Journal of Applied Mathematics and Computational Mechanics 2026, 25(3), 112-124
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2026.3.08 e-ISSN 2353-0588

NUMERICAL COMPUTATION AND ANALYSIS FOR THE NONLINEAR TIME-FRACTIONAL CONVECTION-DIFFUSION PROBLEM

***Eyaya Fekadie Anley*[1,2*], *Chen Sun*[1]**

[1] *School of Physics and Electronics, Hunan University*
Changsha, China
[2] *Department of Mathematics, Natural and Computational Science, Arba Minch University*
Arba Minch, Ethiopia
eyayafekadie@hnu.edu.cn, chensun@hnu.edu.cn

Received: 6 May 2026; Accepted: 23 August 2026

Abstract. This paper aims to explore the numerical and theoretical solution of the time--fractional convection-diffusion equation with a nonlinear source term over a bounded region. We provide a high-order Crank-Nicolson scheme-based L_{2-1} approximation via piecewise quadratic interpolation polynomials for the Caputo-derivative operator in the temporal direction and a fourth-order compact exponential difference technique for the spatial direction. Following the linearization of the nonlinear term, the method's unconditional stability and convergence with an improved order of accuracy $\mathscr{O}(\tau^{min(2,3-\theta)}+h^4)$ are investigated using Fourier analysis and the energy method, respectively. In order to illustrate the theoretical analysis and validate the suitability of the suggested approach, numerical results are provided.

MSC 2010: *35R11, 65M12, 65N30*
Keywords: *compact exponential scheme, Crank-Nicolson method, convection-diffusion problem*

1. Introduction

In order to connect theoretical models with real-world phenomena, such as chemical and biological processes [1], financial systems [2], pattern dynamics [3, 4], and so on, it is essential to validate and analyze these models through numerical investigation. Thus, effective and precise numerical techniques have become more crucial for completing the solutions of time-fractional convection-diffusion equations (TFCDE). Analytical and exact solutions are used in [5,6] for the nonlinear fractional problems based on the Caputo-derivative sense. Accurate and efficient methods are proposed [7, 8] to solve the fractional reaction-diffusion-convection equations based on the Caputo-Fabrizio derivative. A Crank-Nicolson scheme based L_1 approximation for the Caputo time derivative and the Weighted Shifted difference scheme for

* Corresponding author

the Riesz space derivative is applied [9, 10] to the nonlinear time-space fractional convection-diffusion equation to obtain a $\mathscr{O}(\tau^{2-\theta}+h^2)$ convergence order. In order to solve the linear TFCDE [11], a high-order compact exponential difference method with implicit L_1-approximation is taken into consideration to produce an accuracy of $\mathscr{O}(\tau^{2-\theta}+h^4)$. Till now, the majority of researchers have worked with a 4^{th}-order compact difference scheme for spatial and $(2-\theta)^{th}$-order implicit scheme based L_1-approximation for Caputo-derivative.

This paper considers the theoretical investigation and numerical validation of the nonlinear time-fractional convection-diffusion equation (TFCDE) over a bounded domain. The implementation of a Crank-Nicolson scheme based L_{2-1}-interpolation combined with a compact exponential difference method for the nonlinear TFCDE has not been studied before. Here, we have proposed a compact exponential difference method for the convection-diffusion term and a Crank-Nicolson scheme based L_{2-1}-approximation for Caputo-time derivative, where the spatial order is four and the temporal order is $\min(2,3-\theta)$ arising from the truncation error of the Crank--Nicolson scheme and L_{2-1}-approximation. Following the application of the linearization technique for the problem's nonlinearity, we employed the energy method for the theoretical analysis of convergence order and the von-Neumann Fourier analysis method for the stability of the suggested scheme to guarantee the method's robustness.

