

## STABILITY OF A COLUMN LOADED WITH A FOLLOWER FORCE DIRECTED TOWARDS THE NEGATIVE POLE, TAKING INTO ACCOUNT TWO STIFFNESS DISCONTINUITIES

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**Abstract.** This paper considers the boundary value problem concerning the natural vibrations of a single-rod column, in which two stiffness discontinuities occur at a given position. The column is subjected to a load of a follower force directed towards the negative pole. The load is generated by loading heads made of circular elements. The stiffness discontinuities of the column rod were modeled using rotational springs with a given stiffness. The boundary value problem of column stability was formulated based on the principle of minimum potential energy. In the case of stability tests, the static stability criterion was used to determine the critical load. Based on the mathematical model, numerical calculations were performed regarding the stability problem. Based on the calculations, the critical force of the system was determined. Numerical calculations were performed for various values of the system parameters, which include: the stiffness value of the rotational springs defining the stiffness discontinuities of the column rod, the location of the stiffness discontinuities along the rod, the parameters of the loading system, including the radii of the loading heads and the length of the bolt connecting the head of the loading system with the column rod.

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**Keywords:** static criterion of stability, column, critical load, specific load, follower force directed towards to negative pole

### 1. Introduction

In mechanical engineering, slender systems subjected to external compressive loads are called columns. Columns are very important elements of support structures. Errors made in the design process can have very serious consequences, because the strength of such elements determines the strength of the entire building. Damage to one of the columns results in a very dynamic process of destruction of the entire structure. Column loads can be broadly divided into conservative and non-conservative. Among conservative loads, we can distinguish the classical Euler load, in which

the direction of the force remains constant and is independent of the column's deflection from its equilibrium position. Another type of conservative load is the specific load formulated and introduced to the literature by Towski (cf. [2, 3]). Among non-conservative loads, we should mention the Beck's load (cf. [4]) or Reut's load [5]. The basic parameter examined in the case of columns is the critical load. The critical load of a column is influenced by its material, mounting conditions, and the method of loading. Sometimes, support systems may be exposed to external periodic forces, which may contribute to the formation of fatigue cracks causing discontinuity of the column stiffness. Another cause of the formation of discontinuity of stiffness in the column is its excessive heating (local or global). The mechanical properties of the column material (including Young's modulus) depend on temperature (comp. [1]). Considering the development of calculation methods, numerical simulations can be carried out to determine changes in the column's operating parameters (e.g. critical load of the column) in the event of a discontinuity of stiffness occurring in a specific location and with a given value. Such simulations are important because they provide information on the column's stability, which is related to the possibility of predicting the consequences of the formation of discontinuity of the column's stiffness.

In the literature, one can find works in which the authors studied the effect of stiffness discontinuities in the form of cracks on the stability and natural vibrations of columns or beam vibrations [6-13]. The simplest way to model a crack is to use an articulated connection with a rotational spring at the location of the crack. The size of the crack is modeled by appropriately selected spring stiffness. In the case of stability tests aimed at determining the critical load, such a simple model reflects the crack well.

In the case of vibrations of mechanical systems, measuring dynamic properties allows for the identification of damage in the form of a crack. This identification is performed both in terms of the crack location and its size. In the case of breathing cracks, applying rotational stiffness at the location of the damage is not sufficient. The breathing crack is characterized by variable stiffness depending on the direction of the bending moment, which varies during vibration. This problem was considered in [12, 13].

In [9], the effect of a crack on the stability of a Timoshenko beam subjected to a follower force was investigated. It was shown that the crack affects the frequencies and modes of transverse vibrations. The crack also influences the nature of instability. Depending on the crack size, either flutter or divergence instability is achieved. Several crack locations along the beam were considered in the study. The authors of [11] investigated the effect of crack size and axial force (compressive and tensile) on the transverse natural frequencies of the beam. A coupling between crack size and axial force was demonstrated for the natural frequencies. It was shown that an axially loaded beam with cracks cannot be treated as a superposition of two separate effects (crack and axial load) for the purpose of determining the natural frequencies.

In [8], the authors noted the nonlinearity resulting from the crack shape. In the case of an edge crack, the crack can close and open, which affects the stiffness of the beam at the crack location. The non-linearity mentioned above has been modelled as a piecewise-linear system. In an attempt to define effective natural frequencies for this piecewise-linear system, the idea of a “bilinear frequency” is utilized.

