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ANALYZING THE SYNCHRONIZED MOTION OF TWO SYMMETRIC WAVES IN A NOVEL TWO-MODE EXTENSION OF THE (2 + 1)-DIMENSIONAL KADOMTSEV-PETVIASHVILI MODEL

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Abstract. In this study, we introduced a novel two-mode extension of the (2 + 1)-dimensional Kadomtsev-Petviashvili model, where the wave dynamics are governed by moving two synchronized waves with equal amplitudes but distinct velocities. Our investigation employed several analytical approaches to identify the constraints required for the existence of a solution. Consequently, synchronized two wave-solutions, including bright-soliton, cusp-type, and periodic-concave patterns were derived. Furthermore, a graphical analysis is performed to examine the influence of key parameters, such as phase velocity and nonlinearity, on the dynamic behavior of the obtained two wave-solutions. Additionally, the stability of the obtained solutions is investigated through applying the Lyapunov direct method. Such analysis is particularly valuable in optical communication systems, where synchronized soliton-like modes can sustain long-distance transmission, and in plasma physics, where controlled wave synchronization aids in energy confinement. Thus, the presented findings not only enrich the theoretical understanding of nonlinear wave dynamics but also highlight their practical significance in advancing modern scientific and engineering applications.

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1. Introduction

Hirota and Satsuma explored the coupled Korteweg-de Vries equation, which describes the interaction of two long waves with distinct dispersion parameters [1]. Their study revealed that in the absence of wave overlap, where neither wave influences the other, the system reduces to the standard KdV equation, effectively

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describing a single wave. Building on these findings, Korsunsky extended the KdV model by formulating a second-order temporal version, expressed as follows [2]:

$$\begin{aligned}\theta_{\tau\tau}+(c_1+c_2)\theta_{\chi\tau}+c_1c_2\theta_{\chi\chi} &+ \big((\alpha_1+\alpha_2)\frac{\partial}{\partial\tau}+(\alpha_1c_2+\alpha_2c_1)\frac{\partial}{\partial\chi}\big)\theta\theta_\chi\\ &+ \big((\beta_1+\beta_2)\frac{\partial}{\partial\tau}+(\beta_1c_2+\beta_2c_1)\frac{\partial}{\partial\chi}\big)\theta_{\chi\chi\chi}=0.\end{aligned} \tag{1}$$

Here, χ and τ denote the space and time variables, respectively, while $\theta(\chi,\tau)$ represents the field function. Moreover, c_1 and c_2 refer to the phase velocities, α_1 and α_2 refer to the nonlinearity effects, and β_1 and β_2 are the dispersion parameters [2].

Then, Korsunsky applied the transformations

$$\begin{aligned} x &= \frac{\chi-\frac{c_1+c_2}{2}\tau}{\sqrt{\beta_1+\beta_2}},\\ t &= \frac{\tau}{\sqrt{\beta_1+\beta_2}},\\ \psi &= (\alpha_1+\alpha_2)\theta,\end{aligned} \tag{2}$$

and required some constraints. For instance, both $|\dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}|$ and $|\dfrac{\beta_1-\beta_2}{\beta_1+\beta_2}|$ are dominated by one, while as $c_2 \le c_1$. Accordingly, (1) is converted into

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right)\psi+\left(\frac{\partial}{\partial t}-\alpha s\frac{\partial}{\partial x}\right)\psi\psi_x+\left(\frac{\partial}{\partial t}-\beta s\frac{\partial}{\partial x}\right)\psi_{xxx}=0, \tag{3}$$

where $\psi=\psi(x,t)$. Drawing from the formulation in (3), Wazwaz [3, 4], along with the authors of [5–7], introduced the scaled version of the (1 + 1)-dimensional two--mode equation (TME), expressed as follows:

$$\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}\right)w+\left(\frac{\partial}{\partial t}-\alpha s\frac{\partial}{\partial x}\right)N(w,w_x,...)+\left(\frac{\partial}{\partial t}-\beta s\frac{\partial}{\partial x}\right)L(w_{kx})=0, \tag{4}$$

where $k \ge 2$, L and N are representing the linear and nonlinear terms of the model, and $w=w(x,t)$.

