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ELLIPTIC BOUNDARY CONDITIONS FOR THE LAPLACIAN ON BILINEAR FORMS

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Abstract. A system of natural boundary conditions for second order differential operators of the Laplace type that act in the full bundle T^{*2} of covariant two-tensors (bilinear forms) is constructed and investigated on a compact Riemannian manifold of dimension $n \geq 2$ with the boundary. The system consists of 32, if $n > 2$, and 16, if $n = 2$, boundary conditions. All of them are proved to be self-adjoint and elliptic. As a result, for each particular condition of the system, the operator has a discrete spectrum and there is an orthonormal basis in $L^2(T^{*2})$ composed of smooth sections satisfying this condition.

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1. Introduction

An original system of natural elliptic boundary conditions for Laplace-type operators acting on T_2^* – the bundle of 2-tensors (bilinear forms) over an oriented Riemannian manifold M with boundary ∂M – is constructed and studied. The universal construction method developed by Branson and the second author (described in an unpublished manuscript and sketched in [1] or [2]) is used. A system of 32 (for $n > 2$) or 16 (for $n = 2$) boundary conditions is identified. Their ellipticity in the Gilkey-Smith sense (see, e.g., [3]) is established. This ensures the existence of both a discrete spectrum and an orthonormal basis of smooth eigenvectors in $L^2(T_2^*)$, facilitating standard harmonic analysis for such boundary value problems.

This system is equally applicable to other natural 2-tensor classes, including kernels of linear differential operators, which are fundamental in geometry, physics, and engineering [4–9]. Specifically, the integration of conformal Killing forms [5] and covariant differential operators [10] within this framework provides the essential mathematical tools for ensuring stable and unique reconstructions in medical X-ray and PET imaging, leveraging Carleman estimates for geodesic X-ray transforms [11] in anisotropic media.

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The Laplace operator constructed here acts on sections of the full bundle of bilinear forms, and no assumption of any symmetry is required. The same also applies to systems of natural boundary conditions. The basic properties of Δ are established in Section 4. In particular, we provide the Weitzenböck formula (Theorem 4.1), formal self-adjointness, and strong ellipticity (Theorem 4.2). The construction of the boundary condition system is contained in Section 5. Each condition is shown to be self-adjoint and elliptic for the strongly elliptic operator Δ (Theorem 5.1). As a result, there exists an orthonormal basis of $L^2(T^{*2})$ composed of smooth eigenvectors satisfying this condition (Corollary 5.1).

In the case when M is closed (i.e. compact and without the boundary), self-adjoint and strongly elliptic operators acquire a number of important properties [3].

A bordered manifold occurs much less frequently in the literature. The paper fills up this gap partially. The study of bordered manifolds is of considerable interest, both for their potential applications in physics and engineering and for the wide variety of natural boundary conditions they accommodate. Generally, in the case of bordered manifolds, the situation is more subtle and compound. Satisfactory conclusions, similar to that from the case of the closed manifold, always requires adopting and accepting appropriate boundary conditions.

The objects and morphisms are assumed to be smooth, i.e., of class C^∞. For any bundle E over M, denote by $C^\infty(E)$ the module of smooth section of E over the ring $C^\infty(M)$ of smooth functions on M. If $p \in M$, the fiber of E over p is denoted by E_p.

2. Basic definitions and notations

Let M be an oriented compact Riemannian manifold of dimension $n \geq 2$ with a nonempty boundary ∂M. Denote by T and T^* the tangent and cotangent bundle of M, respectively. Let g be the Riemannian metric on M, and let ∇ be its Levi--Civita connection (covariant derivative). Recall that, by the definition, for any $p \in M$, $\langle \cdot, \cdot \rangle = g_p(\cdot, \cdot)$ is then a scalar product (a Riemannian metric) in the tangent space T_p, and ∇ is the unique *torsion-free* and *Riemannian connection*, i.e., the unique connection satisfying the following two conditions: for any fields X, Y on M, $\nabla_X Y - \nabla_Y X = [X,Y]$ where $[X,Y]$ denotes the Lie bracket and $\nabla g = 0$.

