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FRACTIONAL DUAL-PHASE-LAG HEAT CONDUCTION PROBLEM IN A HOMOGENEOUS SPHERE WITH AN INTERNAL HEAT SOURCE

Urszula Siedlecka *

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
urszula.siedlecka@pcz.pl

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Abstract. In this paper, a solution of the dual-phase-lag heat conduction equation is presented. The considerations are related to the heat conduction in a homogeneous sphere. In our research, we assume that there is no heat exchange with the surroundings through the sphere's surface, but that it is heated by an internal heat source. The solution to the problem was derived analytically, but due to its complexity, numerical methods were used to determine the inverse Laplace transform, namely the Stefhest method. The considerations are summarized with numerical examples in which the influence of the Caputo derivative order on the temperature distribution in the considered sphere was investigated.

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Keywords: *heat transfer, dual-phase-lag model, time-fractional derivative, Laplace transform*

1. Introduction

In recent years, it has been observed that describing phenomena not only using ordinary or partial differential equations, but also using fractional differential equations [1-4], has become increasingly common. And one of the quite widely described phenomena is heat conduction [5-8]. The research also aims to describe the heat flow phenomenon as precisely as possible, and one such method seems to be the introduction of a phase-lag model, and in particular, a dual-phase-lag model [9-14]. An additional advantage of the phase-lag model is the elimination of the classical theory based on Fourier's law, which is the starting point for many authors' considerations, a certain nonphysical property, namely, the infinite rate of heat propagation.

In this work, we present a dual-phase-lag time-fractional heat conduction model for a homogeneous spherical body with heating in the center and adiabatic conditions

* Corresponding author

on the surface. An analytical solution to the problem was derived, but the inverse Laplace transform appearing in the final solution was determined numerically. Theoretical considerations were supported by an example in which the influence of fractional order time derivatives on the temperature distribution in the considered sphere was investigated.

The structure of the paper is as follows: Section 2 presents the mathematical model of the problem, including the basic equations that describe it. The solution to the heat transfer problem is presented in Section 3. Section 4 presents numerical examples of determining the temperature distribution in the medium under consideration. The paper concludes with Section 5, which presents the main conclusions.

2. Formulation of the problem

The subject of this study is the heating of a homogeneous sphere of radius R by an internal heat source. Furthermore, we assume that there is no heat exchange with the surroundings on the outer surface of the sphere.

The starting point for our considerations is the classical Fourier law of heat flow, which employs dual-phase-lag time parameters. We consider the fractional dual-phase lag equation in the following form (a detailed description of the derivation of this equation can be found, e.g., in the works [15]):

$$\rho C_p\left(\frac{\partial^\beta}{\partial t^\beta}T(r,t)+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta}{\partial t^\beta}T(r,t)\right)=\nabla\cdot\left(k\nabla T(r,t)\right)+\tau_T\frac{\partial^\alpha}{\partial t^\alpha}\nabla\cdot\left(k\nabla T(r,t)\right)+g(r,t)+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}g(r,t), \quad (1)$$

where τ_q and τ_T are the thermodynamic properties of the material called the thermal relaxation and thermalization times, respectively; k is the thermal conductivity of the material; C_p is a specific heat of the medium; ρ is the density of the material; r is the radial coordinate of the spherical region; t is the time; the operators Nabla $\nabla\varphi \equiv \mathrm{grad}(\varphi)$ and $\nabla\cdot\boldsymbol{\Phi} \equiv \mathrm{div}(\boldsymbol{\Phi})$, represent the gradient and divergence operators, and T is the temperature. The function $g(r,t)$ is an internal heat source, given by the formula

$$g(r,t)=Q_1\,H\left(\frac{R}{5}-r\right), \quad (2)$$

where Q_1 [W/m^3] is a constant value of the volumetric heat source, and H is the Heaviside step function. Colloquially speaking, function (2) describes uniform heating inside the sphere at 20 % of the inner radius length.