Analogously to (4), Alquran and Jaradat [8], suggested the following (2 + 1)--dimensional TME

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}-s^2\frac{\partial^2}{\partial x^2}-s^2\frac{\partial^2}{\partial y^2}\right)w &+ \left(\frac{\partial}{\partial t}-\alpha_1 s\frac{\partial}{\partial x}-\alpha_2 s\frac{\partial}{\partial y}\right)N(w,w_x,w_y,...)\\ &+ \left(\frac{\partial}{\partial t}-\beta_1 s\frac{\partial}{\partial x}-\beta_2 s\frac{\partial}{\partial y}\right)L(w_{kx},w_{ny})=0,\end{aligned} \tag{5}$$

where $k \ge 2$, $n \ge 2$, and $w=w(x,y,t)$.

The primary contribution of this paper lies in the formulation and analysis of a $(2+1)$-dimensional two-mode version derived from the $(2+1)$-dimensional Kadomtsev-Petviashvili (KP) equation. The mathematical representation of the KP equation is given by [9, 10]:

$$w_t + w_{xxx} + \lambda w w_x + \delta\ \partial_x^{-1} w_{yy} = 0, \quad w = w(x,y,t). \tag{6}$$

The KP model is an integrable equation that plays a significant role in various applications, including plasma physics and nonlinear optics. Numerous studies have investigated its diverse solution structures and physical properties using different analytical and numerical approaches [11–14].

Adopting the structure of (5), the (2+1)-dimensional two-mode Kadomtsev-Petviashvili (TMKP) equation is expressed in the following form.

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2} - s^2\frac{\partial^2}{\partial y^2}\right) w \quad &+ \quad \left(\frac{\partial}{\partial t} - \alpha_1 s\frac{\partial}{\partial x} - \alpha_2 s\frac{\partial}{\partial y}\right)\{\lambda w w_x\} \\ &+ \quad \left(\frac{\partial}{\partial t} - \beta_1 s\frac{\partial}{\partial x} - \beta_2 s\frac{\partial}{\partial y}\right)\{w_{xxx} + \delta\ \partial_x^{-1} w_{yy}\} = 0. \end{aligned} \tag{7}$$

The TMKP equation has potential applications in optical fiber communication and plasma wave dynamics and describes the motion of two synchronized waves with equal amplitudes but different velocities. In such systems, synchronized propagation can enhance signal stability, improve energy transport efficiency, and reduce modal dispersion. These findings confirm the practical relevance of the model in contexts where precise wave measurement, control, and stability are critical.

Accordingly, two main objectives are addressed in this study. First, we construct several noteworthy forms of synchronized waves within the newly proposed two-mode model by employing efficient explicit schemes. Second, we carry out a stability analysis to verify the validity of the obtained solutions.

Over the years, a wide variety of computational methods have been developed to derive exact solutions for nonlinear problems. These include the first integral method, the generalized projective Riccati method, the auxiliary equation method, the newly extended mapping method, among others [15–19]. In this study, we employ some of these effective techniques.

The structure of the paper is as follows. Section 2 transforms the TMKP equation into an ODE through a characteristic line approach. The resulting equation is then solved using the tanh(coth) expansion technique and a rational function method based on sine and cosine functions. Section 3 provides a graphical analysis to illustrate the synchronized motion of the obtained two-wave solutions and their dynamical behavior. In Section 4, we apply the Lyapunov direct method to investigate the stability analysis of the proposed model and to verify the validity of the attained solutions. Finally, the overall findings are summarized in the concluding section.