The authors of [14] demonstrated that the spectral characteristics of a beam with a breathing crack can be used as a qualitative indicator for determining crack defects in beam structures. The paper examined the influence of excitation frequency, excitation force amplitude, and crack parameters on the vibration response of a beam with a breathing crack.

Paper [15] investigated parametric vibrations of beams, taking into account geometrically nonlinear deformations and breathing cracks. The crack breathing was described using contact parameters and crack functions. Based on the results of numerical calculations, the occurrence of bifurcation phenomena in the considered resonant vibrations was demonstrated. The authors observed Neimark-Sacker and saddle-node bifurcations. Neimark-Sacker bifurcations lead to the generation of quasi-periodic and chaotic vibrations.

In [6], Kukla considered a column subjected to a generalized load with a force directed towards the positive pole and a load with a follower force directed towards the positive pole. Along the column, he considered two discontinuities in the bending stiffness modeled by rotational springs. The solution to the free vibration and stability problems was obtained using the Green function method. The aim of the conducted research was to determine the effect of cracks on the column vibration and to identify damage based on the free vibration frequencies.

This work addresses the problem of testing the stability of a column with discontinuities loaded with a follower force directed to the negative pole. The load was implemented by using circular heads in the considerations. Two locations of column stiffness discontinuities were also taken into account in the tests.

## 2. Formulation of the boundary value problem of column stability

The system considered in this paper is shown in Figure 1. It is a single-bar column subjected to a follower force directed towards the negative pole. In the system, the load is implemented by means of heads made of circular elements of radius  $R$ . Another implementation of the considered load is possible by using heads built using linear elements (rigid beams) (cf. [1]).

In the presented case, the fixed point (pole) is located above the end of the column. Additionally, two locations of hinges with rotational springs of stiffness  $C_1$  and  $C_2$  were considered along the column, which were used to model stiffness discontinuities. The location of stiffness discontinuities was determined by the parameters  $\zeta_1$  and  $\zeta_2$ :

$$\zeta_{l1} = \frac{l_1}{l}; \quad \zeta_{l2} = \frac{l_2}{l-l_1} \quad (1a, b)$$

where  $l_1$  and  $l_2$  are the lengths of individual column sections, taking into account the locations of stiffness discontinuities (the introduced hinges divide the column into three elements with lengths  $l_1$ ,  $l_2$  and  $l_3$ , respectively – in accordance with Figure 1).

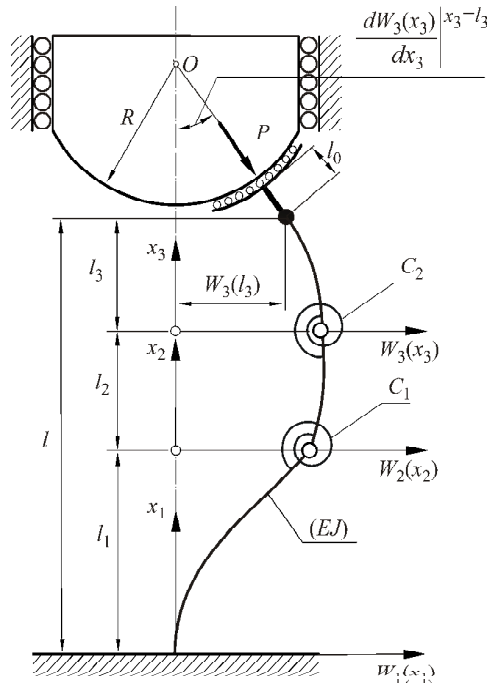


Fig. 1. Considered column with two stiffness discontinuities

Considering that the column under consideration is subjected to a conservative compressive load, it loses stability due to buckling. For this reason, the static stability criterion can be used to determine the critical load. According to this criterion, the column is stable if the potential energy of the slender system is characterized by a minimum in the equilibrium position. The static stability criterion can be presented as:

$$\delta V = 0 \quad (2)$$

where:

$V$  – potential energy,  
 $\delta$  – variation symbol.