The covariant derivative transmits naturally to $T^* = T^{*1}$ and next, by the Leibniz rule, to the bundle T^{*k} of k-tensors, $k \geq 0$. Finally, it is inherited by any sub-bundle. All the mentioned covariant derivatives (connections) will be denoted by the same symbol ∇. In particular, ∇ is well defined both in the bundle of skew-symmetric (alternating) tensors and in the bundle of symmetric tensors.

The metric isomorphism between the bundles T and T^* enables transmitting the scalar product to T^* and next, naturally, to the bundle $T^*$of covariant k-tensors on M.

For any $k = 1, 2, ...$, the bundles Λ^k and S^k of alternating and symmetric k-tensors being the subbundles of T^{*k} inherit the scalar product. It will be denoted by $\langle \cdot, \cdot \rangle$. We will also use, on Λ^k and on S^k, the alternative scalar product $\langle \cdot | \cdot \rangle = \dfrac{1}{k} \langle \cdot, \cdot \rangle$.

Note that $\Lambda^k = S^k$ for $k = 1$, and accept the obvious convention that $\Lambda^0 = S^0 = T^{*0} = M \times \mathbb{R}$.

The basic notions and theorems of Riemannian geometry and their applications can be found in [12].

3. Differential operators

In any tensor bundle on M, the *second order covariant derivative* is the differential operator defined by

$$\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y},$$

for fixed $X, Y \in C^\infty(T)$.

The *curvature operator* R_{XY} is defined by:

$$R_{XY} = \nabla^2_{XY} - \nabla^2_{YX}.$$

R_{XY} is an operator of order zero in any tensor bundle so, it can be treated for fixed X and Y, as the bundle endomorphism and, in particular,

$$R_{XY} : \Lambda^k \longrightarrow \Lambda^k \quad \text{and} \quad R_{XY} : S^k \longrightarrow S^k, \quad k \geq 0.$$

We see that, according to the convention: $(\nabla\varphi)(X, X_1, \ldots, X_k) = (\nabla_X \varphi)(X_1, \ldots, X_k)$, the covariant derivative ∇ may be treated as a map: $\nabla : C^\infty(T^{*k}) \longrightarrow C^\infty(T^* \otimes T^{*k})$, so, in particular, it can be alternating or symmetrized.

Its local shape is the following:

$$\nabla = \sum_{i=1}^{n} e_i^\star \otimes \nabla_{e_i} \tag{1}$$

for any local frames: $e_1, \ldots, e_n$ in T and the dual frame $e_1^\star, \ldots, e_n^\star$ in $T^\star$.

For $k \geq 0$ (with the obvious conventions for $k = 0$) define the operators of *alternating* and *symmetric* derivations $\mathrm{d}^a : C^\infty(\Lambda^k) \longrightarrow C^\infty(\Lambda^{k+1})$ and $\mathrm{d}^s : C^\infty(S^k) \longrightarrow C^\infty(S^{k+1})$ as the alternating and symmetric parts of ∇, respectively:

$$\mathrm{d}^a \varphi(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} (\nabla_{X_j} \varphi)(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

for $\varphi \in C^\infty(\Lambda^k)$,

$$\mathrm{d}^s \psi(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (\nabla_{X_j} \psi)(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

for $\psi \in C^\infty(S^k)$, $X_1, \ldots, X_{k+1} \in C^\infty(T)$.

By (1), we get the following local expressions for d^a and d^s directly:

PROPOSITION 3.1

$$\mathrm{d}^a \varphi = \sum_{j=1}^{n} e_j^* \wedge \nabla_{e_j} \varphi \quad \text{and} \quad \mathrm{d}^s \psi = \sum_{j=1}^{n} e_j^* \odot \nabla_{e_j} \psi \tag{2}$$

for $\varphi \in C^\infty(\Lambda^k)$ and $\psi \in C^\infty(S^k)$.

Note that for $k = 0$, i.e., for the action of operators on functions, the formulas (2) are still valid with the multiplications $\wedge$ and $\odot$ being replaced by the usual multiplication by functions.

The *trace operator* $\mathrm{tr} : T^{*k} \longrightarrow T^{*k-2}$ is defined, for $k \geq 2$ as the metric contraction with respect of the first two arguments and $\mathrm{tr} = 0$ for $k = 0, 1$.

The operators tr^a and tr^s are just restrictions of tr to suitable bundles.