2. Problem formulation

2.1. Spatial discretization

Take the one-dimensional ordinary differential equation over a bounded domain $\Omega_h=[0,l']$ as:

$$-\mathscr{D}\frac{d^2s}{dx^2}+\mathscr{V}\frac{ds}{dx}=r(x), \tag{1}$$

where r is the source term, and a high-order compact exponential difference scheme for (1) has been constructed [12] as,

$$-\rho\delta_x^2 s_l+\mathscr{V}\delta_x s_l=r_l+\rho_1\delta_x r_l+\rho_2\delta_x^2 r_l, \tag{2}$$

with δ_x,δ_x^2 are average difference operators that are given at the space grid point x_l as,

$$\delta_x s_l=\frac{s_{l+1}-s_{l-1}}{2h},\ \delta_x^2 s_l=\frac{s_{l+1}-2s_l+s_{l-1}}{h^2}.$$

To get the coefficient values, we have used the following coefficient values:

$$\rho=\frac{\mathscr{V}h}{2}\coth\left(\frac{\mathscr{V}h}{2\mathscr{D}}\right),\ \rho_1=\frac{\mathscr{D}-\rho}{\mathscr{V}},\ \rho_2=\frac{\mathscr{D}(\mathscr{D}-\rho)}{\mathscr{V}^2}+\frac{h^2}{6}. \tag{3}$$

Combining (2) and (3), one finds that

$$\left(-\rho\delta_x^2+\mathscr{V}\delta_x\right)s_l=\left(1+\rho_1\delta_x+\rho_2\delta_x^2\right)r_l+O(h^4), \tag{4}$$

in which the Taylor series error for this high-order compact exponential difference scheme, which is given as $O(h^4)$. The formulation expressed in (4) gives the diagonally dominant tridiagonal linear system of equation that can be expressed as,

$$\left(1+\rho_1\delta_x+\rho_2\delta_x^2\right)^{-1}\left(-\rho\delta_x^2+\mathscr{V}\delta_x\right)s_l=r_l. \tag{5}$$

By substituting $s(x)$ by $u(x,t)$, and $r(x)$ via $-{}^cD_\tau^\theta u(x,t)+g(u,x,t)$, we have developed the nonlinear time-fractional unsteady convection-diffusion equation, which has never been taken into consideration in the previous literature. The one-dimensional nonlinear time-fractional convection-diffusion equation (TFCDE):

$$\begin{cases}{}^cD_\tau^\theta u(x,t)+\mathscr{V}\dfrac{\partial u(x,t)}{\partial x}-\mathscr{D}\dfrac{\partial^2u(x,t)}{\partial x^2}=g(u,x,t),\\ u(x,0)=\psi(x),(x,t)\in(0,l')\times(0,T],\\ u(0,t)=\phi_1(t),\ \ u(l',t)=\phi_2(t),t\in(0,T],\end{cases} \tag{6}$$

where $u(x,t)$ is the concentration, $\theta\in(0,1)$ is the fractional order, $\mathscr{V}>0$, and $\mathscr{D}>0$ are convection and diffusion coefficients, respectively, with $\psi(x),\phi_1(t),\phi_2(t)$ known smooth functions over a bounded domain. Here $g(u,x,t)$ is a nonlinear function of u that satisfies the Lipschitz condition with respect to the function u as:

$$|g(u,x,t)-g(v,x,t)|\le L|u-v|,\forall u,v, \tag{7}$$

in which L is the Lipschitz constant. From (5), we have the operator

$$\Lambda_x^cD_\tau^\theta u(x,t)=\rho\delta_x^2u(x,t)-\mathscr{V}\delta_xu(x,t)+\Lambda_xg(u,x,t),\ \Lambda_x=\left(1+\rho_1\delta_x+\rho_2\delta_x^2\right). \tag{8}$$

2.2. Temporal discretization

At the grid point $(x_l,t_{n-\frac{1}{2}})$, the discretization of (8) using the Crank-Nicolson method yields,

$$\Lambda_x^cD_\tau^\theta u(x_l,t_{n-\frac{1}{2}})=\rho\delta_x^2u_l^{n-\frac{1}{2}}-\mathscr{V}\delta_xu_l^{n-\frac{1}{2}}+\Lambda_xg(u_l^{n-\frac{1}{2}},x_l,t_{n-\frac{1}{2}})+\mathscr{O}(\tau^2). \tag{9}$$