The potential energy of the system after taking into account the Bernoulli-Euler beam theory can be written as follows:

$$\begin{aligned}
 V = & \frac{1}{2} \sum_i (EJ)_i \int_0^{l_i} \left( \frac{\partial^2 W_i(x_i)}{\partial x_i^2} \right)^2 dx_i - P \frac{1}{2} \sum_i \int_0^{l_i} \left( \frac{\partial W_i(x_i)}{\partial x_i} \right)^2 dx_i + \\
 & - P \frac{1}{2} (l_0 + R) \left( \frac{\partial W_3(x_3)}{\partial x_3} \Big|_{x_3=l_3} \right)^2 + \frac{1}{2} C_1 \left( \frac{\partial W_1(x_1)}{\partial x_1} \Big|_{x_1=l_1} - \frac{\partial W_2(x_2)}{\partial x_2} \Big|_{x_2=0} \right)^2 \\
 & + \frac{1}{2} C_2 \left( \frac{\partial W_2(x_2)}{\partial x_2} \Big|_{x_2=l_2} - \frac{\partial W_3(x_3)}{\partial x_3} \Big|_{x_3=0} \right)^2
 \end{aligned} \quad (3)$$

The geometric boundary conditions of the column are as follows:

$$W_1(0) = 0; \quad \frac{dW_1(x_1)}{dx_1} \Big|_{x_1=0} = 0; \quad (4a,b)$$

$$W_1(l_1) = W_2(0); \quad W_2(l_2) = W_3(0); \quad (4c,d)$$

$$W_3(l_3) + (R + l_0) \frac{dW_3(x_3)}{dx_3} \Big|_{x_3=l_3} = 0 \quad (4e)$$

After appropriate transformations, the natural boundary conditions are determined based on the relationship (2):

$$(EJ) \frac{d^2 W_1(x_1)}{dx_1^2} \Big|_{x_1=l_1} + C_1 \left( \frac{dW_1(x_1)}{dx_1} \Big|_{x_1=l_1} - \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=0} \right) = 0 \quad (5a)$$

$$(EJ) \frac{d^2 W_2(x_2)}{dx_2^2} \Big|_{x_2=0} + C_1 \left( \frac{dW_1(x_1)}{dx_1} \Big|_{x_1=l_1} - \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=0} \right) = 0 \quad (5b)$$

$$(EJ) \frac{d^2 W_2(x_2)}{dx_2^2} \Big|_{x_2=l_2} + C_2 \left( \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=l_2} - \frac{dW_3(x_3)}{dx_3} \Big|_{x_3=0} \right) = 0 \quad (5c)$$

$$(EJ) \frac{d^2 W_3(x_3)}{dx_3^2} \Big|_{x_3=0} + C_2 \left( \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=l_2} - \frac{dW_3(x_3)}{dx_3} \Big|_{x_3=0} \right) = 0 \quad (5d)$$

$$\begin{aligned} & (EJ) \frac{d^3 W_1(x_1)}{dx_1^2} \Big|_{x_1=l_1} - (EJ) \frac{d^3 W_2(x_2)}{dx_2^2} \Big|_{x_2=0} + \\ & + P \left( \frac{dW_1(x_1)}{dx_1} \Big|_{x_1=l_1} - \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=0} \right) = 0 \end{aligned} \quad (5e)$$

$$\begin{aligned} & (EJ) \frac{d^3 W_2(x_2)}{dx_2^2} \Big|_{x_2=l_2} - (EJ) \frac{d^3 W_3(x_3)}{dx_3^2} \Big|_{x_3=0} + \\ & + P \left( \frac{dW_2(x_2)}{dx_2} \Big|_{x_2=l_2} - \frac{dW_3(x_3)}{dx_3} \Big|_{x_3=0} \right) = 0 \end{aligned} \quad (5f)$$

$$(EJ) \frac{d^3 W_3(x_3)}{dx_3^3} \Big|_{x_3=l_3} + (EJ) \frac{d^2 W_3(x_3)}{dx_3^2} \Big|_{x_3=l_3} \frac{1}{R+l_0} = 0 \quad (5g)$$

and the differential equations of displacements:

$$(EJ) \frac{d^4 W_3(x_3)}{dx_3^4} + P \frac{d^2 W_i(x_i)}{dx_i^2} = 0 \quad (6)$$