For $k \geq 1$, the operator $\nabla^* = -\,\mathrm{tr}\,\nabla : C^\infty(T^k) \longrightarrow C^\infty(T^{k-1})$ is formally adjoint to ∇ with respect to the global scalar product $(\cdot,\cdot)$. Its restrictions:

$$\mathrm{d}^{a*} = -\,\mathrm{tr}\,\nabla|_{C^\infty(\Lambda^k)} \quad \text{and} \quad \mathrm{d}^{s*} = -\,\mathrm{tr}\,\nabla|_{C^\infty(S^k)}$$

are called here the *alternating* and *symmetric coderivatives*.

PROPOSITION 3.2

$$\mathrm{d}^{a*} \varphi = -\sum_{j=1}^{n} \iota_{e_j} \nabla_{e_j} \varphi \quad \text{and} \quad \mathrm{d}^{s*} \psi = -\sum_{j=1}^{n} \iota_{e_j} \nabla_{e_j} \psi$$

for $\varphi \in C^\infty(\Lambda^k)$, $\psi \in C^\infty(S^k)$ and for any a local orthonormal frame $e_1 \ldots, e_n$ of T.

PROOF The proof is analogous to the one in [13] (Lemma 1.4.11) and relies on the properties of the covariant derivative and a standard calculus. ∎

The coderivatives d^{a*} and d^{s*} are formally adjoint to d^a and d^s, respectively, for the *global scalar product*: $(\cdot|\cdot) = \frac{1}{k}(\cdot,\cdot)$.

More precisely:

PROPOSITION 3.3

$$(\mathrm{d}^a \varphi_1 | \varphi_2) = (\varphi_1 | \mathrm{d}^{a*} \varphi_2) \quad \text{and} \quad (\mathrm{d}^s \psi_1 | \psi_2) = (\psi_1 | \mathrm{d}^{s*} \psi_2),$$

for $\varphi_1 \in C^\infty(\Lambda^k), \varphi_2 \in C^\infty(\Lambda^{k+1})$ and $\psi_1 \in C^\infty(S^k), \psi_2 \in C^\infty(S^{k+1})$ respectively, if only φ_1 or φ_2 and ψ_1 or ψ_2 are of the compact support not intersecting the boundary.

PROOF See, e.g., [14]. ∎

Recall that the de Rham-Hodge Laplacian Δ^a and the Sampson Laplacian Δ^s in the bundles of alternating and symmetric forms, respectively, are defined as follows

$$\Delta^a = d^{a*}d^a + d^a d^{s*} \quad \text{and} \quad \Delta^s = d^{s*}d^s - d^s d^{s*}. \tag{3}$$

It is known that both Δ^a and Δ^s are second order linear differential operators in their bundles.

Let $e_1,\dots,e_n$ be a local orthonormal frame of sections of T and, let $e_1^*,\dots,e_n^*$ be the dual frame. Define two endomorphisms $\mathscr{R}^a : \Lambda^k \longrightarrow \Lambda^k$ and $\mathscr{R}^s : S^k \longrightarrow S^k$, $k \geq 0$ by:

$$\mathscr{R}^a = \sum_{i,j=1}^{n} e_j^\star \wedge \iota_{e_i} R_{e_i e_j} \quad \text{and} \quad \mathscr{R}^s = \sum_{i,j=1}^{n} e_j^\star \odot \iota_{e_i} R_{e_i e_j}. \tag{4}$$

One can check that the right hand sides do not depend on the choice of frame.

If $k=0$, $\mathscr{R}^s = \mathscr{R}^a = 0$. If $k=1$ the two endomorphisms of $\Lambda^1 = S^1 = T^*$coincide and reduce to the *Ricci endomorphism* of T^*: $\mathrm{Ric} = \sum_{i,j=1}^{n} e_j^\star \iota_{e_i} R_{e_i e_j}$. The significant fact about the endomorphisms (4) is that they are point-wise symmetric in the following sense:

PROPOSITION 3.4 Let $k \geq 0$, $p \in M$.

$$\langle \mathscr{R}^a \varphi_1, \varphi_2 \rangle = \langle \varphi_1, \mathscr{R}^a \varphi_2 \rangle \quad \text{and} \quad \langle \mathscr{R}^s \psi_1, \psi_2 \rangle = \langle \psi_1, \mathscr{R}^s \psi_2 \rangle,$$

for $\varphi_1, \varphi_2 \in \Lambda_p^k$ and $\psi_1, \psi_2 \in S_p^k$.

