} + \sum_{i+j=k} b_ib_{j+2}), \quad k \geq 0. \tag{32}$$

Therefore, $|a_k| \leq b_k$ for $k = 0,1,2,\ldots$, i.e., (31) is the majorant series of (23). From (31) and (32), we have

$$\begin{aligned} B(\omega) = {} & b_0 + b_1\omega + b_2\omega^2 + M\big(B(\omega)\omega^3 + (B(\omega)-b_0)^2\omega \\ & + B^2(\omega)(B(\omega)-b_0)\omega^2 + B(\omega)(B(\omega)-b_0-b_1\omega)\omega\big). \end{aligned} \tag{33}$$

What follows is an implicit function with respect to ω:

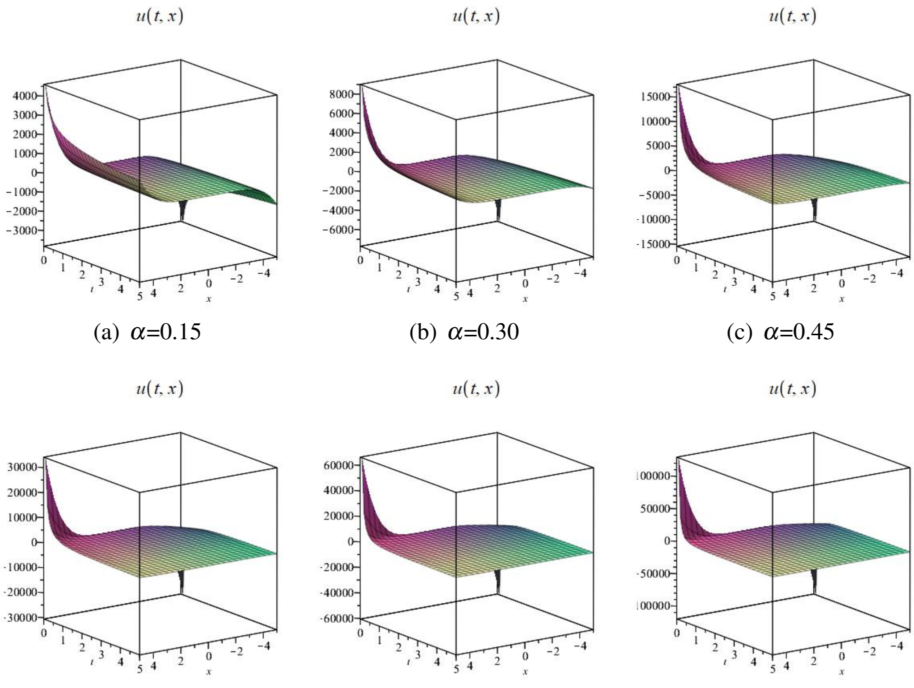
$$\begin{aligned} \Psi(\omega,B) = {} & B - b_0 - b_1\omega - b_2\omega^2 - M\big(B\omega^3 + (B-b_0)^2 \\ & + B^2(B-b_0)\omega^2 + B(B-b_0-b_1\omega)\omega\big). \end{aligned} \tag{34}$$

It is analytic in a neighborhood of $(0,b_0)$, and $\Psi(0,b_0) = 0$, $\frac{\partial}{\partial B}\Psi(0,b_0) = 1$. Therefore, the power series (31) is analytic in this domain based on implicit function theorem. That is, in a neighborhood of the point $(0,|a_0|)$, the power series solution (28) is convergent. □

From (27), we obtain some values of a_n for the given α, which are listed in Table 1. While the dynamical profiles of the power series solution (28) are plotted in Figure 2, which illustrates that for the given initial values $a_0 = a_1 = a_2 = 1$, it varies continuously with fractional order α.

Table 1. The first six coefficients of (28) for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.15$	1	1	1	–2.147658906	0.3244086246	0.7518150253
$\alpha = 0.30$	1	1	1	–2.119598248	0.3123300909	0.7425473856
$\alpha = 0.45$	1	1	1	–2.083589098	0.2977741570	0.7311960493
$\alpha = 0.60$	1	1	1	–2.042266367	0.2816997754	0.7190307747
$\alpha = 0.75$	1	1	1	–2.000000001	0.2650667621	0.7080060378
$\alpha = 0.90$	1	1	1	–1.962835074	0.2481984410	0.7008376236



(a) α=0.15 (b) α=0.30 (c) α=0.45

(d) α=0.60 (e) α=0.75 (f) α=0.90

Fig. 2. Dynamical profiles of the truncated power series solution (28)

4. Conservation laws of Eq. (2)

In this section, for each Lie symmetry (15), we will construct its conservation laws by means of Ibragimov's theory [34, 35].