Therefore, assuming $u(t)\in\mathscr{C}^3[t_0,t_n]$, the Caputo-fractional derivative ${}^cD_\tau^\theta u(t)$ is approximated using the L_{2-1} quadratic interpolation, and ${}_0^cD_t^\theta u(t)$ is approximated using L_1-linear interpolation [13] with $0<\theta<1$, then

$$
\begin{aligned}
{}^{c}D^{\theta}_{\tau}u(t_n) &= {}^{c}_{0}D^{\theta}_{t}u(t_n) + \frac{\tau^{2-\theta}}{\Gamma(2-\theta)}\sum_{k=2}^{n} \nu^{(\theta)}_{n-k}\delta_t^2 u_{k-1} \\
&= \frac{\tau^{-\theta}}{\Gamma(2-\theta)}\left(\upsilon^{(\theta)}_{0}u(t_n) - \sum_{k=1}^{n-1}\left(\upsilon^{(\theta)}_{n-k-1} - \upsilon^{(\theta)}_{n-k}\right)u(t_k) - \upsilon^{(\theta)}_{n-1}u(t_0)\right) + T^n,
\end{aligned} \tag{10}
$$

where T^n is the Taylor series truncation error for the L_{2-1} approximation scheme that can be estimated as

$$
T^n \leq \frac{1}{2\Gamma(2-\theta)}\left(\frac{1}{4} + \frac{\theta-1}{(2-\theta)(3-\theta)}\right)\max_{0\leq t\leq t_n} u^{'''}(t)\tau^{3-\theta}. \tag{11}
$$

Lemma 1 *From the description of* (10)*, we have the coefficient values* $\vartheta^{(\theta)}_0 = \upsilon^{(\theta)}_0 = 1$ *for* $n = 1$ *and for* $n \geq 2$*, the coefficients* $\upsilon^{(\theta)}_k$ *with their properties are obtained in [14] as:*

$$
\upsilon^{(\theta)}_k = \begin{cases} \vartheta^{(\theta)}_0 + \nu^{(\theta)}_0, k = 0, \\ \vartheta^{(\theta)}_k + \nu^{(\theta)}_k - \nu^{(\theta)}_{k-1}, 1 \leq k \leq n-2, \\ \vartheta^{(\theta)}_k - \nu^{(\theta)}_{k-1}, k = n-1, \end{cases} \tag{12}
$$

$$
\nu^{(\theta)}_k = \frac{1}{2-\theta}\left((k+1)^{2-\theta} - k^{2-\theta}\right) - \frac{1}{2}\left((k+1)^{1-\theta} + k^{1-\theta}\right),\ \vartheta^{(\theta)}_0 = (k+1)^{1-\theta} - k^{1-\theta}.
$$

2.3. Full discretization and formulation of the problem

The central value of the Caputo-derivative ${}^{c}D^{\theta}_{\tau}u(x,t)(0 < \theta < 1)$, which is given in (10) at the grid point value (x_l, t_n) and (x_l, t_{n-1}), can be

$$
{}^{c}D^{\theta}_{\tau}u(x_l, t_{n-\frac{1}{2}}) = \frac{1}{\mu}\left[\upsilon^{(\theta)}_0 u_l^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(\upsilon^{(\theta)}_{n-k-1} - \upsilon^{(\theta)}_{n-k}\right)u_l^{k-\frac{1}{2}} - \upsilon^{(\theta)}_{n-1}u_l^0\right] + T^{n-\frac{1}{2}}, \tag{13}
$$

where $\mu = \tau^{\theta}\Gamma(2-\theta)$, and from (9) and (11) the combined truncation error can be estimated as $T^{n-\frac{1}{2}} \leq K\tau^{\min(2,3-\theta)}, K > 0$. Since,

$$
-\sum_{k=1}^{n-2}\left(\upsilon^{(\theta)}_{n-k-2} - \upsilon^{(\theta)}_{n-k-1}\right)u_l^k = -\sum_{k=1}^{n-1}\left(\upsilon^{(\theta)}_{n-k-1} - \upsilon^{(\theta)}_{n-k}\right)u_l^{k-1},
$$

then (13) holds true. By substituting (13) into (9) with (3), the simplification of the formula for the grid values $1 \leq l \leq M+1$, $2 \leq n \leq N$ and the coefficient $\upsilon^{(\theta)}_0 = \vartheta^{(\theta)}_0 + \nu^{(\theta)}_0$, we have the following general formulation:

