

NONLINEAR FREE VIBRATIONS OF THE BEAM TAKING INTO ACCOUNT THE LONGITUDINAL INERTIA OF THE MASS ELEMENT CAUSING THE TENSILE LOAD

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Abstract. This paper presents theoretical and numerical studies of the nonlinear natural vibrations of a beam subjected to a tensile load induced by a mass element. The vibration problem was formulated based on Hamilton's principle, taking into account the Bernoulli-Euler beam theory. Due to the nonlinearity resulting from axial strain defined according to Von Karman's theory, the boundary value problem was derived using the small parameter method. By taking into account the equations related to the small parameter to the appropriate power, the linear component of the natural frequencies, the nonlinear component of the internal force in the beam under tension, and the nonlinear component of the natural frequencies (dependent on the amplitude) were determined. The results of numerical calculations of the first three natural frequencies are graphically presented as a function of the tensile load. The theoretical and numerical studies conducted in this paper are introductory to the research on the dynamic properties of the system. In particular, the dependence of the dynamic properties on the longitudinal inertia of the mass element loading the beam is investigated. The omission of damping in this work was aimed at formulating a boundary value problem with the lowest possible degree of complexity, which would enable a preliminary analysis of the assumed research task.

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1. Introduction

Nowadays, due to the development of computational techniques and high-performance computer hardware, mechanical systems are being designed with optimized strength and durability in mind. The need for optimization is primarily driven by economic and ecological considerations. Mechanical systems are becoming

increasingly smaller, resulting in cost savings in the manufacturing process. However, material savings require designers to more thoroughly check mechanical systems for threats resulting from hazardous phenomena, such as dynamic ones. Very often, simple mechanical systems, such as beams, constitute important elements of larger structures. Their strength, durability, and behavior under periodic loads determine the proper functioning of the entire system. Disrupting the proper functioning of one fundamental element can have disastrous consequences, particularly when considering supporting or load-bearing structures used in construction. Given all the above-mentioned challenges of the mechanical systems design process, computational models must be increasingly detailed. Published scientific papers enrich the state of knowledge about the behavior of the system in the full range of applied loads and under various operating conditions.

Beams, as a fundamental mechanical system, are a very common subject of research. Many authors conduct research on this type of system. In the case of beams, authors are eager to address research issues related to both free and forced vibrations. When studying beam vibrations, linear and nonlinear vibrations are considered. Due to their complexity, nonlinear vibrations present significant challenges in developing a computational models, performing numerical analyses, and designing and conducting experimental studies. It is precisely the influence of non-linearity that provides significant research potential. Development of research in this field contributes to the more optimal design of various mechanical structures. When a beam is loaded with a compressive external force, it is called a column, whereas when a beam is subjected to tension, it is called a tie rod.

Taking nonlinearities into account, new curves of the system's dynamic parameters are obtained, depending on the amplitude. The effect of amplitude is crucial in nonlinear systems. By examining the effect amplitude has on the dynamic parameters of a mechanical system, the designer can prepare a more robust design. This allows them to predict the negative effects of using the mechanical system in certain operating ranges.

A beam equipped with piezoelectric ceramic actuators, which can stabilize the system under a compressive load when properly controlled, was investigated in [1]. The influence of nonlinear beam isolation on forced vibrations was examined in [2]. The isolation was modeled using discrete elements in the form of springs and dampers positioned both transversely and longitudinally to the beam axis. Calculations demonstrated that there is a critical stiffness value at which the fundamental natural frequency is zero. As the longitudinal spring stiffness increases, the higher-order vibration frequencies decrease linearly. Transverse vibrations of a suspended cantilever beam were considered in [3]. Both free and forced vibrations were investigated, taking into account internal resonance induced by gravity. The authors of [4] developed a procedure for optimizing the initial curvature of a mechanical beam to maximize the amplification of the forced transverse vibrations of the beam. Free and forced vibrations of a curved beam were considered in [5]. Six natural frequencies were considered in the calculations, and the results obtained from the mathematical model were compared with the results of numerical simulations

obtained using commercial software. Nonlinear forced vibrations of a rotating beam with variable cross-section were studied by Li and Yao [6]. They took into account the influence of the Coriolis force, static axial strain, and geometric nonlinearity. The authors showed that the fundamental natural frequency increases with increasing rotational speed and the rate of change of the cross-section. In this case, the rate of change of the cross-section significantly affects the nonlinear response of the system to vibrations. The aim of [7] was to analyze the nonlinear vibration behavior of fractional Kelvin-Voigt viscoelastic beams resting on a nonlinear elastic foundation, taking into account harmonic excitation. In the discussed case, the authors performed a parametric analysis to determine the influence of the fractional Kelvin-Voigt viscoelastic model on primary and secondary resonances.

In [8], the authors considered a beam simply supported at both ends without applying additional axial load. The beam displacement at the sliding support was constrained elastically. The constraint on the sliding support displacement was modeled with a translational spring. The authors noted that when considering nonlinear forced vibrations, an additional resonance is created on the resonance curve. The authors defined this additional resonance as internal resonance. Ignoring this resonance can have serious consequences in the design process, as at a certain point along the resonance curve for lateral vibrations, a sudden increase in amplitude occurs due to the aforementioned internal resonance. In [9, 10], a column loaded with a compressive force generated by a mass element was considered. The effect of the mass element's longitudinal inertia on the frequencies of the lateral natural vibrations was investigated. However, the term related to the column's longitudinal stiffness was omitted from the equation for the nonlinear component. Additionally, in [10], the column was subjected to the action of a local heat source, the location and height of which were determined by appropriate coefficients. In [11], a horizontally positioned beam with a mass element placed at its end was considered. Numerical calculations investigated the effect of the mass element's longitudinal inertia on the nonlinear natural frequencies. The beam modeled the piston rod of a hydraulic cylinder. In [9-11], damping was not considered.