In the fractional dual-phase-lag model of heat conduction (equation (1)), the fractional derivatives occur. Here, in this work, the Caputo derivative of orders α and β is used, which is defined (identically for orders α and β) as [16]

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}T(r,t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m}{\partial \tau^m}T(r,\tau)\,d\tau, & m-1<\alpha<m,\ m\in\mathbb{N}\\ \dfrac{\partial^m}{\partial t^m}T(r,t), & \alpha=m\in\mathbb{N}\end{cases}, \quad (3)$$

where Γ is the Gamma function.

Our considerations are carried out assuming that k = const.; therefore

$$\nabla\cdot\left(k\nabla T(r,t)\right)=k\nabla^2 T(r,t), \quad (4)$$

and then the Laplace operator can be written in the following form:

$$\nabla^2 T(r,t)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}T(r,t)\right)=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rT(r,t)\right). \quad (5)$$

Equation (1) is complemented by the following initial-boundary conditions:

$$T(r,t)\big|_{t=0}=T_0, \quad (6)$$

$$\frac{\partial^{\beta}}{\partial t^{\beta}}T(r,t)\bigg|_{t=0}=T_1(r)=\frac{g(r,t)}{\rho C_p}=\frac{Q_1}{\rho C_p}H\left(\frac{R}{5}-r\right), \quad (7)$$

$$T(r,t)\big|_{r\to 0}<\infty, \quad (8)$$

$$\frac{\partial}{\partial r}T(r,t)\bigg|_{r=R}=0\,. \quad (9)$$

3. Solution of the problem

In this section, the solution of the initial-boundary problem described by equation (1) and equations (6)-(9) is derived. The most important steps leading to obtaining the solution are presented.

Using formula (5), equation (1) takes the following form:

$$\begin{aligned}&\frac{\partial^{\beta}}{\partial t^{\beta}}T(r,t)+\tau_q\frac{\partial^{\alpha}}{\partial t^{\alpha}}\frac{\partial^{\beta}}{\partial t^{\beta}}T(r,t)\\&\quad=\frac{k}{\rho C_p}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\,T(r,t)\right)+\tau_T\frac{\partial^{\alpha}}{\partial t^{\alpha}}\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\,T(r,t)\right)\right).\\&\qquad+\frac{1}{\rho C_p}\left(g(r,t)+\tau_q\frac{\partial^{\alpha}}{\partial t^{\alpha}}g(r,t)\right)\end{aligned} \quad (10)$$

To simplify the further calculations, we make the substitution:

$$U(r,t) = r\,T(r,t), \tag{11}$$

then equation (10) takes the form

$$\begin{aligned}\frac{\partial^\beta}{\partial t^\beta}U(r,t) + \tau_q \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta}{\partial t^\beta}U(r,t) = \frac{k}{\rho C_p}\left(\frac{\partial^2}{\partial r^2}U(r,t) + \tau_T \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^2}{\partial r^2}U(r,t)\right)\\ + \frac{r}{\rho C_p}\left(g(r,t) + \tau_q \frac{\partial^\alpha}{\partial t^\alpha}g(r,t)\right)\end{aligned}, \tag{12}$$

where the function $U(r,t)$ satisfies the following initial-boundary conditions:

$$U(r,t)\big|_{t=0} = rT_0, \tag{13}$$

$$\left.\frac{\partial^\beta}{\partial t^\beta}U(r,t)\right|_{t=0} = rT_1(r) = r\frac{Q_1}{\rho C_p}H\left(\frac{R}{5} - r\right), \tag{14}$$

$$U(r,t)\big|_{r\to 0} = 0, \tag{15}$$

$$\left.\frac{\partial}{\partial r}\frac{U(r,t)}{r}\right|_{r=R} = \left.\left(\frac{1}{r}\frac{\partial}{\partial r}U(r,t) - \frac{1}{r^2}U(r,t)\right)\right|_{r=R} = 0. \tag{16}$$