Firstly, we denote equation (2) as

$$F = D_t^\alpha u(t,x) + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \tag{35}$$

and its formal Lagrangian is

$$\mathscr{L} = v(t,x)F = v(t,x)\big(D_t^\alpha u(t,x) + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}\big), \tag{36}$$

where $v(t,x)$ is an undetermined function. The Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha u)} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}, \tag{37}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α and is defined by the right Caputo fractional derivative [25], i.e.,

$$(D_t^\alpha)^* f(t,x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_t^T \frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^n}{\partial s^n} f(s,x)\mathrm{d}s, & n-1<\alpha<n, \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}. \end{cases}$$

So the adjoint equation of (35) is

$$F^* = \frac{\delta \mathscr{L}}{\delta u} = (D_t^\alpha)^* v - av_{xxx} + 3auv_{xx} - 6auvu_x - 3au^2v_x = 0. \tag{38}$$

Then we apply Ibragimov's method with the above adjoint equation to construct conservation laws for symmetries (15). From the following fundamental operator identity:

$$prX + D_t\tau\cdot\mathscr{I} + D_x\xi\cdot\mathscr{I} = W\cdot\frac{\delta}{\delta u} + D_t\mathscr{N}^t + D_x\mathscr{N}^x, \tag{39}$$

where $\mathscr{I}$ is the identity operator, and $W = \eta - \tau u_t - \xi u_x$ is the characteristic of generator X, we obtain the generalized Noether operators as follows:

$$\mathscr{N}^t = \tau\mathscr{I} + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J\Big(W, D_t^n\frac{\partial}{\partial(D_t^\alpha u)}\Big),\ n = [\alpha]+1, \tag{40}$$

$$\mathscr{N}^x = \xi\mathscr{I} + W\Big(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}} + D_x^2\frac{\partial}{\partial u_{xxx}}\Big) + D_xW\Big(\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}}\Big) + D_x^2 W\frac{\partial}{\partial u_{xxx}}, \tag{41}$$

where J is defined by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\theta,x)}{(\theta-\tau)^{\alpha+1-n}}\mathrm{d}\theta\mathrm{d}\tau. \tag{42}$$

We call $C = (C^t, C^x)$ a conserved vector of Eq. (2) if it satisfies conservation equation $[D_tC^t + D_xC^x]_{(2)} = 0$, and we can obtain its components from the new conservation theorem [35] as follows:

$$C^t = \mathscr{N}^t \mathscr{L}, \;\; C^x = \mathscr{N}^x \mathscr{L}. \tag{43}$$

Case 3 $X_1 = \dfrac{\partial}{\partial x}$

The characteristic of X_1 is

$$W = -u_x, \tag{44}$$

and the components of the corresponding conserved vector are

$$C^t = vD_t^{\alpha-1}(W) + J(W, v_t) = -vD_t^{\alpha-1}u_x - J(u_x, v_t), \tag{45}$$

$$C^x = -u_x(av_{xx} + 3avu_x - 3auv_x + 3au^2v) - u_{xx}(3auv - av_x) - avu_{xxx}. \tag{46}$$

Case 4 $X_2 = t\dfrac{\partial}{\partial t} + \dfrac{\alpha}{3}x\dfrac{\partial}{\partial x} - \dfrac{\alpha}{3}u\dfrac{\partial}{\partial u}$

The characteristic of X_2 is

$$W = -\frac{\alpha}{3}u - tu_t - \frac{\alpha}{3}xu_x, \tag{47}$$

and the components of the corresponding conserved vector are

$$C^t = -vD_t^{\alpha-1}\Big(\frac{\alpha}{3}u + tu_t + \frac{\alpha}{3}xu_x\Big) - J(\frac{\alpha}{3}u + tu_t + \frac{\alpha}{3}xu_x, v_t), \tag{48}$$

$$\begin{aligned} C^x = &-(\frac{\alpha}{3}u + tu_t + \frac{\alpha}{3}xu_x)(av_{xx} + 3avu_x - 3auv_x + 3au^2v) \\ &-(\frac{2\alpha}{3}u_x + tu_{xt} + \frac{\alpha}{3}xu_{xx})(3auv - av_x) - av(\alpha u_{xx} + tu_{xxt} + \frac{\alpha}{3}xu_{xxx}). \end{aligned} \tag{49}$$

5. Conclusions

This paper shows that the LSAM is effective in solving nonlinear FPDEs. We obtained all the Lie symmetries of the nonlinear time-fractional Sharma-Tasso--Olever equation and used them to reduce the equation, thereby getting one asymptotic stable solution and one convergent power series solution. Inspired by this, our next step is to apply the LSAM to high-dimensional nonlinear FPDEs and stochastic FPDEs with the Riemann-Liouville fractional derivative. However, the LSAM has not yet been applied to other newly defined fractional derivatives, such as the tempered fractional derivative, which is also a topic worthy of our future research.

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