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NUMERICAL SOLUTION OF A ONE-DIMENSIONAL SINGULARLY PERTURBED REACTION-DIFFUSION EQUATION USING A SPECTRAL-GRID METHOD

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Abstract. A spectral-grid method based on first-kind Chebyshev polynomials is proposed for the numerical solution of a one-dimensional singularly perturbed reaction-diffusion equation. The approximate solution is represented as a finite Chebyshev series, and the original boundary value problem is reduced to a system of linear algebraic equations for the unknown coefficients. The convergence and accuracy of the method are analyzed using the method of test functions. In contrast to conventional approaches that focus primarily on the approximation of the solution, the present study also investigates the behavior of the first and second derivatives, providing a more detailed characterization of the solution, particularly in boundary layer regions. The proposed method enables effective localization of boundary layers within refined mesh elements while maintaining high-order accuracy. Numerical results, presented in tabular and graphical forms, demonstrate the accuracy and efficiency of the method and show good agreement with existing results in the literature.

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1. Introduction

Singularly perturbed differential equations (SPDEs) of higher order arise frequently in the mathematical modeling of physical and engineering processes, particularly in convection-diffusion phenomena and boundary layer theory. Such models also appear in fluid dynamics applications, including the analysis of two-phase gas-solid flows in boundary layers [1]. The presence of a small perturbation parameter multiplying the highest-order derivative typically induces sharp gradients in narrow regions of the domain, known as boundary layers, which makes the numerical approximation of such problems highly challenging.

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A variety of parameter-uniform numerical methods have been developed to address these difficulties. In particular, a Shishkin mesh-based method for fourth-order singularly perturbed problems was proposed in [2], where the original problem is reduced to a weakly coupled system of second-order equations, followed by discretization using classical finite difference schemes. Similarly, exponentially graded meshes combined with quadratic B-spline approximations were employed in [3] for both linear and nonlinear fourth-order convection-diffusion problems, yielding second-order uniform convergence in the discrete maximum norm. An efficient numerical technique has also been proposed for two-parameter singularly perturbed problems with discontinuities in the convection coefficient and source term [4]. Furthermore, higher-order compact schemes, such as the sixth-order method proposed in [5], provide improved accuracy for singularly perturbed reaction-diffusion problems.

Finite element methods (FEM), particularly weak Galerkin formulations, have also been successfully applied to SPDEs. In [6], optimal parameter-uniform convergence in both energy and balanced norms was established using discontinuous high-order polynomials on Shishkin meshes. Parameter-uniform spline-based techniques have also been developed for weakly coupled singularly perturbed reaction-diffusion systems [7]. A robust numerical technique for weakly coupled systems of parabolic singularly perturbed reaction-diffusion equations was proposed in [8].

In addition to finite difference and finite element approaches, spectral methods have gained considerable attention due to their high accuracy for smooth problems. Spectral collocation methods combined with suitable coordinate transformations have been shown to effectively resolve boundary layers in singularly perturbed systems [9]. Spectral collocation methods have also been extended to mixed functional differential equations [10]. Related spectral and collocation approaches based on Jacobi polynomials have also been successfully applied to fractional reaction-subdiffusion and time-fractional sub-diffusion equations [11, 12].

Further developments include rational spectral collocation methods with Sinc transformations for third-order singularly perturbed problems [13], as well as spectral Galerkin and Petrov-Galerkin approaches using Chebyshev polynomials for stability analysis in magnetohydrodynamics [14]. Theoretical studies of fourth-order problems with additional perturbation terms have also been conducted, including asymptotic expansions and stability analysis [15]. Spline-based parameter-uniform schemes have also been developed for fourth-order singularly perturbed differential equations [16]. Additionally, Chebyshev-based spectral methods have been applied to second-order singularly perturbed equations, demonstrating high accuracy and computational efficiency [17].

Recent research has focused on advanced spectral techniques involving Chebyshev polynomials. These include the use of third-kind Chebyshev polynomials for integro-differential equations [18], Petrov-Galerkin formulations based on third- and fourth-kind Chebyshev polynomials for high-order boundary value problems [19], and spectral projection methods for hypersingular integral equations with superconvergence properties [20]. Spectral element collocation methods combining Legendre and Chebyshev techniques have also been developed for complex systems such as Maxwell equations with material interfaces [21]. In addition, Chebyshev-Tau methods have been successfully applied to domain-decomposed problems, including underwater acoustic wave propagation [22]. Numerical approaches have also been developed for singularly perturbed differential-difference systems arising in applications such as biology and physiology [23].