PROOF The case $k=0$ is trivial. If $k=1$, we obtain the classical Ricci tensor which is obviously symmetric. The case of $k=2$ was proved by Lichnerowicz in 1961. For the general case, we refer to [15]. ∎

Recall Weitzenböck's formulas for the two Laplaces under consideration:

THEOREM 3.1

$$\Delta^a = \nabla^*\nabla + \mathscr{R}^a \quad \text{and} \quad \Delta^s = \nabla^*\nabla - \mathscr{R}^s. \tag{5}$$

PROOF For the proof, see, e.g. [14]. ∎

4. Laplacians in bundles of two-tensors

The bundle of two-tensors decomposes (splits) into the direct sum of two of its subbundles:

$$T^{*2} = \Lambda^2 \oplus S^2. \tag{6}$$

These summands are mutually orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$. Furthermore, Λ^2 and S^2 are invariant subbundles under the natural representation (group action) of the orthogonal group $SO(n)$ in T^{*2}. The bundle Λ^2 is generically irreducible (i.e. there is no proper subbundle of Λ^2 which is $SO(n)$-invariant) with the exception that $n=4$ where Λ^2 splits into the direct sum of two invariant subbundles Λ^2_+ and Λ^2_- of self dual and anti-self dual forms, respectively, i.e., we have a decomposition $\Lambda^2=\Lambda^2_+\oplus\Lambda^2_-$. The bundle S^2 splits into two invariant subbundles S^2_0 and λg of trace-free symmetric forms and the 1-dimensional bundle of pure traces, i.e., tensors being multiplicities of the metric tensor g, respectively. Discussion of these specific subbundles, while interesting, is beyond the scope of this paper, where we will focus solely on splitting (6).

The splitting (6) uniquely defines the decomposition of any two-tensor ω,

$$\omega=\omega^a+\omega^s,$$

into its alternating and symmetric parts. The decomposition extends naturally to the sections of the bundles.

Define the *Laplace* operator

$$\Delta : C^\infty(T^{*2}) \longrightarrow C^\infty(T^{*2})$$

in the full bundle T^{*2} by the formula:

$$\Delta\omega=\Delta^a\omega^a+\Delta^s\omega^s, \quad \omega\in C^\infty(T^{*2}). \tag{7}$$

Here, the crucial fact for our further considerations is that both Δ^a and Δ^s respect the splitting (6) (cf., Theorem 3.1). Moreover, both $\mathscr{R}^a$ and $\mathscr{R}^s$ also respect the splitting in the sense that $\mathscr{R}^a$ and $\mathscr{R}^s$ are endomorphisms of the bundles Λ^2 and S^2, respectively.

Define an endomorphism $\mathscr{R}$ of the bundle T^{*2} by

$$\mathscr{R}\omega=\mathscr{R}^a\omega^a-\mathscr{R}^s\omega^s,\ \ \omega\in C^\infty(T^{*2}). \tag{8}$$

It is evident that $\mathscr{R}$ is then an operator of order zero (endomorphism) of the full bundle T^{*2} into itself, keeping the subbundles Λ^2, S^2 invariant.

PROPOSITION 4.1 For $\omega\in C^\infty(T^{*2})$ and a local orthonormal frame $e_1,\dots,e_n$ of the tangent bundle, the endomorphism $\mathscr{R}$ is given locally by

$$\mathscr{R}\,\omega(e_r,e_s)=-\sum_{i=1}^n\Big(R_{e_ie_r}\omega(e_s,e_i)+R_{e_ie_s}\omega(e_i,e_r)\Big).$$

Moreover, $\mathscr{R}$ is a point-wise self-conjugate endomorphism of T^{*2} i.e., for any $\omega_1,\omega_2\in T^{*2}_p$ we have

$$\langle\mathscr{R}\omega_1,\omega_2\rangle=\langle\omega_1,\mathscr{R}\omega_2\rangle. \tag{9}$$