Considering the presented description of the current state of research on beam systems, this work addresses the issue of transverse free vibrations of a beam (tendon) tensioned by a mass element. The scope of the work includes formulating the boundary value problem for the transverse vibrations of the considered beam, taking into account longitudinal inertia, whose influence on the system is a source of nonlinearity. Given that this work is an introduction to broader research issues, damping was not considered. In the considered scope, damping certainly has a significant impact on the obtained results, but it would overly complicate the mathematical model. The scope of the work also does not include conducting experimental studies.

As part of this work, the nonlinear components of internal forces and the nonlinear components of natural frequencies were determined. The impact of nonlinear components on natural frequencies at a selected amplitude value was also planned.

2. Considered system

The system considered in this work is shown in Figure 1. The system is composed of one rod rigidly fixed at one end and loaded at the other by a mass element of mass M and axial mass moment of inertia J_M . The beam is fixed in such a way that the gravitational force of the mass element (Mg) placed at the unfixed end causes it to stretch in the case of static equilibrium. Additionally, the effect of the longitudinal inertia of the mass element on the natural frequencies is considered in the work.

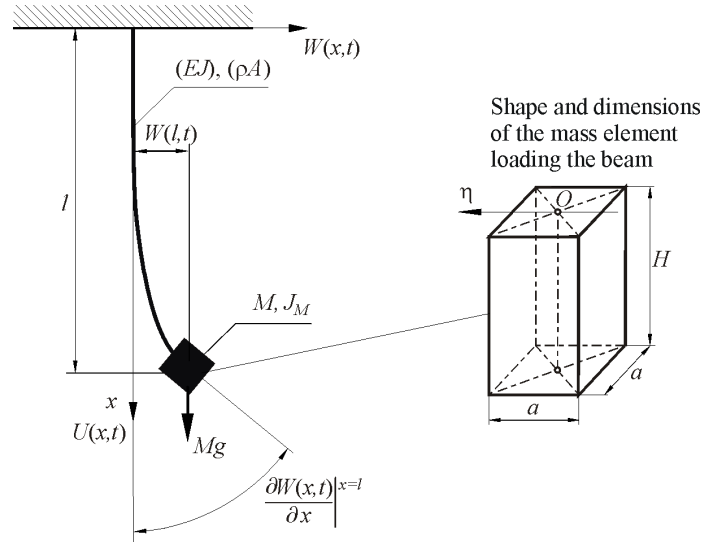


Fig. 1. Schematic diagram of the beam under consideration loaded with a mass element

The considered beam is characterized by bending stiffness EJ , compressive stiffness EA and mass per unit length ρA . A cuboid with a square base (a – dimension of the square side) and height H (Fig. 2) was selected as the mass element loading the beam. The dimension of the square height was related to the dimension of the base side by the coefficient ζ_H :

$$H = a\zeta_H \quad (1)$$

The work takes into account the mass axial moment of inertia of the element loading the beam about the axis $O\eta$ as shown in Figure 1. The point marked in the figure as O is the connection point of the mass element with the beam. The mass axial moment of inertia of the element loading the beam about the axis η is calculated according to the following formula:

$$J_M = \frac{1}{12} M \left[(4\zeta_H^2 + 1) a^2 \right] \quad (2)$$

The research task presented in this work is intended as an introductory study into the research problem under consideration. Therefore, the system model must be as simple as possible during preliminary studies. For this reason, the work does not consider the influence of both internal and external damping, which can be significant in real-world systems. At the stage of preliminary research in this area, damping would overly complicate the mathematical model.

3. Formulation of the boundary value problem

Taking into account the system presented in Figure 1 in the form of a single-member beam loaded with a mass element, the boundary value problem of free vibrations is formulated based on Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0 \quad (3)$$

where T and V are the kinetic and potential energy, respectively. The kinetic and potential energies of a system are described by the formulas:

$$T = \frac{1}{2} \rho A \int_0^l \left[\frac{\partial W(x,t)}{\partial t} \right]^2 dx + \frac{1}{2} M \left[\frac{\partial U(x,t)}{\partial t} \Big|_{x=l} \right]^2 + \frac{1}{2} M \left[\frac{\partial W(x,t)}{\partial t} \Big|_{x=l} \right]^2 + \frac{1}{2} J_M \left[\frac{\partial^2 W(x,t)}{\partial x \partial t} \Big|_{x=l} \right]^2 \quad (4)$$

$$V = \frac{1}{2} EJ \int_0^l \left[\frac{\partial^2 W(x,t)}{\partial x^2} \right]^2 dx - MgU(l,t) + \frac{1}{2} EA \int_0^l \left[\frac{\partial U(x,t)}{\partial x} + \frac{1}{2} \left(\frac{\partial W(x,t)}{\partial x} \right)^2 \right]^2 dx \quad (5)$$

where $W(x,t)$ and $U(x,t)$ are the lateral and longitudinal displacement of the beam under consideration, respectively.