$$\begin{aligned}
& \left(\upsilon_0^{(\theta)}\left(\frac{\rho_2}{h^2}-\frac{\rho_1}{2h}\right)-\mu\left(\frac{\rho}{h^2}+\frac{\mathscr{V}}{2h}\right)\right)u_{l-1}^{n-\frac{1}{2}}+\left(\upsilon_0^{(\theta)}\left(1-\frac{2\rho_2}{h^2}\right)+\frac{2\mu\rho}{h^2}\right)u_l^{n-\frac{1}{2}} \\
+ & \left(\upsilon_0^{(\theta)}\left(\frac{\rho_2}{h^2}+\frac{\rho_1}{2h}\right)-\mu\left(\frac{\rho}{h^2}-\frac{\mathscr{V}}{2h}\right)\right)u_{l+1}^{n-\frac{1}{2}} \\
= & \left(\frac{\rho_2}{h^2}-\frac{\rho_1}{2h}\right)\left[\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_{l-1}^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}u_{l-1}^0+\mu g(u_{l-1}^{n-\frac{1}{2}})\right] \\
+ & \left(1-\frac{2\rho_2}{h^2}\right)\left[\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_l^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}u_l^0+\mu g(u_l^{n-\frac{1}{2}})\right] \\
+ & \left(\frac{\rho_2}{h^2}+\frac{\rho_1}{2h}\right)\left[\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_{l+1}^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}u_{l+1}^0+\mu g(u_{l+1}^{n-\frac{1}{2}})\right],
\end{aligned} \tag{14}$$

with the corresponding initial and boundary conditions:

$$\begin{aligned}
u_l^0 &= \psi(x_l),\ \ l=0,1,2,...,M, \\
u_0^n &= \phi_1(t_n),\ \ u_M^n=\phi_2(t_n), n=1,2,...,N-1.
\end{aligned} \tag{15}$$

The system of (14)-(15) can be formulated as matrix form for $n>1$,

$$Au^{n-\frac{1}{2}} = \left(\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)Bu^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}Bu^0+G^{n-\frac{1}{2}}\right), \tag{16}$$

where the coefficient matrix A has the lower, main, and upper diagonal elements. From (16), the right-hand side matrix $B=tridiag(B_1,B_2,B_3)$ is also a tridiagonal matrix. The column vector matrix is $G^{n-\frac{1}{2}}=\mu Bg(u^{n-\frac{1}{2}})+H^{n-\frac{1}{2}}$, where $g(u^{n-\frac{1}{2}})$ is the nonlinear function that will be linearized. The column vector $H^{n-\frac{1}{2}}$ is,

$$\begin{aligned}
H_1^{n-\frac{1}{2}} &= A_1u_0^{n-\frac{1}{2}}-B_1\left[\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_0^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}u_0^0+\mu g(u_0^{n-\frac{1}{2}})\right], \\
H_M^{n-\frac{1}{2}} &= A_3u_{M+1}^{n-\frac{1}{2}}-B_3\left(\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_{M+1}^{k-\frac{1}{2}}+\upsilon_{n-1}^{(\theta)}u_{M+1}^0+\mu g(u_{M+1}^{n-\frac{1}{2}})\right).
\end{aligned}$$

3. Theoretical analysis of the problem

With the decreasing sequence and properties of coefficients $\upsilon_k^{(\theta)}$ defined in (12) with $k=0$ and $n\geq 2$, $\upsilon_0^{(\theta)}=\vartheta_0^{(\theta)}+\nu_0^{(\theta)}=\left(\frac{1}{2}+\frac{1}{2-\theta}\right)>0$ for $\theta\in(0,1)$, the sum of the absolute values of the lower and upper diagonal elements are less than the absolute value of the main diagonal elements, which implies that the coefficient matrix for the schemes (14), and (15) is strictly diagonally dominant; thus, the matrix is invertible.