2. Some soliton solutions for the TMKP

In this section, we employ two powerful techniques to identify solitary wave solutions for the TMKP. Specifically, we utilize the tanh(coth)-expansion method and a rational function approach, both of which necessitate transforming the given PDE into a corresponding simplified ODE. To begin, we differentiate (7) with respect to the spatial coordinate x, yielding:

$$\begin{aligned} w_{ttx}-s^2w_{xxx}-s^2w_{yyx} \quad &+ \quad \left(\frac{\partial}{\partial t}-\alpha_1 s\frac{\partial}{\partial x}-\alpha_2 s\frac{\partial}{\partial y}\right)\{\lambda w w_x\}_x \\ &+ \quad \left(\frac{\partial}{\partial t}-\beta_1 s\frac{\partial}{\partial x}-\beta_2 s\frac{\partial}{\partial y}\right)\{w_{xxxx}+\delta\, w_{yy}\}=0. \quad (8) \end{aligned}$$

By introducing the new variable $\zeta = x+by-ct$, (8) is converted into:

$$\begin{aligned} (c^2-s^2(1+b^2))f''' \quad &- \quad (c+\alpha_1 s+\alpha_2 bs)\{\lambda f f'\}'' \\ &- \quad (c+\beta_1 s+\beta_2 bs)\left(f^{(5)}+\delta b^2 f'''\right)=0, \qquad (9) \end{aligned}$$

where $f=f(\zeta)=w(x,y,t)$. Furthermore, by integrating (9) three times while assuming the integration constant to be zero, we obtain the following simplified expression:

$$(c^2-s^2(1+b^2))f-\frac{\lambda}{2}(c+\alpha_1 s+\alpha_2 bs)f^2-(c+\beta_1 s+\beta_2 bs)\left(f''+\delta b^2 f\right)=0. \quad (10)$$

Remark 1 The ODE (10) can be interpreted as a quadratic algebraic equation in parameter c. Solving for c yields two distinct wave speeds, indicating that the propagation of (7) consists of two synchronized waves. □

Next, we solve (10) by the proposed methods.

2.1. Method I

Performing the balance order of $f''(\zeta)$ against $f^2(\zeta)$, the suggested tanh-coth solution to (10) is [20,21]

$$f(\zeta)=A+BY+FY^2+\frac{G}{Y}+\frac{R}{Y^2}. \quad (11)$$

The variable $Y=Y(\zeta)$ is the solution of the auxiliary equation $Y'=\mu(1-Y^2)$ and takes the form of $Y=\tanh(\mu\zeta)$ or $Y=\coth(\mu\zeta)$. Implicit differentiation of (11) yields

$$f''(\zeta)=\frac{2\mu^2\left(Y^2-1\right)\left(Y^4\left(BY+3FY^2-F\right)-GY+R\left(Y^2-3\right)\right)}{Y^4}. \quad (12)$$

Substituting both (11) and (12) into (10), and collecting the coefficients of $Y^j:\ j=-4,3,\ldots,3,4$, leads to solving the following system of nonlinear algebraic equations in the unknowns A, B, F, G, R, μ, b, and c.

$$
\begin{aligned}
0 &= -R\left(s\left(\alpha_2 b\lambda R+12b\beta_2\mu^2+12\beta_1\mu^2+\alpha_1\lambda R\right)+c\left(12\mu^2+\lambda R\right)\right),\\
0 &= -2G\left(s\left(\alpha_2 b\lambda R+2b\beta_2\mu^2+2\beta_1\mu^2+\alpha_1\lambda R\right)+c\left(2\mu^2+\lambda R\right)\right),\\
0 &= -s2R\left(\alpha_2 Ab\lambda+\alpha_1 A\lambda+b^3\delta\beta_2+\beta_1 b^2\delta+b^2 s-8b\beta_2\mu^2-8\beta_1\mu^2+s\right)\\
&- sG^2\lambda(\alpha_2 b+\alpha_1)-c\left(2A\lambda R+2b^2\delta R+G^2\lambda-16\mu^2 R\right)+2c^2 R,\\
&\vdots\\
0 &= -2B\left(s\left(\alpha_2 bF\lambda+2b\beta_2\mu^2+2\beta_1\mu^2+\alpha_1 F\lambda\right)+c\left(F\lambda+2\mu^2\right)\right),\\
0 &= -F\left(s\left(\alpha_2 bF\lambda+12b\beta_2\mu^2+12\beta_1\mu^2+\alpha_1 F\lambda\right)+c\left(F\lambda+12\mu^2\right)\right). \qquad (13)
\end{aligned}
$$