Despite these significant advances, classical numerical techniques, such as finite difference methods combined with least-squares formulations, may suffer from reduced accuracy when applied to problems with sharp boundary layers or strong singular perturbations [24]. This limitation motivates the development of high-accuracy and computationally efficient spectral-type methods.

Recent studies have also demonstrated the effectiveness of spectral-grid methods for singularly perturbed boundary value problems with two boundary layers [25]. In this context, the present paper proposes a spectral-grid method based on first-kind Chebyshev polynomials for solving singularly perturbed boundary value problems. The method is designed to achieve high accuracy while effectively capturing boundary layer behavior, thereby addressing the limitations of existing numerical approaches.

2. Statement of the problem

Consider the following second-order singularly perturbed differential equation

$$-\varepsilon u''(\eta)+u(\eta)=f(\eta), \quad 0\le\eta\le 1, \tag{1}$$

subject to the boundary conditions

$$u(0)=u(1)=0. \tag{2}$$

Here, ε is a small positive parameter. To numerically approximate the solution of problem (1)-(2), we employ a spectral-grid method based on Chebyshev polynomials of the first kind as basis functions. In general, to investigate the convergence and accuracy of a numerical method, a test function approach is commonly utilized. According to this approach, an auxiliary function $u_e(\eta)$ satisfying the boundary conditions (2) is selected. Substituting this function into equation (1), the corresponding right-hand side function $f(\eta)=-\varepsilon\dfrac{d^2u_e}{d\eta^2}+u_e(\eta)$ is determined. The considered problem is then solved using the spectral-grid method, and the obtained approximate solution is compared with the selected test function.

To analyze the problem (1)-(2), we choose the following test function, which satisfies the boundary conditions (2):

$$u_e(\eta)=\frac{e^{-\frac{1-\eta}{\sqrt{\varepsilon}}}+e^{-\frac{\eta}{\sqrt{\varepsilon}}}}{1+e^{-\frac{1}{\sqrt{\varepsilon}}}}-\cos^2(\pi\eta). \tag{3}$$

For this function, the right-hand side of equation (3) takes the form:

$$f(\eta) = -\cos^2(\pi\eta) - 2\varepsilon\pi^2\cos(2\pi\eta)$$

The derivatives of the function $u_e(\eta)$ are computed using the following expressions:

$$\begin{aligned} u_e'(\eta) &= \frac{1}{\sqrt{\varepsilon}\left(1+e^{-\frac{1}{\sqrt{\varepsilon}}}\right)}\left[e^{-\frac{1-\eta}{\sqrt{\varepsilon}}} - e^{-\frac{\eta}{\sqrt{\varepsilon}}}\right] + \pi\sin(2\pi\eta), \\ u_e''(\eta) &= \frac{e^{-\frac{1-\eta}{\sqrt{\varepsilon}}} + e^{-\frac{\eta}{\sqrt{\varepsilon}}}}{\varepsilon\left(1+e^{-\frac{1}{\sqrt{\varepsilon}}}\right)} + 2\pi^2\cos(2\pi\eta). \end{aligned} \tag{4}$$

3. Solution method

To numerically approximate the problem (1)-(2), we employ a spectral-grid method. First, the interval of integration $[\eta_0, \eta_1]$ is partitioned into a mesh such that $\eta_0 < \eta_1 < \cdots < \eta_N$, which generates N subintervals:

$$[\eta_0, \eta_1],\ [\eta_1, \eta_2],\ \ldots,\ [\eta_j, \eta_{j+1}],\ \ldots,\ [\eta_{N-1}, \eta_N], \quad j = 1, 2, \ldots, N-1,$$

where $\eta_0 = 0$ and $\eta_N = 1$.

On each subinterval, the differential equation (1) can be written in the form

$$-\varepsilon\frac{d^2u_j}{d\eta^2} + u_j(\eta) = f(\eta), \quad j = 1, 2, \ldots, N. \tag{5}$$

The boundary conditions (2) at the endpoints η_0 and η_N take the form

$$u(\eta_0) = 0, \quad u(\eta_N) = 0. \tag{6}$$

At the internal mesh points, continuity of the solution of equation (1) and its first derivative is required. These conditions can be written in the form

$$u_j^{(t)}(\eta_j) = u_{j+1}^{(t)}(\eta_j), \quad t = 0, 1; \quad j = 1, 2, \ldots, N-1, \tag{7}$$

where t denotes the order of the derivative; in particular, $t = 0$ corresponds to the continuity of the solution u, while $t = 1$ corresponds to the continuity of its first derivative u'.