The system under consideration is characterized by the following geometric boundary conditions:

$$U(0,t) = 0 \quad (6a)$$

$$W(0,t) = 0 \quad (6b)$$

$$\left. \frac{\partial W(x,t)}{\partial x} \right|_{x=0} = 0 \quad (6c)$$

The internal force in a rod, by definition, is equal to:

$$S(t) = EA \left[\frac{\partial U(x,t)}{\partial x} + \frac{1}{2} \left(\frac{\partial W(x,t)}{\partial x} \right)^2 \right] \quad (7)$$

Taking into account the kinetic energy (4), potential energy (5), and geometric boundary conditions (6) in Hamilton's principle (3), after performing appropriate transformations, we obtain:

– the differential equation of motion in the longitudinal direction:

$$\frac{\partial}{\partial x} \left[\frac{\partial U(x,t)}{\partial x} + \frac{1}{2} \left(\frac{\partial W(x,t)}{\partial x} \right)^2 \right] = 0 \quad (8)$$

– differential equation of motion in the transverse direction:

$$(EJ) \frac{\partial^4 W(x,t)}{\partial x^4} - S(t) \frac{\partial^2 W(x,t)}{\partial x^2} + (\rho A) \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \quad (9)$$

– natural boundary conditions:

$$(EJ) \frac{\partial^2 W(x,t)}{\partial x^2} \Big|_{x=l} + J_M \frac{\partial^3 W(x,t)}{\partial x \partial t^2} \Big|_{x=l} = 0 \quad (10)$$

$$(EJ) \frac{\partial^3 W(x,t)}{\partial x^3} \Big|_{x=l} - S(t) \frac{\partial W(x,t)}{\partial x} \Big|_{x=l} - M \frac{\partial^2 W(x,t)}{\partial t^2} \Big|_{x=l} = 0 \quad (11)$$

$$M \frac{\partial^2 U(x,t)}{\partial t^2} \Big|_{x=l} + S(t) - P = 0 \quad (12)$$

After integrating the equation of motion twice in the longitudinal direction (8) and taking into account the geometric boundary condition (6a), the following equation was obtained:

$$U(x,t) = \frac{S(t)}{(EA)} x - \frac{1}{2} \int_0^x \left(\frac{\partial W(x,t)}{\partial x} \right)^2 dx \quad (13)$$

Further considerations were carried out using dimensionless quantities:

$$\begin{aligned} \xi = \frac{x}{l}; \quad w(\xi, \tau) = \frac{W(x, t)}{l}; \quad u(\xi, \tau) = \frac{U(x, t)}{l}; \quad \tau = \omega t; \quad \theta = \frac{Al^2}{J}; \\ k^2(\tau) = \frac{S(\tau)l^2}{(EJ)}; \quad \Omega^2 = \frac{\omega^2(\rho A)l^4}{(EJ)}; \quad \lambda = \frac{Pl^2}{(EJ)} \end{aligned} \quad (14a-h)$$

After taking into account the data values with formulas (14), the equations in the longitudinal and transverse directions can be written as:

$$u(\xi, \tau) = \frac{k^2(\tau)}{\theta} \xi - \frac{1}{2} \int_0^\xi \left(\frac{\partial w(\xi, \tau)}{\partial \xi} \right)^2 d\xi \quad (15)$$

$$\frac{\partial^4 w(\xi, \tau)}{\partial \xi^4} - k^2(\tau) \frac{\partial^2 w(\xi, \tau)}{\partial \xi^2} + \Omega^2 \frac{\partial^2 w(\xi, \tau)}{\partial \tau^2} = 0 \quad (16)$$

The boundary conditions also modify to the appropriate dimensionless form:

$$u(0, \tau) = w(0, \tau) = \left. \frac{\partial w(\xi, \tau)}{\partial \xi} \right|_{\xi=0} = 0 \quad (17a-c)$$

$$\left. \frac{(EJ)}{l} \frac{\partial^2 w(\xi, \tau)}{\partial \xi^2} \right|_{\xi=1} + J_M \omega^2 \left. \frac{\partial^3 w(\xi, \tau)}{\partial \xi \partial \tau^2} \right|_{\xi=1} = 0 \quad (18)$$

$$\left. \frac{\partial^3 w(\xi, \tau)}{\partial \xi^3} \right|_{\xi=1} - k^2(\tau) \left. \frac{\partial w(\xi, \tau)}{\partial \xi} \right|_{\xi=1} - M \frac{\omega^2 l^3}{(EJ)} \left. \frac{\partial^2 w(\xi, \tau)}{\partial \tau^2} \right|_{\xi=1} = 0 \quad (19)$$

$$M \frac{\omega^2 l^3}{(EJ)} \left. \frac{\partial^2 u(\xi, \tau)}{\partial \tau^2} \right|_{\xi=1} + k^2(\tau) - \lambda = 0 \quad (20)$$

Given the nonlinearity in the longitudinal displacement equation (15), the final formulation of the boundary value problem for free vibrations uses the small parameter method. The nonlinear quantities of the equations are expanded into a small parameter power series. These expansions are of the following form (comp. [10, 11]):

$$w(\xi, \tau) = \sum_{j=1}^N \varepsilon^{2j-1} w_{2j-1}(\xi, \tau) + O(\varepsilon^{2(N+1)-j}) \quad (21)$$

$$u(\xi, \tau) = u_0(\xi) + \sum_{j=1}^N \varepsilon^{2j} u_{2j}(\xi, \tau) + O(\varepsilon^{2(N+1)}) \quad (22)$$

$$k^2(\tau) = k_0^2 + \sum_{j=1}^N \varepsilon^{2j} k_{2j}^2(\tau) + O(\varepsilon^{2(N+1)}) \quad (23)$$

$$\Omega^2 = \Omega_0^2 + \sum_{j=1}^N \varepsilon^{2j} \Omega_{2j}^2 + O(\varepsilon^{2(N+1)}) \quad (24)$$

where:

$$w_1(\xi, \tau) = w_1^{(1)}(\xi) \cos(\tau) \quad (25)$$

$$w_2(\xi, \tau) = w_2^{(1)}(\xi) \cos(\tau) + w_3^{(3)}(\xi) \cos(3\tau) \quad (26)$$