3.1. Stability analysis

Let U_l^n be the analytical solution for the scheme defined in (15) with,

$$\kappa_l^n = U_l^n - u_l^n,\ 1 \leq l \leq M+1, 1 \leq n \leq N, \tag{17}$$

as the error value for the given vector, $\kappa^n = \left(\kappa_1^n, \kappa_2^n, \kappa_3^n, ..., \kappa_{M+1}^n\right)^T$. The Fourier analysis for $\kappa^n(x)$ can be given as,

$$\kappa^n(x) = \sum_{j=-\infty}^{\infty} \varepsilon^n(j) e^{i2\pi jx}/q, \varepsilon^n(j) = \frac{1}{q}\int_0^q \kappa^n(j) e^{i2\pi jx/q} dx.$$

Theorem 1 *The stated scheme given in* (14)-(15) *is unconditionally stable.* □

PROOF Thus, linearizing the nonlinear problem is important to simplify the stability analysis of the scheme, and the following estimate has been obtained from the error value in (17) using the linearized term $g(u_l^{n-\frac{1}{2}}) \approx g(\frac{3}{2}u_l^{n-1} - \frac{1}{2}u_l^{n-2})$:

$$\begin{aligned}
&\left(\upsilon_0^{(\theta)} + (\upsilon_0^{(\theta)}\rho_1 + \mu\mathscr{V})\delta_x + (\upsilon_0^{(\theta)}\rho_2 - \mu\rho)\delta_x^2\right)\kappa_l^{n-\frac{1}{2}} \\
&= (1+\rho_1\delta_x + \rho_2\delta_x^2)\left(\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)} - \upsilon_{n-k}^{(\theta)}\right)\kappa_l^{k-\frac{1}{2}}\right. \\
&\left.+\upsilon_{n-1}^{(\theta)}\kappa_l^0 + \mu g(\frac{3}{2}\kappa_l^{n-1} - \frac{1}{2}\kappa_l^{n-2})\right) + T_{h\tau}^{n-\frac{1}{2}}.
\end{aligned} \tag{18}$$

The error value estimation of the scheme in (14) have been found by substituting $\kappa_l^n = \varepsilon^n e^{i\zeta lh}, \zeta = 2\pi/l$ in to (18) with $e^{i\zeta h} - 2 + e^{-i\zeta h} = -4\sin^2(\zeta h/2)$, then

$$\begin{aligned}
&\left(\upsilon_0^{(\theta)} - 4(\upsilon_0^{(\theta)}\rho_2 - \mu\rho)\frac{\sin^2(\zeta h/2)}{h^2} + i(\upsilon_0^{(\theta)}\rho_1 + \mu\mathscr{V})\frac{\sin\zeta h}{h}\right)\varepsilon^{n-\frac{1}{2}} \\
= \quad &\left(1 - 4\rho_2\frac{\sin^2(\zeta h/2)}{h^2} + i\rho_1\frac{\sin\zeta h}{h}\right)\left(\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)} - \upsilon_{n-k}^{(\theta)}\right)\varepsilon^{k-\frac{1}{2}}\right. \\
+ \quad &\left.\upsilon_{n-1}^{(\theta)}\varepsilon^0 + \mu g\left(\frac{3}{2}\varepsilon^{n-1} - \frac{1}{2}\varepsilon^{n-2}\right)\right).
\end{aligned} \tag{19}$$

Suppose $n = 1$, and from the weight coefficients obtained in Lemma 1, with $\upsilon_0^{(\theta)} = \frac{1}{2} + \frac{1}{2-\theta} > 0, \mu = \tau^\theta\Gamma(2-\theta)$, with $0 < \theta < 1$, and denote $\beta = \zeta h$, and the summation term defined in (19) vanishes and $\upsilon_{n-1}^{(\theta)} = \upsilon_0^{(\theta)}$, then we have

$$\mathscr{A}\varepsilon^{1/2} = \mathscr{B}\left(\upsilon_0^{(\theta)}\varepsilon^0 + \mu g(\varepsilon^0)\right), \tag{20}$$

$$\mathscr{A} = \upsilon_0^{(\theta)} - 4\left(\upsilon_0^{(\theta)}\rho_2 - \mu\rho\right)X + i\left(\upsilon_0^{(\theta)}\rho_1 + \mu\mathscr{V}\right)\sqrt{Y}, \tag{21}$$

$$\mathscr{B} = 1 - 4\rho_2 X + i\rho_1\sqrt{Y}, \tag{22}$$

and $X = \dfrac{\sin^2(\beta/2)}{h^2}$, $Y = \dfrac{\sin^2(\beta)}{h^2}$. Taking the modulus of (20), we obtain:

$$|\varepsilon^{1/2}| = \frac{|\mathscr{B}|}{|\mathscr{A}|}|\upsilon_0^{(\theta)}\varepsilon^0 + \mu g(\varepsilon^0)|. \tag{23}$$