The approximate solution $u_j(x)$ of problem (5)-(7) is expressed as a truncated series in terms of Chebyshev polynomials of the first kind. For this purpose, each subinterval $[\eta_{j-1}, \eta_j]$ is mapped onto the reference interval $[-1, 1]$ using the transformation

$$\eta = \frac{m_j}{2} + \frac{l_j}{2}x, \quad m_j = \eta_j + \eta_{j-1}, \quad l_j = \eta_j - \eta_{j-1}, \quad j = 1, 2, \ldots, N. \tag{8}$$

Here, $\eta \in [\eta_{j-1}, \eta_j]$ and $x \in [-1,1]$. The quantity $\dfrac{m_j}{2} = \dfrac{\eta_{j-1}+\eta_j}{2}$ represents the midpoint of the jth subinterval, while $l_j = \eta_j - \eta_{j-1}$ denotes its length. Using the transformation in Eq. (8), Eq. (5) can be rewritten in the form

$$-\varepsilon\left(\frac{2}{l_j}\right)^2 \frac{d^2u_j}{dx^2} + u_j(x) = f(x), \quad j = 1,2,\ldots,N. \tag{9}$$

The boundary conditions (6) and the continuity conditions (7) take the form

$$\begin{gathered} u_1(-1) = 0, \quad u_N(1) = 0, \\ \left(\frac{2}{l_j}\right)^t u_j^{(t)}(1) = \left(\frac{2}{l_{j+1}}\right)^t u_{j+1}^{(t)}(-1), \quad t = 0,1; \quad j = 1,2,\ldots,N-1. \end{gathered} \tag{10}$$

The local Chebyshev expansions are coupled by enforcing the continuity of the solution and its first derivative at the interfaces. Thus, the spectral-grid approximation is globally C^1, while the second derivative is determined separately on each subinterval. The approximate solution of the problem (9)-(10) and the right-hand side function $f_j(x)$ are sought on each subinterval in the form of truncated series expansions with respect to Chebyshev polynomials of the first kind:

$$\begin{gathered} u_a^{(j)}(x) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(x), \quad f_j\left(\bar{x}_l^{(j)}\right) = \sum_{n=0}^{p_j} b_n^{(j)} T_n\left(x_l^{(j)}\right), \quad \bar{x}_l^{(j)} = \frac{m_j}{2} + \frac{l_j}{2} x_l^{(j)}, \\ x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0,1,\ldots,p_j, \quad j = 1,2,\ldots,N. \end{gathered} \tag{11}$$

Here, $u_a^{(j)}(x)$ denotes the approximate solution, $T_n(x)$ are the Chebyshev polynomials of the first kind, $x_l^{(j)}$ are the corresponding collocation points, p_j is the number of polynomials used to approximate the solution on the j-th subinterval, and $a_n^{(j)}$ are unknown coefficients.

For the known function $f_j(x)$, the coefficients $b_n^{(j)}$ in expansion (11) are determined using the inverse transformation formula

$$b_n^{(j)} = \frac{2}{p_j c_n} \sum_{l=0}^{p_j} \frac{1}{c_l} f_j\left(\bar{x}_l^{(j)}\right) T_n\left(x_l^{(j)}\right), \qquad n = 0,1,2,\ldots,p_j, \tag{12}$$

where $c_0 = c_{p_j} = 2$ and $c_m = 1$ for $m = 1,\ldots,p_j - 1$. The endpoint weights c_0 and c_{p_j} account for the discrete orthogonality of the Chebyshev polynomials at the Chebyshev-Lobatto points, while the interior points have unit weight.

Furthermore, $M = \sum\limits_{j=1}^{N}(p_j+1)$ denotes the total number of Chebyshev polynomials used in the spectral-grid approximation, and N is the number of subintervals.