To help solve the above system, we may assume some valid restrictions on certain embedded parameters such as

$$\alpha_1=\alpha_2=\beta_1=\beta_2=k. \qquad (14)$$

Accordingly, the following findings are attained:

Case 1:

$$
\begin{aligned}
A &= \frac{4\mu^2}{\lambda},\\
B &= G=R=0,\\
F &= -\frac{12\mu^2}{\lambda},\\
c &= \frac{1}{2}\left(b^2\delta-4\mu^2\pm\sqrt{(b^2\delta-4\mu^2)^2+4(b^2+1)s^2+4k(b+1)s(b^2\delta-4\mu^2)}\right).
\end{aligned}
\qquad (15)
$$

Therefore, the first solution of TMKP is

$$w_1(x,y,t)=\frac{4\mu^2}{\lambda}\left(1-3\tanh^2\left(\mu\left(x+by-ct\right)\right)\right). \qquad (16)$$

Case 2:

$$
\begin{aligned}
A &= \frac{4\mu^2}{\lambda},\\
B &= G=F=0,\\
R &= -\frac{12\mu^2}{\lambda},\\
c &= \frac{1}{2}\left(b^2\delta-4\mu^2\pm\sqrt{(b^2\delta-4\mu^2)^2+4(b^2+1)s^2+4k(b+1)s(b^2\delta-4\mu^2)}\right).
\end{aligned}
\qquad (17)
$$

The second solution of TMKP is

$$w_2(x,y,t) = \frac{4\mu^2}{\lambda}\left(1-3\coth^2\left(\mu\left(x+by-ct\right)\right)\right). \tag{18}$$

Case 3:

$$\begin{aligned} A &= -F = \frac{12\mu^2}{\lambda}, \\ B &= G = R = 0, \\ c &= \frac{1}{2}\left(b^2\delta+4\mu^2 \pm \sqrt{\left(b^2\delta+4\mu^2\right)^2+4\left(b^2+1\right)s^2+4k(b+1)s\left(b^2\delta+4\mu^2\right)}\right). \end{aligned} \tag{19}$$

The third solution of TMKP is

$$w_3(x,y,t) = \frac{12\mu^2}{\lambda}\,\operatorname{sech}^2\left(\mu\left(x+by-ct\right)\right). \tag{20}$$

Case 4:

$$\begin{aligned} A &= -R = \frac{12\mu^2}{\lambda}, \\ B &= G = F = 0, \\ c &= \frac{1}{2}\left(b^2\delta+4\mu^2 \pm \sqrt{\left(b^2\delta+4\mu^2\right)^2+4\left(b^2+1\right)s^2+4k(b+1)s\left(b^2\delta+4\mu^2\right)}\right). \end{aligned} \tag{21}$$

The fourth solution of TMKP is

$$w_4(x,y,t) = -\frac{12\mu^2}{\lambda}\,\operatorname{csch}^2\left(\mu\left(x+by-ct\right)\right). \tag{22}$$

Case 5:

$$\begin{aligned} A &= -\frac{8\mu^2}{\lambda}, \\ B &= G = 0, \\ F &= R = -\frac{12\mu^2}{\lambda}, \\ c &= \frac{1}{2}\left(b^2\delta-16\mu^2\right) \\ &\pm \frac{1}{2}\left(\sqrt{\left(b^2\delta-16\mu^2\right)^2+4\left(b^2+1\right)s^2+4k(b+1)s\left(b^2\delta-16\mu^2\right)}\right). \end{aligned} \tag{23}$$

The fifth solution of TMKP is

$$w_5(x,y,t) = -\frac{4\mu^2}{\lambda}\left(2+3\tanh^2(\mu(x+by-ct))+3\coth^2(\mu(x+by-ct))\right). \quad (24)$$