$$u_2(\xi, \tau) = u_2^{(0)}(\xi) + u_2^{(2)}(\xi) \cos(2\tau) \quad (27)$$

$$k_2^2(\tau) = k_2^{(0)}(\xi) + k_2^{(2)}(\xi) \cos(2\tau) \quad (28)$$

By substituting the power series expansions of the small parameter (21)-(24) (taking into account the relations (25)-(28)) into the differential equations (15) and (16) and into the boundary conditions (17)-(20), we obtain differential equations and their corresponding boundary conditions grouped with respect to the same powers of the small parameter. Based on the equations related to the zero power of the small parameter, the static internal force in the beam can be determined. Therefore, we can write:

$$k_0^2 = \lambda \quad (29)$$

The solution of the equations corresponding to the first power of the small parameter ε^1 leads to the determination of the linear component of the natural frequency ω_0 . The solutions of the differential equations related to the first power of the small parameter can be represented by the function:

$$w_1^{(1)}(\xi) = A \cosh(\alpha \xi) + B \sinh(\alpha \xi) + C \cos(\beta \xi) + D \sin(\beta \xi) \quad (30)$$

where:

$$\alpha = \sqrt{\frac{k_0^2}{2} + \sqrt{\frac{k_0^4}{4} + \Omega_0^2}}; \quad \beta = \sqrt{-\frac{k_0^2}{2} + \sqrt{\frac{k_0^4}{4} + \Omega_0^2}} \quad (31a,b)$$

The solutions (30) are substituted into the boundary conditions of the transversal displacements, thus obtaining the system of equations:

$$[a_{ij}]\{A, B, C, D\} = 0 \quad (32)$$

The determinant of the coefficient matrix of this system, set equal to zero, is a transcendental equation from which the linear components of the system's vibration frequency are determined:

$$|a_{ij}| = 0 \quad (33)$$

equations, along with boundary conditions related to the square of the small parameter ε^2 , are used to determine the nonlinear components of the internal force in the beam during vibrations. The relationships enabling the calculation of the nonlinear component of the internal force are as follows:

$$k_2^{(2)} = \frac{-M\omega_0^2 \frac{l^3}{(EJ)} \int_0^1 \left(\frac{d w_1^{(1)}(\xi)}{d\xi} \right)^2 d\xi}{1 - 4M\omega_0^2 \frac{l^3}{(EJ)} \frac{1}{\theta}} \quad (34)$$

The next step in calculating the dynamic parameters of the system under consideration involves determining the nonlinear component of the natural frequency ω_2 . The nonlinear components of the natural frequency are calculated based on equations related to the cube of the small parameter ε^3 . In order to derive the appropriate relationship, the condition of orthogonality of the eigenfunctions should be used, taking into account the appropriate boundary conditions. The formulas used to calculate the nonlinear components of the natural frequency are as follows:

$$\omega_2^2 = \frac{(EJ) w_1^{(1)}(1) \frac{1}{2} k_2^{(2)} \frac{d w_1^{(1)}(\xi)}{d\xi} \Big|_{\xi=1} + (EJ) \int_0^1 \left(w_1^{(1)}(\xi) \frac{1}{2} k_2^{(2)} \frac{d^2 w_1^{(1)}(\xi)}{d\xi^2} \right) d\xi}{Ml^3 \left(w_1^{(1)}(1) \right)^2 + J_M l \left(\frac{d w_1^{(1)}(\xi)}{d\xi} \Big|_{\xi=1} \right)^2 - (\rho A) l^4 \int_0^1 \left(w_1^{(1)}(\xi) \right)^2 d\xi} \quad (35)$$

4. Results of numerical calculations

When considering beam systems loaded with axial forces, a crucial element of the research is determining how the natural frequency is affected by the internal force present in the system. Therefore, the relationship between the system load (in this case, tensile axial load) and the natural frequency is determined. The curves plotted on the load-natural frequency plane are presented in Figures 2 and 3 (referring to the first natural frequency), Figures 4 and 5 (referring to the second natural frequency), and Figures 6 and 7 (referring to the third natural frequency). The system parameters considered in the calculations are the beam length and diameter. The results are presented using the dimensionless parameter ζ_M for the mass of the element loading the beam. The parameter ζ_M is defined as the ratio of the mass of the element loading the beam to the reference mass. The reference mass refers to the beam diameter because the gravity force of the reference mass causes tensile stresses in the beam equal to $\sigma = 300$ MPa. In the presentation of the results, the dimensionless parameter of the natural frequency given by formula (14g) was used.

Considering that the maximum tensile stress in the beam caused by the gravity force of the mass element is 300 MPa, in order to present the results of numerical calculations in dimensionless form, a reference beam diameter of $d_{ref} = 20$ mm was introduced. Additionally, the following dimensionless parameters were introduced for the beam diameter and length:

$$\zeta_d = \frac{d}{d_{ref}}; \quad \zeta_l = \frac{d}{l} \tag{36a,b}$$

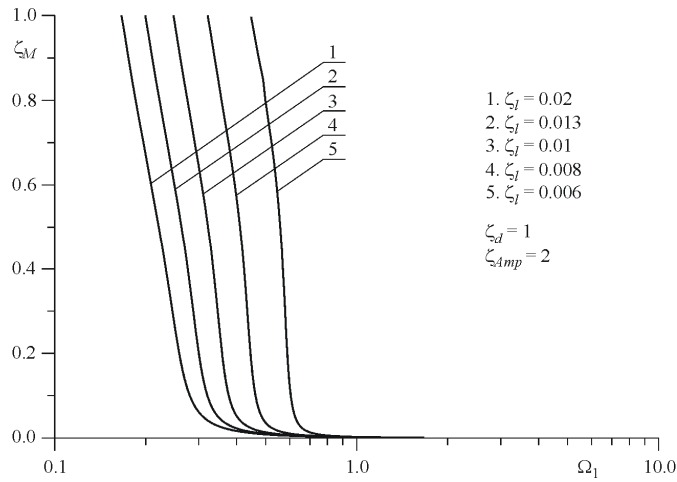


Fig. 2. The parameter of the first natural frequency depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