To determine a solution of the initial-boundary problem (12)-(16), we apply the Laplace transform technique. For this purpose we use the transform $\mathcal{L}\{U(r,t)\}(s) = \bar{U}(r,s)$ and several properties of the Laplace transform [17]:

$$\mathcal{L}\left\{\frac{\partial^\beta U(r,t)}{\partial t^\beta}\right\}(s) = s^\beta \bar{U}(r,s) - s^{\beta-1}U(r,0) = s^\beta \bar{U}(r,s) - s^{\beta-1}rT_0, \tag{17}$$

$$\begin{aligned}\mathcal{L}\left\{\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta}{\partial t^\beta}U(r,t)\right\}(s) &= s^\alpha \mathcal{L}\left\{\frac{\partial^\beta}{\partial t^\beta}U(r,t)\right\}(s) - s^{\alpha-1}\left.\frac{\partial^\beta}{\partial t^\beta}U(r,t)\right|_{t=0}\\ &= s^{\alpha+\beta}\bar{U}(r,s) - s^{\alpha+\beta-1}rT_0 - s^{\alpha-1}r\frac{Q_1}{\rho C_p}H\left(\frac{R}{5} - r\right)\end{aligned}, \tag{18}$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial r^2}U(r,t)\right\}(s) = \frac{\partial^2}{\partial r^2}\bar{U}(r,s), \tag{19}$$

$$\mathcal{L}\left\{\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^2}{\partial r^2}U(r,t)\right\}(s) = s^\alpha \frac{\partial^2}{\partial r^2}\bar{U}(r,s) - s^{\alpha-1}\underbrace{\frac{\partial^2}{\partial r^2}U(r,0)}_{=0}, \tag{20}$$

$$\mathcal{L}\left\{g(r,t)+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}g(r,t)\right\}(s)=\bar{g}(r,s)+\tau_q\left(s^\alpha\bar{g}(r,s)-s^{\alpha-1}g(r,0)\right). \quad (21)$$

Because the considered function $g(r,t)$ (described by equation (2)) does not depend on time t, hence $\partial^\alpha g(r,t)/\partial t^\alpha=0$ and the formula (21) reduces to the form

$$\mathcal{L}\left\{g(r,t)+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}g(r,t)\right\}(s)=\frac{1}{s}Q_1H\left(\frac{R}{5}-r\right). \quad (22)$$

After inserting the above transforms into equation (12), then this equation takes the form

$$\begin{aligned}&s^\beta\bar{U}(r,s)-s^{\beta-1}rT_0+\tau_q\left(s^{\alpha+\beta}\bar{U}(r,s)-s^{\alpha+\beta-1}rT_0-s^{\alpha-1}r\frac{Q_1}{\rho C_p}H\left(\frac{R}{5}-r\right)\right)\\&=\frac{k}{\rho C_p}\left(1+\tau_Ts^\alpha\right)\frac{\partial^2\bar{U}(r,s)}{\partial r^2}+\frac{1}{s}r\frac{Q_1}{\rho C_p}H\left(\frac{R}{5}-r\right)\end{aligned} \quad (23)$$

or written as

$$\frac{\partial^2\bar{U}(r,s)}{\partial r^2}-a^2(s)\bar{U}(r,s)+b(s)H\left(\frac{R}{5}-r\right)r+c(s)r=0, \quad (24)$$

$$a(s)=\sqrt{\frac{\rho C_p}{k}s^\beta\frac{1+\tau_qs^\alpha}{1+\tau_Ts^\alpha}},\ b(s)=\frac{Q_1}{k}\frac{1}{s}\frac{1+\tau_qs^\alpha}{1+\tau_Ts^\alpha},\ c(s)=T_0\frac{\rho C_p}{k}s^{\beta-1}\frac{1+\tau_qs^\alpha}{1+\tau_Ts^\alpha}. \quad (25)$$

To obtain the solution of the differential equation (24) that satisfies the boundary conditions (15) and (16), one can use, e.g., the Fourier transform method or just use the Mathematica package for calculations. This solution can be written as follows:

$$\begin{aligned}\bar{U}(r,s)&=\frac{c(s)}{a^2(s)}r+\frac{b(s)}{a^2(s)}r\times\\&\times\left(\frac{\sinh(a(s)r)}{a(s)r}\frac{\left(1-a^2(s)\frac{R^2}{5}\right)\sinh\left(a(s)\frac{4R}{5}\right)-a(s)\frac{4R}{5}\cosh\left(a(s)\frac{4R}{5}\right)}{a(s)R\cosh(a(s)R)-\sinh(a(s)R)}\right.\\&\left.+\begin{cases}\frac{1}{a(s)r}\sinh\left(a(s)\left(r-\frac{R}{5}\right)\right)+\frac{R}{5r}\cosh\left(a(s)\left(r-\frac{R}{5}\right)\right), & \text{if } r>\frac{R}{5}\\1, & \text{if } r\le\frac{R}{5}\end{cases}\right).\end{aligned} \quad (26)$$