Case 6:

$$\begin{aligned}
A &= \frac{24\mu^2}{\lambda},\\
B &= G = 0,\\
F &= R = -\frac{12\mu^2}{\lambda},\\
c &= \frac{1}{2}\left(b^2\delta+16\mu^2\right)\\
&\pm \frac{1}{2}\left(\sqrt{(b^2\delta+16\mu^2)^2+4(b^2+1)s^2+4k(b+1)s(b^2\delta+16\mu^2)}\right).
\end{aligned} \quad (25)$$

Using the identity, $\tanh^2(\zeta)+\coth^2(\zeta)-2 = 4\ \operatorname{csch}^2(2\zeta)$, the sixth solution of TMKP is

$$w_6(x,y,t) = -\frac{48\mu^2}{\lambda}\operatorname{csch}^2(2\mu(x+by-ct)). \quad (26)$$

2.2. Method II

Here, we seek rational form solutions to the TMKP equation. In particular, we assume that the solution of (10) can take one of the following [22, 23]:

$$f(\zeta) = \frac{A+B\sin(\mu\zeta)}{G+R\cos(\mu\zeta)}, \quad (27)$$

or

$$= \frac{A+B\cos(\mu\zeta)}{G+R\sin(\mu\zeta)}. \quad (28)$$

By applying (27), we obtain a polynomial expansion in terms of $\sin^m(\mu\zeta)\cos(\mu\zeta)^n$ for $m = 0,1$ and $n = 0,1,2$, and $\sin^3(\mu\zeta)$.

Equating each coefficient to zero leads to the following algebraic system.

$$
\begin{aligned}
0 &= 2BR^2\left(cb^2\delta + s\left(b^2\delta(b\beta_2+\beta_1) + \left(b^2+1\right)s\right) - c^2\right),\\
0 &= B^2\lambda R(s(\alpha_2 b+\alpha_1)+c),\\
0 &= B^2G\lambda\left(s(\alpha_2 b+\alpha)+c\right) + 2cAR^2\left(\mu^2 - b^2\delta\right)\\
&+ 2AR^2\left(c^2 - s\left(\left(b^2\delta-\mu^2\right)(b\beta_2+\beta_1) + \left(b^2+1\right)s\right)\right),\\
0 &= sBR\left(A\lambda(\alpha_2 b+\alpha_1) + G\left(2b^2\delta+\mu^2\right)(b\beta_2+\beta_1) + 2\left(b^2+1\right)Gs\right)\\
&+ cBR\left(\left(A\lambda + 2b^2\delta G + G\mu^2\right) + 2c^2G\right),\\
&\vdots
\end{aligned}
\tag{29}
$$

The above system contains multiple free parameters, making it challenging to solve directly. To simplify the process, we introduce additional reasonable constraints. For consistency, we adopt the same constraint used in the earlier tanh-coth approach, setting $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = k$. As a result, two distinct solutions for the TMKP, denoted as w_7 and w_8, are obtained and defined as follows:

$$
\begin{aligned}
w_7(x,y,t) &= -\frac{3\mu^2}{\lambda}\csc^2\left(\frac{1}{2}\mu\zeta\right),\\
w_8(x,y,t) &= -\frac{6\mu^2}{\lambda}\frac{1}{\cos(\mu\zeta)+1},
\end{aligned}
\tag{30}
$$

where

$$
\zeta = x + by - \frac{1}{2}t\left(b^2\delta \mp \sqrt{(\mu^2-b^2\delta)^2 + 4(b^2+1)s^2 + 4k(b+1)s(b^2\delta-\mu^2)} - \mu^2\right).
\tag{31}
$$

Alternatively, applying Eq. (28) yields similar results, providing several additional solutions for the TMKP equation denoted by w_9 and w_{10}, which are defined as follows:

$$
\begin{aligned}
w_9(x,y,t) &= -\frac{3\mu^2}{\lambda}\sec^2\left(\frac{1}{2}\mu\zeta\right),\\
w_{10}(x,y,t) &= -\frac{6\mu^2}{\lambda}\frac{1}{\sin(\mu\zeta)+1}.
\end{aligned}
\tag{32}
$$