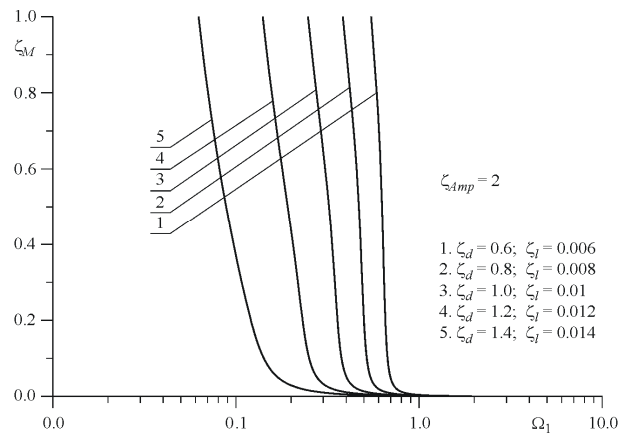


Fig. 3. The parameter of the first natural frequency depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

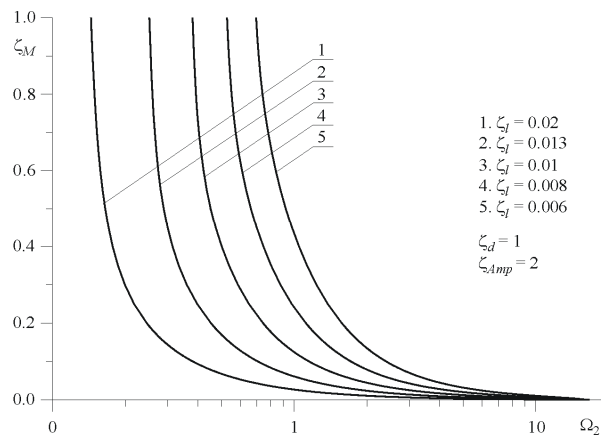


Fig. 4. The parameter of the second natural frequency depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

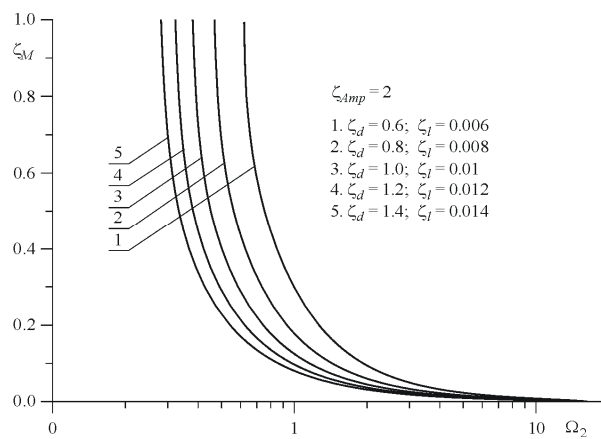


Fig. 5. The parameter of the second natural frequency depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

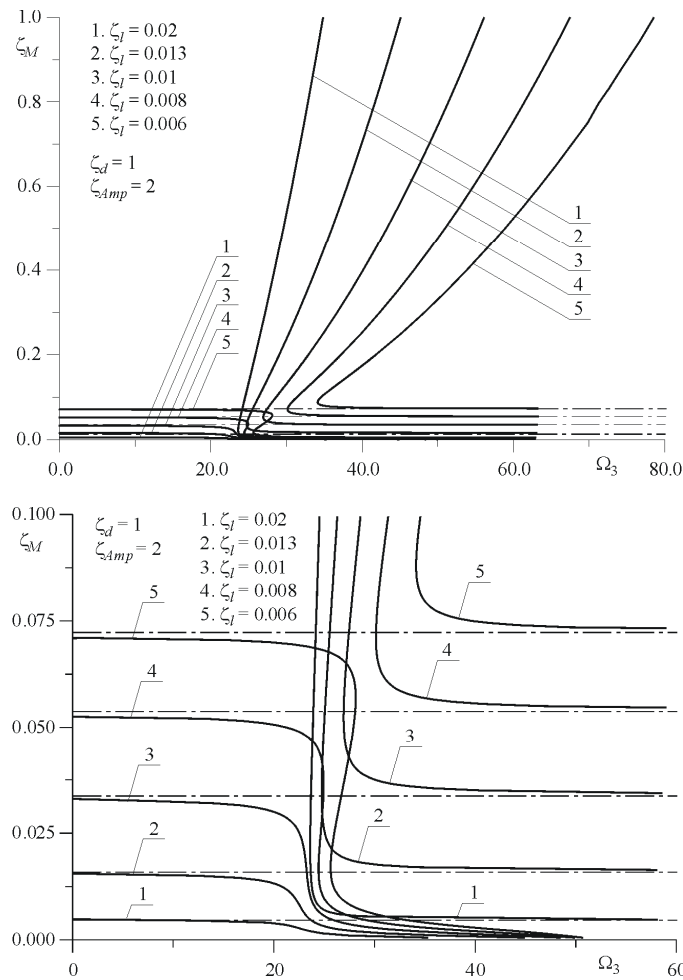


Fig. 6. The parameter of the third natural frequency depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

Taking into account the presented results of numerical calculations, it is possible to determine how the mass of the element loading the beam affects the first three natural frequencies of the system. Considering low values of the ζ_M parameter, a sudden decrease in frequency is noticeable with a slight increase in the mass of the element loading the beam. A sharp decrease in frequency occurs in each of the presented results. At the first vibration frequency, the area of the ζ_M coefficient corresponding to the rapid decrease in frequency is smaller compared to the second natural frequency. At the first frequency, this area occurs from $\zeta_M = 0$ to approximately $\zeta_M = 0.1$. At the second vibration frequency, it ranges from $\zeta_M = 0$ to approximately $\zeta_M = 0.4$. The largest changes in frequency are observed at the third frequency. These changes are caused by longitudinal resonance.