Because

$$T(r,t) = \frac{U(r,t)}{r} = \mathcal{L}^{-1}\left\{\frac{\bar{U}(r,s)}{r}\right\}(t), \tag{27}$$

and using the relationships $c(s)/a^2(s) = T_0/s$, $b(s)/a^2(s) = Q_1/\left(\rho C_p s^{\beta+1}\right)$ and $\mathcal{L}^{-1}\{1/s\}(t) = 1$, then one obtains

$$T(r,t) = T_0 + \frac{Q_1}{\rho C_p}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta+1}}\times\right.$$
$$\times\left(\frac{\sinh(a(s)r)}{a(s)r}\frac{\left(1-a^2(s)\frac{R^2}{5}\right)\sinh\left(a(s)\frac{4R}{5}\right)-a(s)\frac{4R}{5}\cosh\left(a(s)\frac{4R}{5}\right)}{a(s)R\cosh(a(s)R)-\sinh(a(s)R)}\right. \tag{28}$$
$$\left.\left.+\begin{cases}\frac{1}{a(s)r}\sinh\left(a(s)\left(r-\frac{R}{5}\right)\right)+\frac{R}{5r}\cosh\left(a(s)\left(r-\frac{R}{5}\right)\right), & \text{if } r > \frac{R}{5}\\ 1, & \text{if } r \le \frac{R}{5}\end{cases}\right)\right\}(t)$$

If an analytical inverse Laplace transform is unavailable or excessively complex to compute (as in the case occurring in equation (28)), then one can employ the numerical inversion techniques to approximate the time-domain function $T(r,t)$. Common methods include the Stehfest, Talbot, and de Hoog algorithms [18-20]. Here, in numerical simulations, the Stehfest algorithm has been chosen.

4. Numerical examples

The solution of the fractional dual-phase heat conduction problem presented in Section 3 was used in numerical calculations of the temperature distribution in the homogeneous sphere, and more precisely, the effect of the time-fractional order of the Caputo derivatives α and β. The analysis concerns the heat conduction in the sphere with heating inside and with an adiabatic boundary condition on the surface. We assume the following geometrical and physical data: $R = 0.25$ m, $k = 16.0$ W/(m K), $\rho C_p = 4.848\cdot 10^6$ J/(m^{3}K), $Q_1 = 2{,}000{,}000$, $\tau_q = 5$ s and $\tau_T = 1$ s. The results are presented for dimensionless time defined by the formula $\bar{t} = tk/R^2$ and for non-dimensional radii $\bar{r} = r/R$. In Figure 1, the temperature distributions along the radius of the sphere at the selected moments of time $\bar{t} \in \{0.25, 0.5, 1, 2\}$ and for values $\alpha = \beta$ equal to: $\alpha = \beta \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ are presented.