3. Graphical analysis and concluding remarks

The application of the proposed methods yielded ten distinct solutions for the newly introduced TMKP model. These solutions can be classified into three categories based on their physical propagation patterns. Specifically, w_1 and w_3 correspond to the propagation of two synchronized bright-soliton waves (see Figure 1), whereas w_2, w_4, w_5, and w_6 describe the motion of two cusp-type waves, characterized by discontinuities along the characteristic wave plane $x+by-ct=0$ (see Figure 2). The remaining four solutions, w_7 through w_{10}, correspond to periodic--concave wave patterns with discontinuities occurring at $\mu(x+by-ct)=(2n+1)\pi$ for w_7 and w_8, and at $\mu(x+by-ct)=\frac{1}{2}(4n+3)\pi$ for w_9 and w_{10}.

On the other hand, we investigated how the principal parameters of the TMKP influence the propagation behavior of the obtained solutions. In particular, our analysis focused on the role of the phase velocity, denoted by s, and the dispersion parameter, k. For clarity, we designate the two synchronized waves as the left-wave (LW) and the right-wave (RW). The results reveal a notable trend: as either the phase velocity or the dispersion parameter increases, the LW and RW gradually converge, reducing the spatial separation between them. This phenomenon is clearly illustrated in Figures 3 and 4, where the narrowing gap between the two waves becomes increasingly evident.

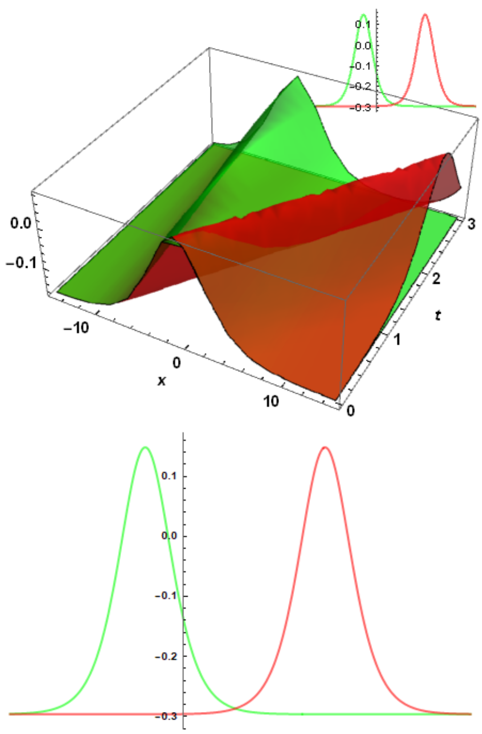

Fig. 1. 3D and 2D visualization of moving two synchronized bright-soliton waves depicted for $w_1(x,y,t)$. Where $\delta=6$, $\lambda=3$, $b=1$, $\mu=\frac{1}{4}$, and $y=1$

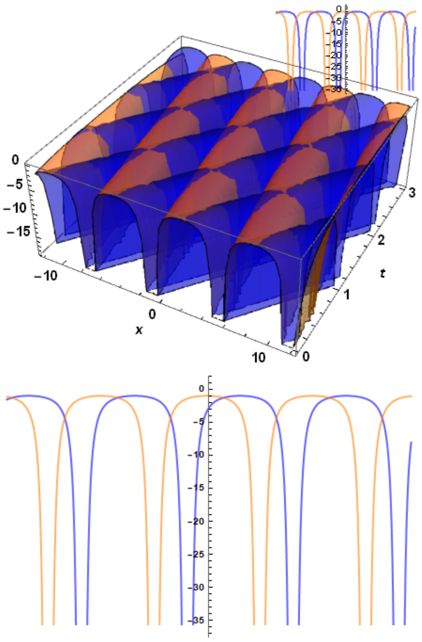

Fig. 2. 3D and 2D visualization of moving two synchronized periodic-concave waves depicted for $w_7(x,y,t)$. Where $\delta = 6$, $\lambda = 3$, $b = 1$, $\mu = \frac{1}{4}$, and $y = 1$