PROOF From (8) and (4), straightforward calculations in a local orthonormal frame $e_1, \dots, e_n$ yield:

$$
\begin{aligned}
&(\mathscr{R}^a\omega^a - \mathscr{R}^s\omega^s)(e_r, e_s) \\
&= \sum_{i,j=1}^{n} \Big((e_j^* \wedge \iota_{e_i} R_{e_i e_j}\omega^a)(e_r, e_s) - (e_j^* \odot \iota_{e_i} R_{e_i e_j}\omega^s)(e_r, e_s) \Big) \\
&= \sum_{i,j=1}^{n} \Big(\delta_{jr}(\iota_{e_i} R_{e_i e_j}\omega^a)(e_s) - \delta_{js}(\iota_{e_i}) R_{e_i e_j}\omega^a)(e_r) \\
&\qquad - \delta_{jr}(\iota_{e_i}(R_{e_i e_j}\omega^s)(e_s) - \delta_{js}(\iota_{e_i} R_{e_i e_j}\omega^s)(e_r) \Big) \\
&= \sum_{i=1}^{n} \Big((\iota_{e_i} R_{e_i e_r}\omega^a)(e_s) - (\iota_{e_i} R_{e_i e_s}\omega^a)(e_r) \\
&\qquad - (\iota_{e_i} R_{e_i e_r}\omega^s)(e_s) - (\iota_{e_i} R_{e_i e_s}\omega^s)(e_r) \Big) \\
&= -\sum_{i=1}^{n} \Big(R_{e_i e_r}\omega(e_s, e_i) + R_{e_i e_s}\omega(e_i, e_r) \Big).
\end{aligned}
$$

Formula (9) is a direct consequence of Proposition 3.4 and the orthogonality of splitting (6). ■

In this way, the following important relationship can be established:

THEOREM 4.1 (THE WEITZENBÖK FORMULA)

$$\Delta = \nabla^*\nabla + \mathscr{R}. \tag{10}$$

PROOF Formula (10) is a direct consequence of (7), (5), and (8). ■

Conclude the section with the following

THEOREM 4.2 The Laplacian Δ is a formally self-adjoint strongly elliptic operator in the bundle T^{*2}.

PROOF The first part of the thesis, i.e., the formal self-adjointness of Δ means that the relation

$$(\Delta\omega_1, \omega_2) = (\omega_1, \Delta\omega_2), \tag{11}$$

holds for any sections $\omega_1, \omega_2 \in C^\infty(T^{*2})$ with a compact support that does not intersect the boundary. However, (11) is a simple consequence of the Weitzenböck formula (10), the obvious self-adjointness of the Bochner Laplacian $\nabla^*\nabla$, and the point-wise self-conjugacy of the operator $\mathscr{R}$ confirmed in Proposition 4.1.

To establish the ellipticity of the differential operator Δ, we evaluate its principal symbol. Fix $p \in U$. Take $\xi \in T_p^*$ and $e \in T_p^{*2}$. Let f be a smooth function in a neighborhood of p such that $f(p)=0$ and $\mathrm{d}f(p)=\xi$. Let $\omega \in C^\infty(T^{*2})$ be any section such that $\omega(p)=e$. Then (cf. e.g., [16] for the definition), the *leading symbol* $\sigma_\Delta(p,\xi)$ of Δ at p in the direction ξ is the endomorphism of T_p^{*2} defined by

$$\sigma_\Delta(p,\xi)(e)=\frac{1}{2!}\Delta(f^2\omega)(p).$$

By (7) and (3), we infer that in our case

$$\sigma_\Delta(p,\xi)\,e = -\|\xi\|^2 e, \tag{12}$$

for all $p\in M$, which implies the strong ellipticity of Δ, since the symbol is negative definite for all $\xi \neq 0$. ■

5. Elliptic boundary conditions

Recall that M is a compact and oriented n-dimensional Riemannian manifold ($n\geq 2$) with a nonempty boundary ∂M. The orientation and the Riemannian metric define a unique volume form Ω_M on M and then a unique volume form $\Omega_{\partial M}$ on ∂M by $\Omega_{\partial M}=\iota_\nu \Omega_M$, where, on the boundary, ν is the outer field of vectors normal to ∂M.

To form a set of natural boundary conditions, we need the following *half-geodesic coordinate system* at the boundary (cf. [3]).