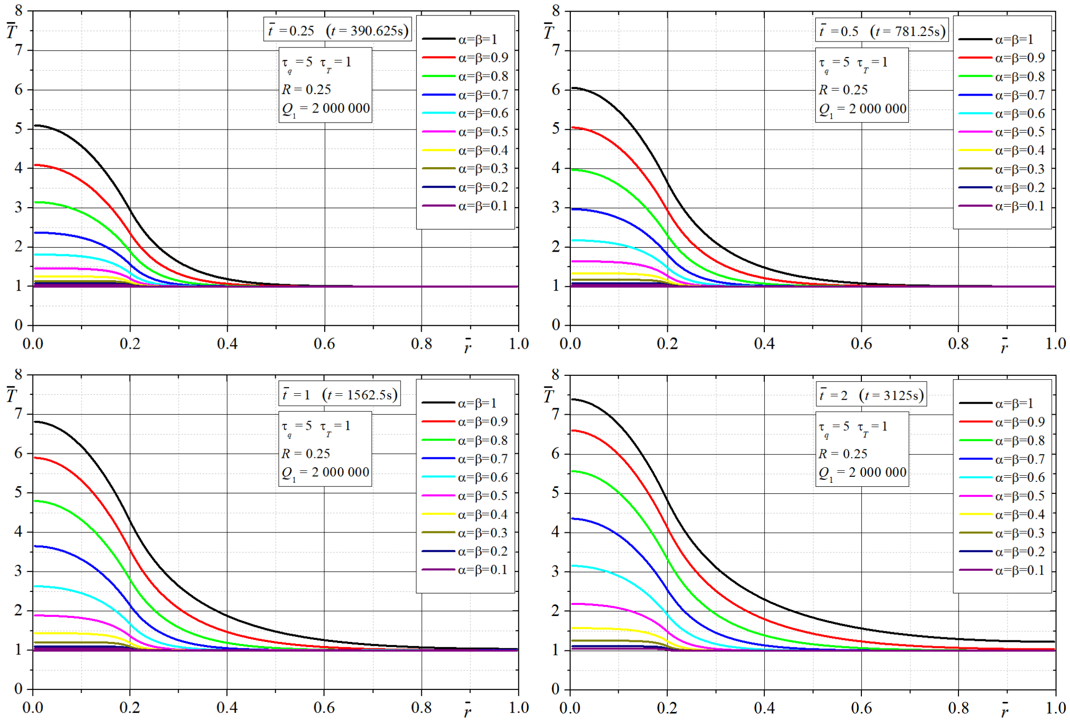

Fig. 1. Temperature distribution along the radius of the sphere for different values of α = β and for the non-dimensional time $\bar{t} \in \{0,25;0,5;1,0;2,0\}$

It turned out that in the case of α = β, the average temperature in the sphere is consistent with the relationship:

$$T_{avg}(t) = T_0 + \frac{t^{\beta}}{\Gamma(\beta+1)} \frac{Q_1}{\rho C_p} \frac{1}{5^3}. \tag{29}$$

The selected values of the average temperature in the sphere for different values of α = β and for the dimensional time $t \in \{390.625; 781.25; 1562.5; 3125\}$ are presented in Table 1.

Figure 2 shows the temperature changes along the radius of the sphere for the same α and $\bar{t}$ values as in Figure 1, but assuming that β = 1.

For both solutions presented in Figures 1 and 2, it can be seen that the highest temperature increase occurs in the center of the sphere. Both figures show identical solutions for the case when α = β = 1. According to equation (29), the average temperature of the sphere increases to infinity when the simulation time also tends to infinity (taking into account the assumed boundary condition (9) and the definition of heat source (2)), whereas for β = 1, such a temperature increase is linear.

Table 1. The average temperature in the sphere for different values of $\alpha = \beta$ and for the dimensional time $t \in \{390.625; 781.25; 1562.5; 3125\}$

$\alpha = \beta$	Time [–]			
	390.625	781.25	1562.5	3125
1.0	21.28906250	22.57812500	25.15625000	30.31250000
0.9	20.73795287	21.37706875	22.56970115	24.79523191
0.8	20.41955925	20.73049508	21.27186581	22.21444700
0.7	20.23678529	20.38465884	20.62488013	21.01512076
0.6	20.13257693	20.20094904	20.30458179	20.46165967
0.5	20.07359512	20.10407922	20.14719024	20.20815843
0.4	20.04047298	20.05340442	20.07046755	20.09298249
0.3	20.02203036	20.02712256	20.03339179	20.04111011
0.2	20.01185614	20.01361913	20.01564427	20.01797055
0.1	20.00630013	20.00675231	20.00723694	20.00775637