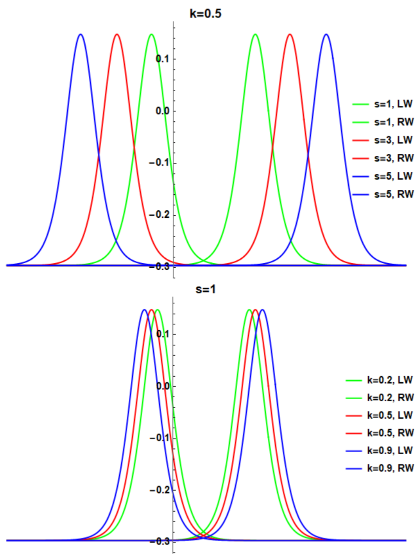

Fig. 3. The impact of the phase velocity s and dispersion parameter k on the propagation of the function $w_1(x,y,t)$ are examined with $\delta = 6$, $\lambda = 3$, $b = 1$, $\mu = \frac{1}{4}$, $y = 1$ and $t = 2$. Motion of two synchronized bright-soliton waves

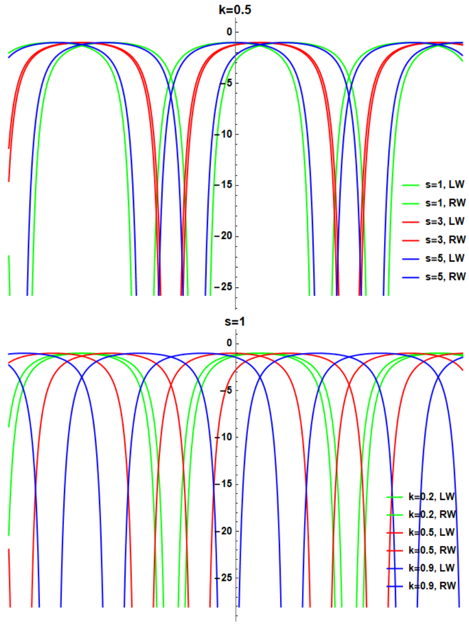

Fig. 4. The impact of the phase velocity s and dispersion parameter k on the propagation of the function $w_7(x,y,t)$ are examined with $\delta = 6$, $\lambda = 3$, $b = 1$, $\mu = \frac{1}{4}$, $y = 1$ and $t = 2$. Motion of two synchronized periodic-concave waves.

Remark 2 Within Method II, if the proposed rational forms are substituted by

$$f(\zeta) = \frac{A + B\sinh(\mu\zeta)}{G + R\cosh(\mu\zeta)}, \tag{33}$$

or

$$= \frac{A + B\cosh(\mu\zeta)}{G + R\sinh(\mu\zeta)}, \tag{34}$$

the same conclusions as previously established are obtained. In fact, the resulting solutions coincide with w_3, w_4 and w_6. □

4. Lyapunov direct method for stability analysis of TMKP equation

In this section, we shall consider the Lyapunov stability analysis of the TMKP equation using an appropriate choice for a Lyapunov function. Note that applying the Lyapunov direct method is appropriate for the TMKP equation because it provides a parameter-independent way to assess the stability of equilibrium solutions of the PDE. In particular, a Lyapunov functional is constructed whose time derivative is shown to be identically zero, implying that the equilibria are stable in the Lyapunov sense, and that this stability is not altered by variations in the system parameters.

This makes the method particularly suitable for the problem, as it avoids reliance on linearization or spectral assumptions along with the justification for such an approach to generalize to nonlinear infinite dimensional systems.