Under the physical assumption $0 < \mathscr{V} < \mathscr{D}$, recall the fitted coefficients as, $\rho = \dfrac{\mathscr{V}h}{2}\coth\left(\dfrac{\mathscr{V}h}{2\mathscr{D}}\right)$. Since $\coth z > 1$ for $z > 0$, we have $\rho > \dfrac{\mathscr{V}h}{2}$, and also using the inequality $\coth z \leq \dfrac{1}{z} + \dfrac{z}{3}$ for $z > 0$, we obtain $\rho < \mathscr{D} + \dfrac{\mathscr{V}^2h^2}{12\mathscr{D}}$. Because $\mathscr{V} < \mathscr{D}$, then $0 < \mathscr{D} - \lambda < \mathscr{D}$. Furthermore, we also have the following fitted coefficients $\rho_1 = \dfrac{\mathscr{D} - \rho}{\mathscr{V}} > 0$, $\rho_2 = \dfrac{\mathscr{D}(\mathscr{D} - \rho)}{\mathscr{V}^2} + \dfrac{h^2}{6} > 0$. Now compute the bound for $|\mathscr{A}|$:

$$|\mathscr{A}|^2 = \left(\upsilon_0^{(\theta)} - 4(\upsilon_0^{(\theta)}\rho_2 - \mu\rho)X\right)^2 + \left(\upsilon_0^{(\theta)}\rho_1 + \mu\mathscr{V}\right)^2 Y.$$

From the positive contribution of diffusion, we have $\upsilon_0^{(\theta)}\rho_2 - \mu\rho \geq 0$, $\mu\rho \leq \upsilon_0^{(\theta)}\rho_2$, which gives $|\mathscr{A}| \geq \upsilon_0^{(\theta)} - 4\upsilon_0^{(\theta)}\rho_2 X$. Since $0 \leq X \leq \dfrac{1}{h^2}$, we obtain $4\rho_2 X \leq \dfrac{4\rho_2}{h^2}$. Using $\rho_2 = \dfrac{\mathscr{D}(\mathscr{D} - \rho)}{\mathscr{V}^2} + \dfrac{h^2}{6} \leq \dfrac{\mathscr{D}^2}{\mathscr{V}^2} + \dfrac{h^2}{6}$, and $\mathscr{V} < \mathscr{D}$, the constant term $\dfrac{4\rho_2}{h^2}$ remains bounded. At the end, $|\mathscr{A}|$ is bounded as $|\mathscr{A}| \geq \upsilon_0^{(\theta)}(1 - K_1)$, for some constant value $0 < K_1 < 1$. In the same way, compute the bound for $|\mathscr{B}|$: Thus, $|\mathscr{B}|^2 = (1 - 4\rho_2 X)^2 + \rho_1^2 Y$. As the values $\rho_1 > 0, \rho_2 > 0$ are positive, we get