The solution of differential equations can be presented in the form:

$$W_i(x_i) = A_i \cos(k_i x_i) + B_i \sin(k_i x_i) + C_i x_i + D_i \quad (7)$$

After substituting the solutions for both geometric and natural boundary conditions, a system of homogeneous equations is obtained:

$$[a_{ij}] \text{col}\{A_i, B_i, C_i, D_i\} = 0 \quad (8)$$

The determinant of the coefficient matrix of the system of equations (8) equal to zero is an equation from which the critical load can be determined:

$$|a_{ij}| = 0 \quad (9)$$

### 3. Results of numerical calculations

The results of numerical calculations are presented using dimensionless quantities:

$$\lambda_{cr} = \frac{P_{cr} l^2}{EJ}; \quad c_i = \frac{C_i l}{EJ}; \quad \zeta_R = \frac{R}{l}; \quad \zeta_{l0} = \frac{l_0}{l} \quad (10a-d)$$

As part of the numerical calculations, the critical load was determined depending on the system parameters. The results of the numerical calculations are presented in Figures 2-5.

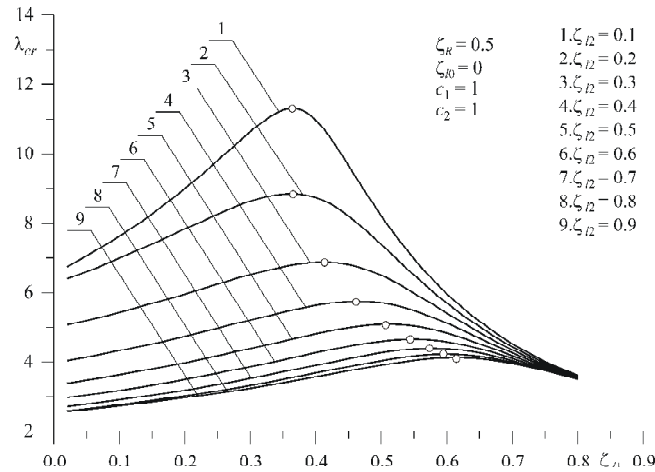


Fig. 2. Critical load parameter depending on the position of two discontinuities (parameters  $\zeta_{l1}$  and  $\zeta_{l2}$ ) at  $\zeta_R = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = c_2 = 1$

Figures 2-5 present the critical load depending on the location of the stiffness discontinuity for both the first and the second discontinuity (parameters  $\zeta_{l1}$  and  $\zeta_{l2}$ ). The calculations were performed for four combinations of spring stiffness: a)  $c_1 = 1$ ,  $c_2 = 1$ , b)  $c_1 = 1$ ,  $c_2 = 5$ , c)  $c_1 = 5$ ,  $c_2 = 1$ , d)  $c_1 = 5$ ,  $c_2 = 5$ . For easier comparison of results, the scales on the graphs (Figs. 2-5) are the same. As shown on the basis of the numerical calculations, the location and size of the stiffness discontinuity have a significant effect on the critical load. It is possible to determine such places along the column, in the location of which the occurrence of stiffness discontinuity significantly reduces the critical load. In most cases, the presented curves concerning the critical load are characterized by the occurrence of an extremum (maximum). The extremes of the  $\lambda_{cr}(\zeta_{l1})$  curves are marked in Figures 2-5 by points. The highest critical load of the considered column is obtained in the case when the location of the discontinuity of the bending stiffness  $c_2$  is near the discontinuity  $c_1$ , which corresponds to the lowest value of the  $\zeta_{l2}$  parameter. It was noticed that for the stiffness  $c_1 = 1$  and  $c_2 = 5$  in a large area of the location of the discontinuity of the stiffness related to  $c_1$  (with the parameter  $\zeta_{l1} > 0.55$ ), the critical load is only slightly dependent on the  $\zeta_{l2}$  parameter.

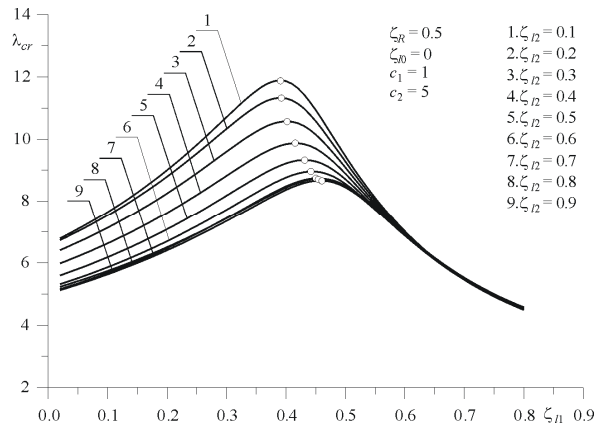


Fig. 3. Critical load parameter depending on the position of two discontinuities (parameters  $\zeta_{l1}$  and  $\zeta_{l2}$ ) at  $\zeta_R = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = 1$ ,  $c_2 = 5$