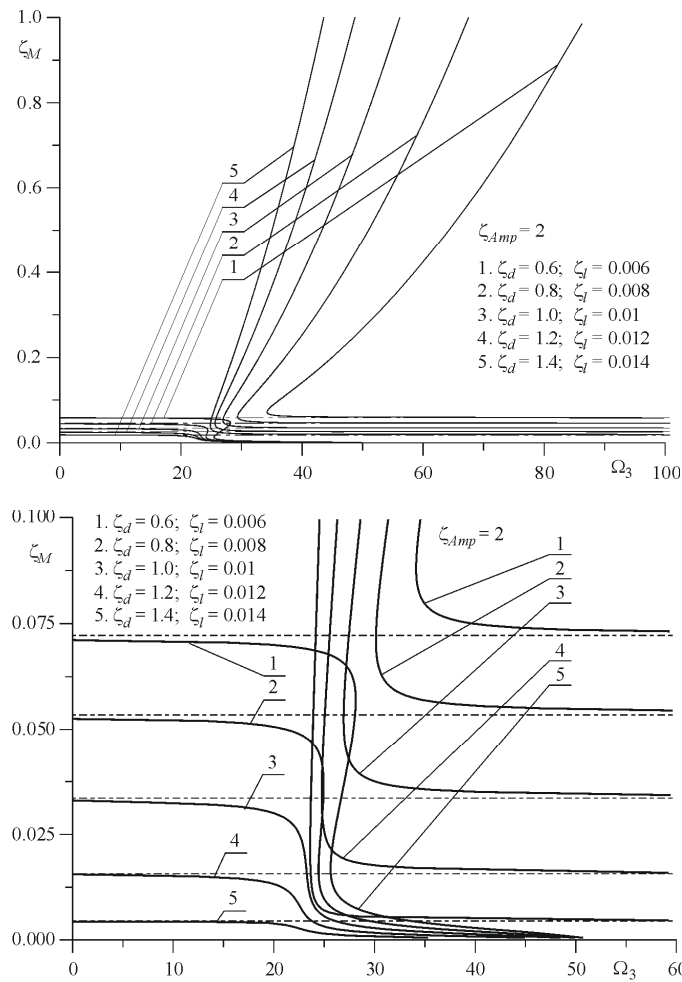


Fig. 7. The parameter of the third natural frequency depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

This resonance is caused by transverse vibrations of the beam causing longitudinal movement of the mass element. Longitudinal resonance leads to the creation of large internal forces in the beam, causing significant changes in the natural frequency. Longitudinal resonance occurs at a specific value of the ζ_M coefficient, which depends on the beam's diameter and length. These two parameters directly influence the beam's stiffness. At the value of the ζ_M coefficient corresponding to longitudinal resonance, the vibration frequency changes asymptotically. Below the value of the ζ_M parameter corresponding to longitudinal resonance, the vibration frequency approaches zero. Above this value, however, it increases asymptotically to infinity. This behavior of the system when passing through the longitudinal resonance region is caused by a change in the phase shift angle of the longitudinal

displacement by an angle of $\pi/2$ compared to the excitation induced by transverse vibrations. The occurrence of asymptotic changes in the frequency in the presented form is caused by the omission of damping in the mathematical model. The value of the vibration amplitude influences the character of the presented curves in the resonance region. The smaller the vibration amplitude, the more violent the curves on the $\zeta_M(\Omega_3)$ plane in the resonance region.

Given that the linear and nonlinear components of the natural frequency were calculated in this work, the value of the coefficient ζ_{ω_i} is additionally presented depending on the beam load parameter. The parameter ζ_{ω_i} determines the share of the nonlinear component ω_{i2} in the value of the natural frequency ω_i . The index i refers to the i -th natural frequency, comprising both its linear (ω_{i0}) and nonlinear (ω_{i2}) components. The coefficient ζ_{ω_i} is given by the formula:

$$\zeta_{\omega_i} = \frac{\omega_i - \omega_{i2}}{\omega_i}; \text{ where: } \omega_i = \sqrt{\omega_{i0}^2 + \varepsilon^2 \omega_{i2}^2} \quad (37a,b)$$

The results of the numerical calculations were performed with the value of the small parameter corresponding to the amplitude parameter $\zeta_{Amp} = 2$. The amplitude parameter was defined as follows:

$$\zeta_{Amp} = \frac{Amp}{r} = Amp \sqrt{\frac{A}{J}} \quad (38)$$

where: Amp – amplitude of vibrations of the system, r – minimum radius of gyration of the beam cross-section.