Consider the differential equation given in (10) with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = k$, $-1 < k < 1$. Then, we can rewrite the (10) in the form

$$\begin{aligned} f'' &= \left(\frac{c^2 - s^2(1+b^2)}{c+\beta_1 s+\beta_2 bs} - \delta b^2\right) f - \frac{\lambda(c+\alpha_1 s+\alpha_2 bs)}{2(c+\beta_1 s+\beta_2 bs)} f^2, \\ &= \left(\frac{c^2 - s^2(1+b^2)}{c+ks(1+b)} - \delta b^2\right) f - \frac{\lambda}{2} f^2. \end{aligned} \tag{35}$$

Now consider the positive definite function $V : \mathscr{D}(f) \to \mathbb{R}$ given by

$$V(f(\zeta)) = \frac{1}{2}\int_{\mathscr{D}(f)} (f')^2 d\zeta. \tag{36}$$

We shall show that V is a Lyapunov function to (10) with Lyapunov derivative given by

$$\begin{aligned} \dot{V}(f(\zeta)) &= \int_{\mathscr{D}(f)} f' f'' d\zeta, \\ &= \int_{\mathscr{D}(f)} f' \left(\left(\frac{c^2 - s^2(1+b^2)}{c+ks(1+b)} - \delta b^2\right) f - \frac{\lambda}{2} f^2\right) d\zeta, \\ &= \left[\frac{1}{2}\left(\frac{c^2 - s^2(1+b^2)}{c+ks(1+b)} - \delta b^2\right) f^2 - \frac{\lambda}{6} f^3\right]\Bigg|_{\mathscr{D}(f)}. \end{aligned} \tag{37}$$

If the solution of (10) is an even function, i.e., $f(-\zeta) = f(\zeta)$, or if it is periodic, then the Lyapunov derivative given in (37) vanishes [24–27]. This implies that the solution is Lyapunov stable within the defined domain of the function f.

For instance, consider solution w_1 given in (16). It is clear that this solution is even on its domain. Figures 1 and 3 show that the solution is stable for all x, y, t. Note that the stability is not affected by the change of the values of the parameters δ, λ, b, and μ.

Moreover, in solution w_7 given in (30), it is clear that the solution is even, periodic on $\zeta \in (0, \frac{2\pi}{\mu})$, and not continuous at $\zeta = 0, \frac{2\pi}{\mu}$. The same stability result as above will be concluded. Figures 2 and 4 are consistent with the theoretical findings.

Finally, since the solutions $w_1, \cdots, w_9$ are even functions, the system is Lyapunov stable within its domain, with distinct singularities determined by the domain of each solution. Moreover, the propagation is unaffected by the parameter values δ, λ, b, and μ. Note that w_{10} is not an even solution; however, it is periodic, and its stability analysis follows similarly. In conclusion, the system (10), as well as the TMKP, is Lyapunov stable and exhibits a limit cycle that is unaffected by model's parameters.

5. Conclusions

In this work, we introduced for the first time the two-mode version of the $(2+1)$-dimensional KP designed to describe nonlinear media where wave propagation occurs in the form of moving two synchronized waves. By exploring bright--soliton, cusp-type, and periodic-concave patterns, this study bridges theoretical modeling with practical scenarios where wave synchronization is essential. Beyond their mathematical interest, synchronized two wave solutions have tangible implications in applied physics and engineering. In optical fibers, simultaneous propagation of two stable modes can reduce dispersion effects and support high-capacity data transmission. Similarly, in plasma environments, synchronized waves contribute to controlled energy transfer and improved confinement stability.

Stability analysis is a fundamental step to determine which solutions are physically viable and can endure in realistic environments. To address this, we employed the Lyapunov indirect method to confirm that the propagation of synchronized two waves can yield robust and controllable wave structures. Such stability is particularly valuable in optical communication systems and in plasma physics.

For future work, the new TMKP offers multiple avenues for further exploration. These include deriving conservation laws to understand invariant quantities, applying the Painleve test to examine integrability, and exploring additional types of novel solutions that could enrich the understanding of two-mode wave propagation. Such studies would not only deepen the theoretical understanding of the model but also provide practical insights for modern applications in physics and engineering.

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