The representation (11) of the approximate solution $u_a^{(j)}(x)$ allows for efficient computation of its derivatives, which can be expressed as

$$
\begin{gathered}
u_a^{(j)}(y)=\sum_{n=0}^{p_j} a_n^{(j)} T_n(y), \\
u_a^{(1)}(y)=\sum_{n=0}^{p_j} a_n^{(1)} T_n(y), \quad a_n^{(1)}=\frac{2}{c_n} \sum_{\substack{q=n+1 \\ q+n \equiv 1 \,(\operatorname{mod} 2)}}^{p_j} q\, a_q, \quad n \geq 0, \\
u_a^{(2)}(y)=\sum_{n=0}^{p_j} a_n^{(2)} T_n(y), \quad a_n^{(2)}=\frac{2}{c_n} \sum_{\substack{q=n+2 \\ q \equiv n \,(\operatorname{mod} 2)}}^{p_j} q\left(q^2-n^2\right) a_q, \quad n \geq 0.
\end{gathered} \tag{13}
$$

Next, substituting the truncated series representations of the solution and its derivatives into the differential equation (9), and combining them with the additional conditions (10), we obtain the following system of algebraic equations. This system is used to determine the unknown coefficients $a_n^{(j)}$ $(n=0,1,2,\ldots,p_j;\ j=1,2,\ldots,N)$:

$$
\begin{cases}
-\varepsilon\left(\dfrac{2}{l_j}\right)^2 \dfrac{1}{c_n} \displaystyle\sum_{\substack{q=n+2 \\ q=n \,(\operatorname{mod} 2)}}^{p_j} q\left(q^2-n^2\right) a_q \\
\quad +\left(\dfrac{2}{l_j}\right) \dfrac{2}{c_n} \displaystyle\sum_{\substack{q=n+1 \\ q+n \equiv 1 \,(\operatorname{mod} 2)}}^{p_j} q\, a_q+a_n=b_n^{(j)}, \quad n=0,1,\ldots,p_j-1, \\
\displaystyle\sum_{n=0}^{p_1}(-1)^n a_n^{(1)}=0, \quad \sum_{n=0}^{p_N} a_n^{(N)}=0, \\
\displaystyle\sum_{n=0}^{p_j} a_n^{(j)}=\sum_{n=0}^{p_{j+1}}(-1)^n a_n^{(j+1)}, \\
\dfrac{2}{l_j} \displaystyle\sum_{n=0}^{p_j} n^2 a_n^{(j)}=\dfrac{2}{l_{j+1}} \sum_{n=0}^{p_{j+1}}(-1)^{n-1} n^2 a_n^{(j+1)}, \quad j=1,2,\ldots,N.
\end{cases} \tag{14}
$$

The number of unknowns for determining

$$
x^T=\left(a_0^{(1)}, a_1^{(1)}, \ldots, a_{p_1}^{(1)}, a_0^{(2)}, a_1^{(2)}, \ldots, a_{p_2}^{(2)}, \ldots, a_0^{(N)}, a_1^{(N)}, \ldots, a_{p_N}^{(N)}\right)
$$

is obtained from the formula $M=\sum_{j=1}^{N}(p_j+1)=(p_1+p_2+\cdots+p_N+N)$. The number of equations in system (14) is also equal to $M=(p_1+p_2+\cdots+p_N+N)$. Moreover, the number of main equations in system (14) is $\sum_{j=1}^{N}(p_j-1)$, to which two boundary conditions and $2(N-1)$ continuity conditions are added. Thus,

$$\sum_{j=1}^{N}(p_j-1)+2(N-1)+2=\sum_{j=1}^{N}(p_j+1)=M,$$

and hence the resulting matrix A is of size $M \times M$. It is convenient to write system (14) in the form of the linear algebraic system

$$A\mathbf{x}=\mathbf{b}, \tag{15}$$

where $\mathbf{x}$ is the vector of unknown coefficients. The right-hand-side vector $\mathbf{b}$ contains the coefficients $b_n^{(j)}$ obtained from the expansion of the right-hand side function, as well as the zero entries corresponding to the boundary conditions $u(\eta_0)=0$ and $u(\eta_N)=0$. In particular, $\mathbf{b}^T=\left(b^{(1)},b^{(2)},\ldots,b^{(N)}\right)$, where $b^{(j)}=\left(b_0^{(j)},b_1^{(j)},\ldots,b_{p_j}^{(j)}\right)$. Here, the superscript T denotes vector transposition.