Let $x=(y,r)$ be a local coordinate system on M near ∂M such that $y=(y_1,\dots,y_{n-1})$ is a local coordinate system on ∂M, and r is the normal distance to the boundary. Assume that $\partial M=\{x: r(x)=0\}$ and that $\dfrac{\partial}{\partial r}$ is the inward unit normal vector. Let $e_1=\dfrac{\partial}{\partial x_1},\dots,e_{n-1}=\dfrac{\partial}{\partial x_{n-1}},e_n=\dfrac{\partial}{\partial r}$ and $e_1^*=dx_1,\dots,e_{n-1}^*=dx_{n-1},e_n^*=dr$. Normalize the choice of coordinate by requiring that the curves $x(r)=(y_0,r)$, for r being small enough, are unit speed geodesics for any $y_0\in\partial M$. Since M is compact, the inward geodesic flow identifies a neighborhood of ∂M in M with the collar $\partial M\times[0,\delta)$ for some $\delta>0$. The collaring gives a splitting of $TM=T(\partial M)\oplus T\mathbb{R}$ and a dual splitting $T^*M=T^*(\partial M)\oplus T^*\mathbb{R}$. To reflect this splitting, we let for $\xi\in T^*M$ that $\xi=(\zeta,z)$ where $\zeta\in T^*\partial M$, $z\in T^*\mathbb{R}$. Let – in this coordinate system – $\nu=-\dfrac{\partial}{\partial r}$ and $\nu^*=-dr$. Note that then ν^* is a smooth field of covectors in a neighborhood of ∂M, so, in particular, over ∂M.

Let us recall the decomposition (6) and then perform a further orthogonal decomposition of Λ^2 and S^2 onto orthogonal direct sums of subbundles.

For $n>2$, the bundle of alternating 2-tensors decomposes into the direct sum of its two subbundles:

$$\Lambda^2=\Lambda^{2^T}\oplus\Lambda^{2^N}, \tag{13}$$

where Λ^{2^N} is the bundle of alternating two-forms that are proportional to ν^*, and Λ^{2^T} is the orthogonal complement of Λ^{2^N} in Λ^2. Note that for $n = 2$, $\Lambda^{2^T} = 0$. So, in this case, (13) reduces to:

$$\Lambda^2 = \Lambda^{2^N}. \tag{14}$$

For $n \geq 2$, the bundle of symmetric 2-tensors decomposes into the direct sum of its three subbundles:

$$S^2 = S^{2^T} \oplus S^{2^N} \oplus S^{2^{NN}}. \tag{15}$$

Here, $S^{2^{NN}}$ is the bundle of symmetric two-forms that are proportional to $\nu^* \odot \nu^*$, S^{2^N} is a subbundle of the orthogonal complement of $S^{2^{NN}}$ in S^2 consisting of forms that are proportional to ν^*, S^{2^T} is the orthogonal complement of $S^{2^N} \oplus S^{2^{NN}}$ in S^2.

It is obvious that the summands of the splittings (13) - (13) are invariant – at each point of ∂M – for the action of $\mathrm{SO(n-1)}$, wherein $\mathrm{SO(n-1)}$ is treated as the subgroup of $\mathrm{SO(n)}$, keeping ν^* invariant.

Take $\omega = \omega^a + \omega^s \in C^\infty(T^{*2})$. By (13)-(15) we have then that on ∂M:

For $n > 2$:

$$\omega^a = \omega^{aT} + \omega^{aN} = \omega^a{}_0 + \omega^a_1 \wedge \nu^*. \tag{16}$$

For $n = 2$:

$$\omega^a = \omega^{aN} = \omega^a_1 \wedge \nu^*. \tag{17}$$

For $n \geq 2$:

$$\omega^s = \omega^{sT} + \omega^{sN} + \omega^{sNN} = \omega^s_0 + \omega^s_1 \odot \nu^* + \omega^s_2 \odot \nu^* \odot \nu^*, \tag{18}$$

where ω^a_i, $i = 0, 1$, are alternating $(2-i)$-forms and ω^s_j, $j = 0, 1$, are symmetric $(2-j)$-forms, which do not contain ν^* in their coordinate expansions, and where ω^s_2 is simply a function.