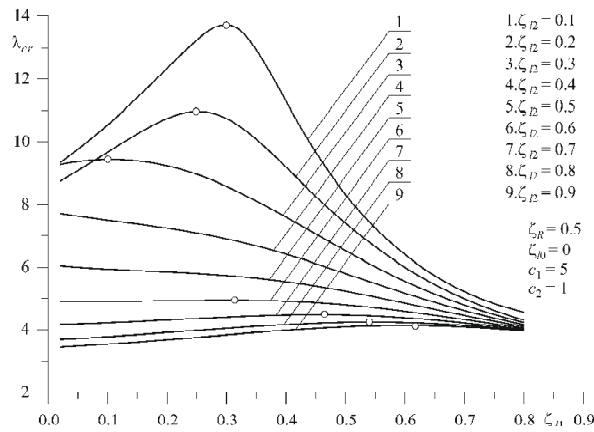


Fig. 4. Critical load parameter depending on the position of two discontinuities (parameters  $\zeta_{l1}$  and  $\zeta_{l2}$ ) at  $\zeta_R = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = 5$ ,  $c_2 = 1$

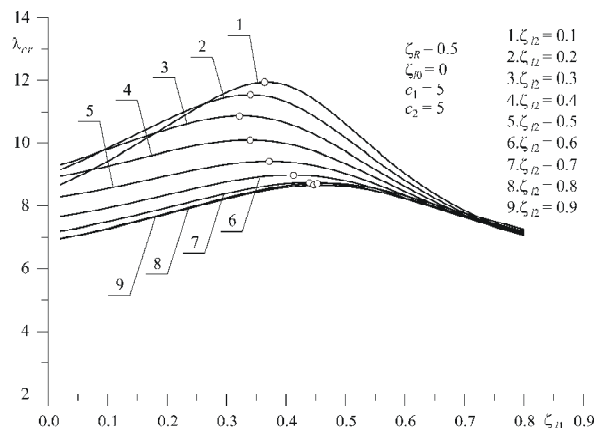


Fig. 5. Critical load parameter depending on the position of two discontinuities (parameters  $\zeta_{l1}$  and  $\zeta_{l2}$ ) at  $\zeta_R = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = c_2 = 5$

Figures 6-8 show how the value of the critical load of the considered column is influenced by the parameter defining the value of the radius of the loading head zetaR. In this case, the calculations were performed for  $\zeta_{l1} = \zeta_{l2} = 0.5$  and for different sizes of discontinuity (Fig. 6:  $c_1 = 1$ ;  $c_2 \in \langle 1,5 \rangle$ ; Fig. 7:  $c_1 \in \langle 1,5 \rangle$ ;  $c_2 = 1$ ; Fig. 8:  $c_2 = c_1 \in \langle 1,5 \rangle$ ).

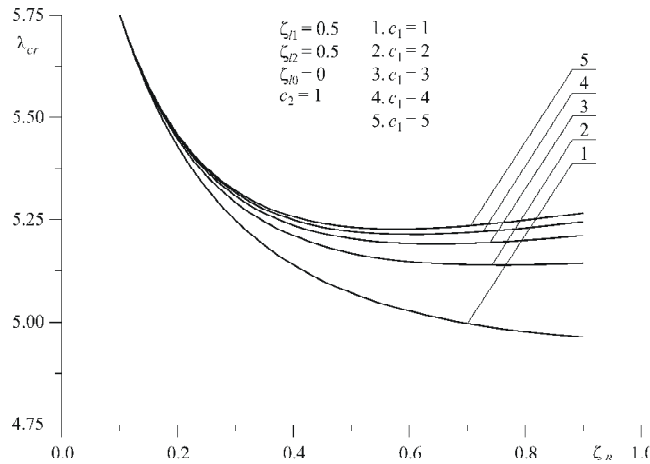


Fig. 6. Critical load parameter depending on the radius of the loading head (parameter  $\zeta_R$ ) at  $\zeta_{l1} = \zeta_{l2} = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = 1-5$ ,  $c_2 = 1$