$$|\mathscr{B}| \leq 1 + 4\rho_2 X + \rho_1\sqrt{Y},$$

and $\mathscr{V} < \mathscr{D}$ both ρ_1 and ρ_2 are bounded independent of the step-size h. Thus, $|\mathscr{B}|$ is bounded as: $|\mathscr{B}| \leq 1 + K_2$. From the Lipschitz condition stated in (7), we have obtained the estimation result as $|\upsilon_0^{(\theta)}\varepsilon^0 + \mu g(\varepsilon^0)| \leq \left(\upsilon_0^{(\theta)} + \mu L\right)|\varepsilon^0|$. By combining all the bounds, we obtain $|\varepsilon^{1/2}| \leq \dfrac{1 + K_2}{\upsilon_0^{(\theta)}(1 - K_1)}\left(\upsilon_0^{(\theta)} + \mu L\right)|\varepsilon^0|$. Substituting $\mu = \tau^\theta\Gamma(2 - \theta)$, we obtain $|\varepsilon^{1/2}| \leq K\left(1 + \dfrac{\tau^\theta\Gamma(2 - \theta)L}{\upsilon_0^{(\theta)}}\right)|\varepsilon^0|$, where K depends only on $\mathscr{D}/\mathscr{V}$. $\tau^\theta\Gamma(2-\theta)L \leq \upsilon_0^{(\theta)}$, then we have, $|\varepsilon^{1/2}| \leq K|\varepsilon^0|$. Since K approaches 1, we get $|\varepsilon^{1/2}| \leq |\varepsilon^0|$ which holds true. Assume $|\varepsilon^{r-\frac{1}{2}}| \leq |\varepsilon^0|$ holds true for $r = 1, 2, 3, ..., n-1$. From (21), we have $\mathscr{B} = 1 - 4\rho_2 X + i\rho_1\sqrt{Y}$, satisfies $|\mathscr{B}| \leq 1$. In addition, $\mathscr{A} = \upsilon_0^{(\theta)} - 4\left(\upsilon_0^{(\theta)}\rho_2 - \mu\rho\right)X + i\left(\upsilon_0^{(\theta)}\rho_1 + \mu\mathscr{V}\right)\sqrt{Y}$, then we have given

$$\mathscr{A}\varepsilon^{n-\frac{1}{2}} = \mathscr{B}\Xi^{n-\frac{1}{2}}, \tag{24}$$

where,

$$\Xi^{n-\frac{1}{2}} = \sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)} - \upsilon_{n-k}^{(\theta)}\right)\varepsilon^{k-\frac{1}{2}} + \upsilon_{n-1}^{(\theta)}\varepsilon^{0} + \mu g\left(\frac{3}{2}\varepsilon^{n-1} - \frac{1}{2}\varepsilon^{n-2}\right).$$

By taking the modulus of (24), we have $|\varepsilon^{n-\frac{1}{2}}| = \dfrac{|\mathscr{B}(\theta)|}{|\mathscr{A}(\theta)|}|\Xi^{n-\frac{1}{2}}|$. Assume the positivity of diffusion coefficient, $\upsilon_0^{(\theta)}\rho_2 - \mu\rho \geq 0 \Longleftrightarrow \tau^{\theta}\Gamma(2-\theta)\rho \leq \upsilon_0^{(\theta)}\rho_2$. From the assumption $\mathscr{V} < \mathscr{D}$, diffusion ensures, $|\mathscr{A}| \geq \upsilon_0^{(\theta)}(1-K_0)$ for some $0 < K_0 < 1$ independent of n, h, τ. Hence, $\dfrac{|\mathscr{B}|}{|\mathscr{A}|}| \leq \dfrac{1}{\upsilon_0^{(\theta)}(1-K_0)} = K_*$. From the properties of the weight coefficients (see Lemma 1), and the telescoping identity results

$$\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)} - \upsilon_{n-k}^{(\theta)}\right) + \upsilon_{n-1}^{(\theta)} = \upsilon_0^{(\theta)}. \tag{25}$$

Assume induction hypothesis $|\varepsilon^{k-\frac{1}{2}}| \leq |\varepsilon^0|, k \leq n-1$. Then from the assumption of Lipschitz continuity and Eq. (25), we obtain $|\Xi^{n-\frac{1}{2}}| \leq \left(\upsilon_0^{(\theta)} + \mu L\right)|\varepsilon^0|$. Thus, the final estimate becomes $|\varepsilon^{n-\frac{1}{2}}| \leq K_*\left(\upsilon_0^{(\theta)} + \tau^{\theta}\Gamma(2-\theta)L\right)|\varepsilon^0|$. $\tau^{\theta}\Gamma(2-\theta)L \leq \upsilon_0^{(\theta)}$, then $|\varepsilon^{n-\frac{1}{2}}| \leq K_* 2\upsilon_0^{(\theta)}|\varepsilon^0|$, and we conclude that $|\varepsilon^{n-\frac{1}{2}}| \leq |\varepsilon^0|, \forall n \geq 1$. Therefore, the scheme is unconditionally stable. ■