A double application of the Green formula to our Laplacian $\Delta = \nabla^*\nabla + \mathscr{R}$ leads directly to the following:

PROPOSITION 5.1 For $\omega, \eta \in C^\infty(T^{*2})$:

$$\begin{aligned}(\Delta\omega, \eta) - (\omega, \Delta\eta) = &\int_{\partial M} (\langle \omega^a, \nabla_\nu \eta^a \rangle - \langle \nabla_\nu \omega^a, \eta^a \rangle)\Omega_{\partial M} \\ &+ \int_{\partial M} (\langle \omega^s, \nabla_\nu \eta^s \rangle - \langle \nabla_\nu \omega^s, \eta^s \rangle)\Omega_{\partial M}.\end{aligned} \tag{19}$$

By the orthogonality of splittings (16)-(18), we get the following decomposition for the scalar products that appear in the integrands on the right hand sides of (19):

For $n > 2$:

$$
\begin{aligned}
\langle \omega^a, \nabla_\nu \eta^a \rangle = & \\
& + \left\langle \omega^{aT}, \nabla_\nu \eta^{aT} \right\rangle \\
& + \left\langle \omega^{aN}, \nabla_\nu \eta^{aN} \right\rangle .
\end{aligned}
$$

For $n = 2$:

$$
\begin{aligned}
\langle \omega^a, \nabla_\nu \eta^a \rangle = & \\
& + \left\langle \omega^{aN}, \nabla_\nu \eta^{aN} \right\rangle .
\end{aligned}
$$

For $n \geq 2$:

$$
\begin{aligned}
\langle \omega^s, \nabla_\nu \eta^s \rangle = & \\
& + \left\langle \omega^{sT}, \nabla_\nu \eta^{sT} \right\rangle \\
& + \left\langle \omega^{sN}, \nabla_\nu \eta^{sN} \right\rangle \\
& + \left\langle \omega^{sNN}, \nabla_\nu \eta^{sNN} \right\rangle .
\end{aligned}
$$

In summary, we have:

For $n > 2$ and a given section $\omega \in C^\infty(T^{*2})$, the right hand sides of the last equality can be presented (symbolically) in the form of two column matrix:

$$
\begin{bmatrix}
\omega^{aT} & \nabla_\nu \omega^{aT} \\
\omega^{aN} & \nabla_\nu \omega^{aN} \\
\omega^{sT} & \nabla_\nu \omega^{sT} \\
\omega^{sN} & \nabla_\nu \omega^{sN} \\
\omega^{sNN} & \nabla_\nu \omega^{sNN}
\end{bmatrix} .
$$

For $n = 2$ and a given section $\omega \in C^\infty(T^{*2})$, the corresponding matrix is analogous, except for the first row which must be removed.

Now, an individual boundary condition will be established if we control exactly one of the two entries of each row to force it to be zero. Exhausting all the described possibilities, we get $32 = 2^5$ for $n > 2$, and $16 = 2^4$ for $n = 2$ conditions in our natural system of boundary conditions.

Therefore, the system is uniquely determined by the system of following matrices ($2^5 = 32$ of five-row matrices for $n > 2$ or $2^4 = 16$ of four-row matrices for $n = 2$)

$$
\begin{bmatrix}
0 & * \\
0 & * \\
\vdots & \vdots \\
0 & *
\end{bmatrix}, \cdots,
\begin{bmatrix}
0 & * \\
* & 0 \\
\vdots & \vdots \\
0 & *
\end{bmatrix}, \cdots,
\begin{bmatrix}
* & 0 \\
* & 0 \\
\vdots & \vdots \\
* & 0
\end{bmatrix} . \tag{20}
$$

For example, if $n \geq 2$, the first matrix defines the condition: $\omega^{aT} = 0$, $\omega^{aN} = 0$, $\omega^{sT} = 0$, $\omega^{sN} = 0$, $\omega^{sNN} = 0$ on ∂M or, equivalently, $\omega = 0$ on ∂M, so the first matrix defines the well-known Dirichlet boundary condition. The last matrix defines the condition: $\nabla_\nu \omega^{aT} = 0$, $\nabla_\nu \omega^{aN} = 0$, $\nabla_\nu \omega^{sT} = 0$, $\nabla_\nu \omega^{sN} = 0$, $\nabla_\nu \omega^{sNN} = 0$ on ∂M or, equivalently, $\nabla_\nu \omega = 0$ on ∂M, so the last matrix defines the Neumann boundary condition. These two conditions, together with all remaining ones, defined by the matrices standing between the fringe two, create the full set of 32 natural boundary conditions. Note that under each of them, Δ is self-adjoint with respect to the global scalar product. If $n = 2$, the matrices (20) have four rows, and similar reasoning leads to the set of 16 boundary conditions.