4. Discussion of results

The singularly perturbed problem defined by (1)-(2) is solved using the spectral--grid method (14). The spectral-grid method can be implemented in several variants: 1) Uniform grid with uniform approximation: the computational domain is divided into subintervals of equal length, and the same number of Chebyshev polynomials is used on each subinterval; 2) Uniform grid with non-uniform approximation: the grid is uniform, while different numbers of Chebyshev polynomials are used on different subintervals; 3) Non-uniform grid with uniform approximation; 4) Non-uniform grid with non-uniform approximation. The use of these variants demonstrates the flexibility and adaptability of the spectral-grid method. In particular, boundary layers arising in singularly perturbed problems can be effectively localized within sufficiently small subintervals, allowing the concentration of Chebyshev polynomials in those regions (variant 4). This approach ensures high accuracy of the method for arbitrary small values of the perturbation parameter. To assess the convergence and accuracy of the proposed method, the test function approach is employed. In this study, equation (3) is selected as the test function. This test function was selected according to [6], and it was used to compare the numerical results obtained by the spectral-grid method with those presented in [6]. Tables 1-3 present the maximum absolute errors computed at the Chebyshev collocation points. The maximum absolute error is defined as

$$\Delta=\max_{\substack{j=1,\ldots,N\\ l=0,\ldots,p_j}}\left|u_e\left(x_l^{(j)}\right)-u_a^{(j)}\left(x_l^{(j)}\right)\right|, \quad j=1,2,\ldots,N, \quad t=0,1,2, \tag{16}$$

where t denotes the order of the derivative.

The collocation points corresponding to the Chebyshev polynomials of the first kind are defined as $x_l^{(j)}=\cos\left(\dfrac{\pi l}{p_j}\right)$, $l=0,1,\ldots,p_j$, $j=1,2,\ldots,N$. It is assumed that $0 < \varepsilon \ll 1$ is a small parameter, and $M = \sum_{j=1}^{N}(p_j+1)$ denotes the total number of Chebyshev polynomials. Here, p_j $(j = 1,2,\ldots,N)$ represents the number of polynomials used for approximation on the j-th subinterval, and Δ denotes the maximum absolute error.

In Table 1, the computed values of the maximum absolute error Δ are presented for the case of an interval divided into three non-uniform subintervals ($N = 3$).

Table 1. Maximum absolute error Δ of the spectral-grid method consisting of three non-uniform elements ($N = 3$)

ε	$M = \sum_{j=1}^{N}(p_j+1)$	Spectral method Δ	Spectral-grid method			
			p_1+1 and element length	p_2+1 and element length	p_3+1 and element length	Δ
10^{-6}	64	9.2×10^{-3}	22,[0,0.05] 25,[0,0.05] 27,[0,0.05]	20,[0.05,0.95] 14,[0.05,0.95] 10,[0.05,0.95]	22,[0.95,1] 25,[0.95,1] 27,[0.95,1]	2.4×10^{-5} 1.6×10^{-6} 2.3×10^{-7}
	128	2.2×10^{-8}	44,[0,0.05] 50,[0,0.05] 54,[0,0.05]	40,[0.05,0.95] 28,[0.05,0.95] 20,[0.05,0.95]	44,[0.95,1] 50,[0.95,1] 54,[0.95,1]	3.7×10^{-16} 7.3×10^{-20} 4.7×10^{-19}
10^{-8}	64	8.1×10^{-1}	22,[0,0.005] 25,[0,0.005] 27,[0,0.005]	20,[0.005,0.995] 14,[0.005,0.995] 10,[0.005,0.995]	22,[0.995,1] 25,[0.995,1] 27,[0.995,1]	2.4×10^{-5} 1.6×10^{-6} 2.3×10^{-7}
	128	1.7×10^{-1}	44,[0,0.005] 50,[0,0.005] 54,[0,0.005]	40,[0.005,0.995] 28,[0.005,0.995] 20,[0.005,0.995]	44,[0.995,1] 50,[0.995,1] 54,[0.995,1]	3.7×10^{-16} 7.3×10^{-20} 4.1×10^{-19}

In Table 1, p_1+1, p_2+1, and p_3+1 represent the numbers of Chebyshev polynomials employed for approximating the solution on the first, second, and third subintervals, respectively.