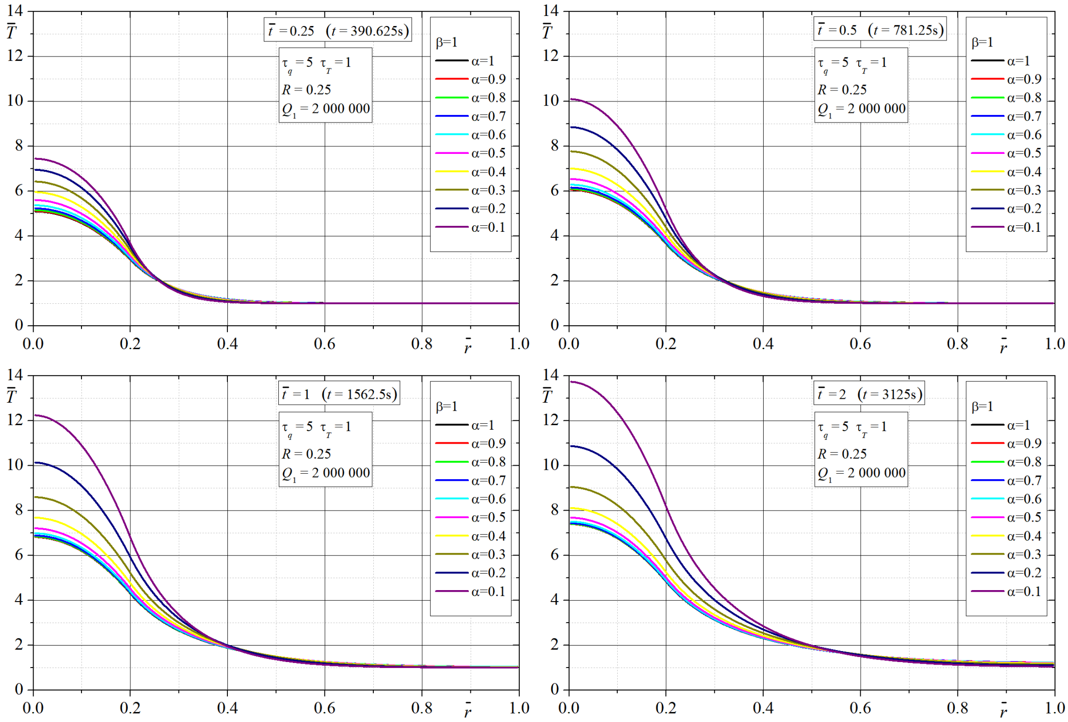

Fig. 2. Temperature distribution along the radius of the sphere for $\beta = 1$, different values of α and for the non-dimensional time $\bar{t} \in \{0,25; 0,5; 1,0; 2,0\}$

The influence of the parameter β on the temperature distribution (see Figure 1) is very significant, especially when $\beta \ll 1$. In a certain sense, it can be concluded that this parameter "dampens" the influence of the heat source on heating the interior of the domain. In the case of the results presented in Figure 2 (i.e. for $\beta = 1$), one can notice different ways of heat propagation from the center of the sphere depending on the parameter α; however, the average values of the temperature distribution along the sphere radius (i.e. the integral mean) for the fixed simulation times are identical (despite the visible higher temperature in the center of the sphere when $\alpha < 1$).

5. Conclusions

In this paper, a solution of the time-fractional dual-phase heat conduction problem in a homogeneous sphere has been presented. The considerations were carried out, assuming that the sphere is heated inside but there is no heat exchange with the surroundings. The presented form of the analytical solution (28) contains a fragment expressed by the inverse Laplace transform. Due to the impossibility of determining it in an analytical form due to complexity, a numerical method was used to calculate it. The considerations were supported by two examples in which the influence of the Caputo derivative orders on the temperature distribution in the considered sphere was examined. As was expected, the fractional orders α and β of derivatives occurred in equation (1) actually have importance for the temperature distribution in a homogeneous sphere.

In further research, it is planned to find a solution for this fractional dual-phase-lag heat conduction model, in which the third kind (Robin) boundary condition is assumed on the external surface of the sphere – it will cause heating or cooling of the sphere through heat exchange with the environment. Another idea for future work is to find a solution to this model in which there are fractional derivatives of order greater than 1.

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