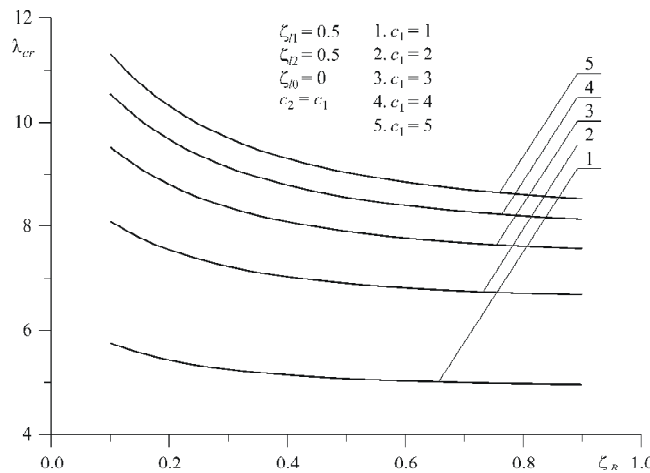


Fig. 7. Critical load parameter depending on the radius of the loading head (parameter  $\zeta_R$ ) at  $\zeta_{l1} = \zeta_{l2} = 0.5$ ,  $\zeta_{l0} = 0$ ,  $c_1 = c_2 = 1-5$

It has been shown that increasing the radius of the loading head decreases the value of the critical force. The dynamics of reducing the critical force with increasing the  $\zeta_R$  parameter depends on the size of the discontinuity. In the case where  $c_1 = 1$ ;  $c_2 \in \langle 1,5 \rangle$  (Fig. 6) a greater dynamics in reducing the critical load at lower

values of the  $\zeta_R$  parameter is obtained when the stiffness  $c_2$  is the lowest among those considered. Considering that  $c_2 = 1$  and  $c_1 \in \langle 1,5 \rangle$  (Fig. 7), a more rapid decrease in the critical load is obtained with a higher value of  $c_1$ .

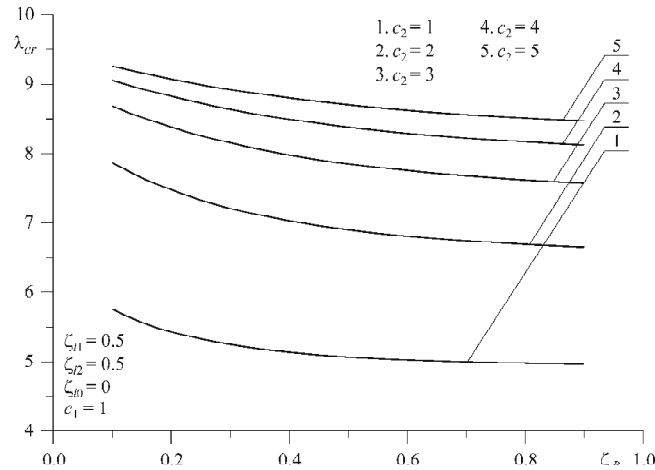


Fig. 8. Critical load parameter depending on the radius of the loading head (parameter  $\zeta_R$ ) at  $\zeta_{11} = \zeta_{12} = 0.5$ ,  $\zeta_0 = 0$ ,  $c_1 = 1$ ,  $c_2 = 1-5$

#### 4. Conclusions

The work involved formulating a boundary value problem concerning the stability of a column subjected to a follower force directed towards the negative pole. Simulations were carried out to determine the effect of two stiffness discontinuities on the bending of the system on the critical load. Both the size of the discontinuities and their location along the column were investigated. As it was shown, stiffness discontinuities have a very significant – destructive – effect on the entire system in the case of the considered column. The results of the conducted tests can be particularly useful when designing support systems. The causes of column discontinuities can be very different. One of the most common are fatigue cracks if the system is exposed to periodically variable loads. Discontinuity can also be caused by heating the column. The results allow prediction of the system's behavior in the event of stiffness discontinuities, which may arise in certain cases. For a complete study of the considered system, calculations should also be carried out to determine the effect of external load on the natural vibration frequencies. Such calculations evaluate structural strength and diagnostic indicators of stiffness discontinuities. Diagnostics of support systems is very useful and increases the safety of use, especially of constructed objects. Disturbance of the support structure, even locally, carries very serious consequences in the form of construction disasters.

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