Table 2. Comparison of the CPU time and maximum absolute error of the spectral-grid method (SGM) and the spectral method (SM)

ε	M	Method	CPU Times(s)		Maximum Absolute Error
			Linear System Solution	Total	
10^{-6}	64	SGM	0.471s	0.521s	2.3×10^{-7}
		SM	0.434s	0.508s	9.2×10^{-3}
	128	SGM	2.838s	3.169s	7.3×10^{-20}
		SM	2.722s	3.019s	2.2×10^{-8}
10^{-8}	64	SGM	0.478s	0.526s	2.3×10^{-7}
		SM	0.453s	0.511s	8.1×10^{-1}
	128	SGM	2.871s	3.248s	7.3×10^{-20}
		SM	2.738s	3.207s	1.7×10^{-1}

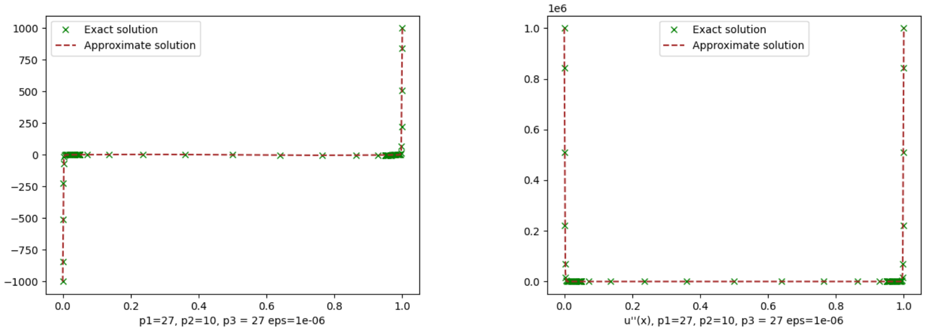
From the results presented in Tables 1 and 2, it can be observed that increasing the number of grid elements and refining the grid near the boundary leads to significant clustering of Chebyshev collocation points. This clustering results in a substantial reduction of the maximum absolute error. The spectral-grid method allows flexible selection of both the number and the lengths of grid elements in accordance with the properties of the problem under consideration. For instance, in the studied second-order singularly perturbed problem, when the parameter ε takes very small values, the solution exhibits sharp variations in the boundary layers. In such cases, if the number and distribution of grid elements are not chosen appropriately, the desired accuracy cannot be achieved even when the total number of Chebyshev polynomials M is large. Although the second derivative of the boundary-layer component contains a factor of order ε^{-1}, this factor is multiplied by ε in the differential operator $-\varepsilon u'' + u$. Thus, this factor is compensated at the continuous level. However, for sufficiently small value of ε, the discretized system may still become ill-conditioned due to the sharp boundary layers and the differentiation matrices. Moreover, the spectral-grid method makes it possible to analyze the behavior of higher-order derivatives of the solution to the second-order singularly perturbed differential equation (1) subject to the boundary conditions (2). Numerical experiments demonstrate that the spectral-grid method ensures stability and high accuracy even for very small values of the perturbation parameter, for example, $\varepsilon = 10^{-6}, 10^{-8}$ and $N = 64, 128$.

Table 3 and Figure 1 present the maximum absolute errors for derivatives of different orders at the Chebyshev collocation points. In this case, the exact values of the derivatives given by (3) are evaluated at the collocation nodes x_l using formula (4).

Table 3. Maximum absolute error Δ for derivatives of different orders computed by the spectral-grid method ($N = 3$)

ε	$M = \sum_{j=1}^{N}(p_j+1)$	p_1+1 and element length	p_2+1 and element length	p_3+1 and element length	Maximum absolute error Δ for derivatives	
					First	Second
10^{-6}	64	22,[0,0.05]	20,[0.05,0.95]	22,[0.95,1]	2.4×10^{-2}	7.1×10^{1}
		25,[0,0.05]	14,[0.05,0.95]	25,[0.95,1]	1.9×10^{-3}	5.5×10^{0}
		27,[0,0.05]	10,[0.05,0.95]	27,[0.95,1]	3.1×10^{-4}	8.8×10^{-1}
	128	44,[0,0.05]	40,[0.05,0.95]	44,[0.95,1]	1.0×10^{-12}	3.0×10^{-9}
		50,[0,0.05]	28,[0.05,0.95]	50,[0.95,1]	4.4×10^{-16}	5.1×10^{-13}
		54,[0,0.05]	20,[0.05,0.95]	54,[0.95,1]	4.4×10^{-16}	2.3×10^{-13}
10^{-8}	64	22,[0,0.005]	20,[0.005,0.995]	22,[0.995,1]	2.4×10^{-1}	7.1×10^{3}
		25,[0,0.005]	14,[0.005,0.995]	25,[0.995,1]	1.9×10^{-2}	5.5×10^{2}
		27,[0,0.005]	10,[0.005,0.995]	27,[0.995,1]	3.1×10^{-3}	8.8×10^{1}
	128	44,[0,0.005]	40,[0.005,0.995]	44,[0.995,1]	1.0×10^{-11}	2.9×10^{-7}
		50,[0,0.005]	28,[0.005,0.995]	50,[0.995,1]	2.4×10^{-15}	1.1×10^{-10}
		54,[0,0.005]	20,[0.005,0.995]	54,[0.995,1]	3.9×10^{-15}	3.9×10^{-11}