Taking into account the results concerning the coefficient ζ_{ω_i} depending on the value of the external load ζ_M , it is concluded that in the case of the first frequency, the influence of the nonlinear component ω_{12} on the natural frequency ω_1 is very small (see Figs. 8 and 9). For lower values of the ζ_M parameter, a slight decrease in the influence of the nonlinear component on the natural frequency is observed with increasing load. Then, the influence of the nonlinear component increases with increasing load. For the considered system parameters, this influence does not exceed 0.2% at the first natural frequency. In the case of the second natural frequency (see Figs. 10 and 11), the influence of the nonlinear component ω_{22} on the frequency ω_2 is greater and reaches the value of 2%. The nature of changes in the considered coefficient ζ_{ω_2} is also different from that in the case of the first frequency ζ_{ω_1} . First, the influence of the nonlinear component on the frequency increases and then decreases with the increase of the load parameter ζ_M . At the highest value of the considered external load $\zeta_M = 1$, this influence is zero. The second vibration frequency ω_2 near $\zeta_M = 1$ is independent of the nonlinear component of the natural frequency ω_{22} , which is equivalent to the statement that the vibration amplitude has no influence on the second natural frequency at the load corresponding to $\zeta_M = 1$.

The greatest influence of the nonlinear component on the natural frequency is noticeable at the third frequency near the occurrence of longitudinal resonance (Figs. 12 and 13).

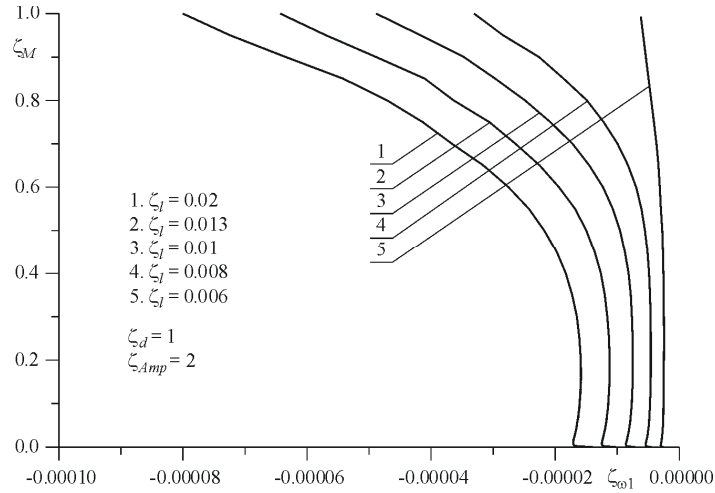


Fig. 8. The influence parameter of the nonlinear component of the first natural frequency $\zeta_{\omega 1}$ depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

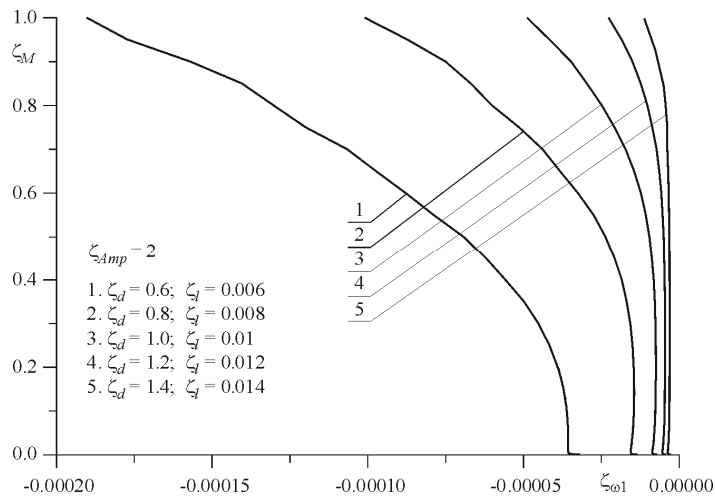


Fig. 9. The influence parameter of the nonlinear component of the first natural frequency $\zeta_{\omega 1}$ depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

Calculations regarding the effect of the nonlinear component on natural frequencies are presented for a selected amplitude value defined by the coefficient $\zeta_{Amp} = 2$. Calculation results for other amplitude levels are not presented because

the value of the nonlinear component of the natural frequency ω_2 is independent of the amplitude value. This paper focuses on presenting the trends in the effect of the nonlinear component on natural frequencies for different parameter values of the system under consideration.

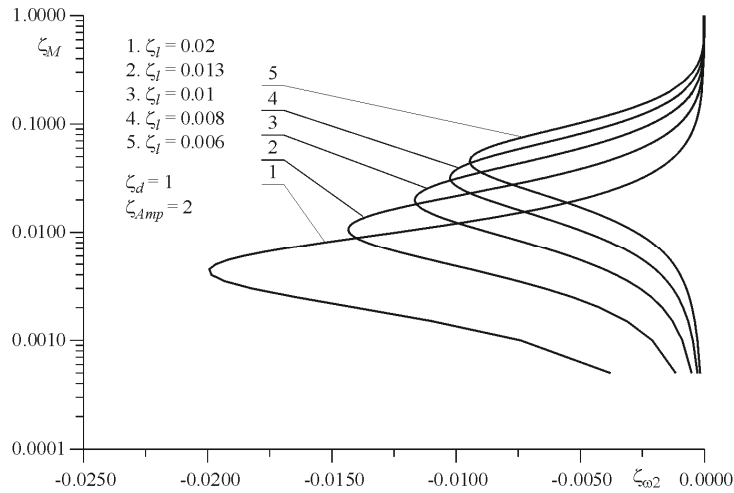


Fig. 10. The influence parameter of the nonlinear component of the second natural frequency $\zeta_{\omega 1}$ depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