Now, we are ready to confirm the ellipticity of our boundary conditions.

Recall that a self-adjoint boundary condition is *elliptic* with respect to the cone $C = \mathbb{C} \setminus (\mathbb{R}_- \cup \mathbb{R}+)$ for the elliptic operator Δ, if (cf. [3]) the ordinary differential equation $\sigma_\Delta(\zeta, \frac{\partial}{\partial r})\varphi = \lambda\varphi$, where $(\zeta, \lambda) \neq (0,0) \in T^*(\partial M) \times C$, has a unique solution satisfying the boundary condition and such that $\varphi(r) \to 0$ as $r \to \infty$.

THEOREM 5.1 Each of the 32, if $n > 2$, and 16, if $= 2$, boundary conditions defined by system of matrices (20) is elliptic with respect to the cone $C = \mathbb{C} \setminus (\mathbb{R}_- \cup \mathbb{R}+)$ for the elliptic operator Δ.

PROOF In the light of Theorem 4.2 and formula (12), the ellipticity is governed here by the differential equation.

$$\frac{\partial^2 f}{\partial r^2} = \mu f \tag{21}$$

where $\mu = \lambda + ||\zeta||^2$ and $\lambda \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \mathbb{R}_-)$. By the assumption on λ, μ is not real, so for exactly one of its two square roots, call it τ, the solutions are of the form $f(r) = const \cdot \exp(\sqrt{-1}\,\tau r)$ with $\lim_{r\to\infty} f(r) = 0$.

The problem of uniqueness of the solution to (21) can be reduced, by decomposition of the bilinear form into its alternating and symmetric parts, to the study of the uniqueness problem in each of these cases. We refer to [17] for the alternating case and to [18] for the symmetric case, respectively. ■

The obtained ellipticity of the boundary conditions assures then not only the discreetness of spectrum but also the existence of a basis consisting of smooth eigenvectors for Δ, satisfying the boundary condition. This is a direct consequence of [3, Lemma 1.9.1].

More precisely, we can state that the following

COROLLARY 5.1 For each of the 32, if $n > 2$, and 16, if $= 2$, boundary conditions defined by system of matrices (20):

(a) there exists a complete orthonormal system $(\phi_n)_{n=1}^{\infty}$ for $L^2(T^{*2})$ with

$\Delta\phi_n = \lambda_n\phi_n$,
(b) ϕ_n are smooth (i.e. of class C^∞) and satisfy the boundary condition,
(c) $\lambda_n \in \mathbb{R}$ and $\lim_{n\to\infty}|\phi_n| = \infty$.

6. Conclusions

The class of bilinear forms, i.e., tensors of degree two, occupies a unique place in the class of all tensors. Its importance stems primarily from its wide-ranging applications in geometry, physics, and engineering. Especially on manifolds with a non--empty boundary, where various boundary conditions naturally imposed by external forces must be satisfied.

There are a number of bundles of bilinear forms on M that appear naturally in geometry and physics and a number of differential operators in these bundles. Operators of the Laplace type play a distinguished role there.

By Theorem 5.1, the Laplacian Δ has a discrete spectrum on any compact bordered manifold M, if only any boundary condition of our system defined by matrices (20) is satisfied.

Note that this system of boundary conditions is also applicable to other natural classes of 2-tensors. This is especially true regarding classes defined by the kernels of natural linear differential operators. Such classes play a significant role in geometry and physics [4, 6, 19]. An example is the class of trace free Killing 2-tensors [20]. Killing and conformal Killing 2-tensors [20, 21] appear in several other areas of mathematics, e.g., in connection with geometric inverse problems [22], integrable systems [11], and Einstein-Weyl geometry [4, 23]. The case of a Kählerian manifold M is particularly interesting. The conformal Killing tensors of degree two on such M were studied in [9]. Another example are so called *Hamiltonian* (1,1)-forms strictly related to the J-invariant conformal Killing 2-forms on M [24].

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