Table 4 presents a comparison of the maximum absolute errors obtained by the spectral-grid method and the sixth-order compact finite difference method [5]. In order to investigate the sensitivity of the maximum absolute error with respect to the perturbation parameter, results are provided for different numbers of Chebyshev polynomials M and for values of the parameter ε ranging from 2^{-4} to 10^{-7}.



(a) First derivative (b) Second derivative

Fig. 1. Graphs of the first and second derivatives of the solution

Table 4. Maximum absolute errors Δ obtained by the spectral-grid method and the sixth-order compact finite difference method [5]

Sixth-order compact finite difference method Δ					
ε	$M=16$	$M=32$	$M=64$	$M=128$	$M=256$
$1/16$	3.1×10^{-7}	4.8×10^{-9}	7.6×10^{-11}	1.1×10^{-12}	6.5×10^{-14}
$1/32$	2.6×10^{-7}	4.1×10^{-9}	6.4×10^{-11}	1.0×10^{-12}	8.6×10^{-14}
$1/64$	4.7×10^{-7}	7.5×10^{-9}	1.1×10^{-10}	1.8×10^{-12}	4.8×10^{-14}
$1/128$	3.6×10^{-6}	6.3×10^{-8}	1.0×10^{-9}	1.5×10^{-11}	2.3×10^{-13}
Spectral-grid method Δ					
ε	$M=16$	$M=32$	$M=64$	$M=128$	$M=256$
$1/16$	2.1×10^{-12}	6.4×10^{-32}	6.5×10^{-80}	1.9×10^{-194}	4.4×10^{-317}
$1/32$	2.0×10^{-12}	6.3×10^{-32}	6.5×10^{-80}	1.9×10^{-194}	6.2×10^{-317}
$1/64$	1.5×10^{-11}	2.0×10^{-29}	4.4×10^{-74}	6.5×10^{-182}	9.3×10^{-317}
$1/128$	2.1×10^{-9}	5.8×10^{-25}	7.6×10^{-65}	4.7×10^{-163}	1.0×10^{-316}

From Table 4, it can be observed that the spectral-grid method provides highly accurate results for various values of the small parameter ε, and the corresponding errors are significantly smaller than those obtained by the sixth-order compact finite difference method. As the number of polynomials M increases, the error decreases for both methods. However, in the spectral-grid method, the error decreases much more rapidly and reaches extremely small values. For an analytic solution, the Chebyshev coefficients decay exponentially with the polynomial degree. Consequently, the truncation error behaves as $\Delta \approx Ce^{-\alpha p_j}$, which explains the exponential convergence observed as p_j increases. This demonstrates the high accuracy and fast convergence properties of the proposed method.

5. Conclusions

In this study, a one-dimensional singularly perturbed reaction-diffusion problem has been approximated using a spectral-grid method based on first-kind Chebyshev polynomials. The convergence behavior and accuracy order were analyzed using the method of test functions, where suitable test functions satisfying both the differential equation and the boundary conditions were employed. In contrast to many existing studies that focus primarily on the solution, the present work additionally investigates the behavior of the first and second derivatives, providing a more comprehensive description of the solution, particularly in boundary layer regions. The method combines domain decomposition with high-order polynomial approximation, enabling accurate resolution of sharp gradients through locally refined meshes. Numerical results, presented in both tabular and graphical forms, demonstrate that the proposed method achieves high accuracy and efficiency, and comparisons with existing approaches confirm its effectiveness and superiority.

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