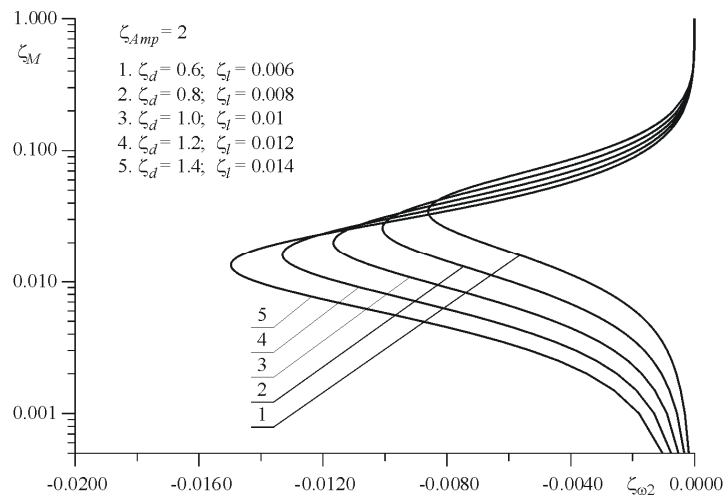


Fig. 11. The influence parameter of the nonlinear component of the second natural frequency $\zeta_{\omega 2}$ depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

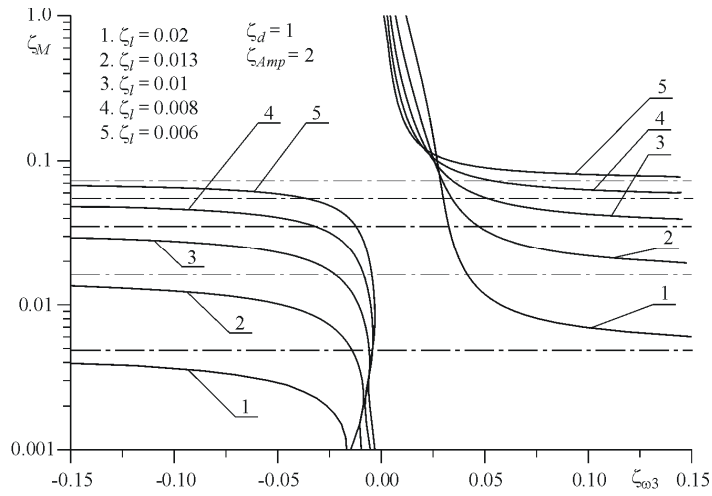


Fig. 12. The influence parameter of the nonlinear component of the third natural frequency $\zeta_{\omega 3}$ depending on the load parameter ζ_M at the diameter $d = 20$ mm and at different lengths of the beam under consideration

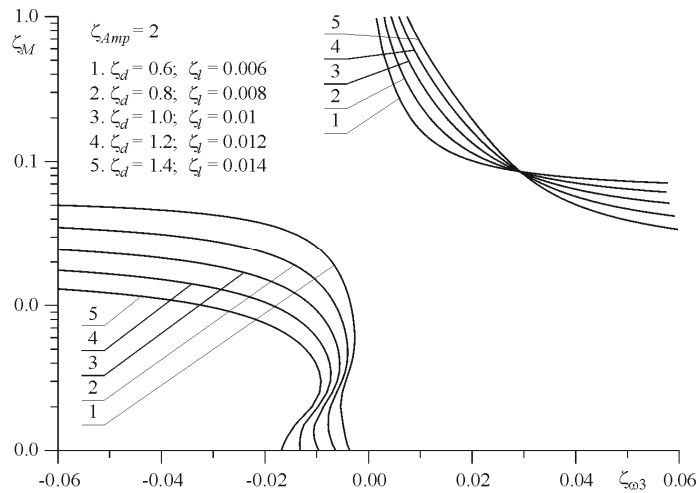


Fig. 13. The influence parameter of the nonlinear component of the third natural frequency $\zeta_{\omega 3}$ depending on the load parameter ζ_M at the length $l = 2$ m and at different diameters of the beam under consideration

Taking into account the obtained numerical calculation results, a significant disturbance in the value of the third natural frequency is found for system parameters that cause longitudinal resonance. It is shown that longitudinal resonance, caused by the longitudinal inertia of the mass element attached to the system, significantly affects the internal force in the beam, which is directly related to the value of the natural frequency.

Taking into account forced vibrations, disturbances in the course of resonance curves are very often described by other authors as internal resonance (cf. [8, 12-15]). Some authors in their works took into account the longitudinal inertia of the beam (comp. [12]) and the longitudinal inertia of the mass attached to the unrestrained end of the beam (comp. [15]).

5. Conclusion

This work involved research on a beam rigidly fixed at one end and loaded with a mass element at the other. The beam is oriented so that the mass element's gravity causes it to stretch. The presented research problem involves examining the effect of the mass element's longitudinal inertia on the beam's natural frequencies. To achieve the research objective, the boundary value problem for the system under consideration was formulated based on Hamilton's principle and, due to the non-linearity of the small parameter method, based on numerical calculations. The first three natural frequencies were determined, taking into account their linear and non-linear components. Additionally, detailed studies were conducted on the value of the coefficient determining the influence of the nonlinear component on the tested natural frequencies. It was shown that at the first two natural frequencies, the influence of the nonlinear component is small. For the first frequency, the influence of the nonlinear component does not exceed 2‰, and for the second frequency, 2%. The conducted research demonstrated that the value of the third natural frequency is strongly dependent on the longitudinal inertia of the mass element loading the beam. This is particularly evident in the load factor range, which corresponds to the region where longitudinal resonance occurs. Neglecting the influence of the longitudinal inertia of the mass element loading the beam when designing such mechanical systems can contribute to their failure, particularly when the system is exposed to periodic excitations.

To fully investigate the phenomenon of longitudinal resonance induced by lateral vibrations, both external and internal damping of the system must be considered. Taking damping into account will certainly mitigate the negative impact of the resonance phenomenon on the system's dynamic parameters (dynamic forces as well as nonlinear components of the natural frequency). In the next stage of research, it is also possible to verify the results of numerical calculations obtained from the mathematical model with those obtained through experimental studies.

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