3.2. Convergence analysis

For this study, we need to introduce an Euclidean space of grid function on Ω_h, and let $\chi_h = \{\mathbf{v}|\mathbf{v} \in R^M, \varsigma_0 = \varsigma_M = 0\}$ be the real-valued grid function defined on $\{x_l, 0 \leq l \leq M\}$, then the L^2-norm induced by the weighted Euclidean vector product is given by $\langle \mathbf{u}, \mathbf{v}\rangle = h\sum_{l=1}^{M+1} u_l.v_l$, $\langle \mathbf{v}, \mathbf{v}\rangle^{1/2} = ||\mathbf{v}|| = ||\mathbf{v}||_{l^2} = \sqrt{h\sum_{l=1}^{M} v_l^2}$, $\langle \delta_x^2\mathbf{u}, \mathbf{v}\rangle = h\sum_{l=1}^{M+1}\delta_x^2 u_l.v_l$, and we also denote, $||\delta_x u|| = \sqrt{h\sum_{l=1}^{M}|\delta_x u_{l-\frac{1}{2}}|^2}$, $||\delta_x^2 u|| = \sqrt{h\sum_{l=1}^{M+1}|\delta_x^2 u_{l-\frac{1}{2}}|^2}$.

Theorem 2 *Assume that U_l^n is the analytical solution for* (6) *and u_l^n is the numerical solution of* (15)*, then we have the error estimate,*

$$||U_l^n - u_l^n|| \leq C\left(\tau^{3-\theta} + h^4\right).$$

PROOF Consider the full discretization as follows:

$$\frac{\Lambda_x}{\mu}\left[\upsilon_0^{(\theta)}u_l^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)u_l^{k-\frac{1}{2}}-\upsilon_{n-1}^{(\theta)}u_l^0\right]$$
$$=\rho\delta_x^2u_l^{n-\frac{1}{2}}-\mathscr{V}\delta_xu_l^{n-\frac{1}{2}}+\Lambda_xg(u_l^{n-\frac{1}{2}}). \tag{26}$$

If we assume $U_l^{n-\frac{1}{2}}$ to be the exact solution of (26), then taking the inner product with $\varepsilon^{n-\frac{1}{2}}$ on both sides yields

$$\Lambda_x\left[(\upsilon_0^{(\theta)}\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}})-\sum_{k=1}^{n-1}\left((\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)\varepsilon^{k-\frac{1}{2}}-\upsilon_{n-1}^{(\theta)}\varepsilon^0,\varepsilon^{n-\frac{1}{2}})\right]$$
$$=\left(\mu\rho\delta_x^2\varepsilon^{n-\frac{1}{2}}-\mu\mathscr{V}\delta_x\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)+\Lambda_x\left(g(\frac{3}{2}\varepsilon^{n-1}-\frac{1}{2}\varepsilon^{n-2}),\varepsilon^{n-\frac{1}{2}}\right)+\left(T^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right).$$

Since $\Lambda_x=1+\rho_1\delta_x+\rho_2\delta_x^2$, and $\mathscr{V}<\mathscr{D}$, we have $\rho_2>0$. Thus, we have the estimation $(\Lambda_xw,w)=||w||^2-\rho_2||\delta_xw||^2\leq||w||^2$. Furthermore, there exists $C_1>0$ such that $C_1||w||^2\leq(\Lambda_xw,w)\leq C_2||w||^2$. Therefore, we have the relation $\Lambda_x\left(\upsilon_0^{(\theta)}\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)\geq C||\varepsilon^{n-\frac{1}{2}}|^2$. By using the Cauchy-Schwartz inequality,

$$\left(\varepsilon^{n-\frac{1}{2}},\varepsilon^{k-\frac{1}{2}}\right)\leq\frac{1}{2}\left(||\varepsilon^{n-\frac{1}{2}}||^2+||\varepsilon^{k-\frac{1}{2}}||^2\right).$$

Hence

$$\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)\left(\varepsilon^{n-\frac{1}{2}},\varepsilon^{k-\frac{1}{2}}\right)\leq\frac{1}{2}\sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)||\varepsilon^{k-\frac{1}{2}}||^2+\frac{1}{2}\upsilon_0^{(\alpha)}||\varepsilon^{n-\frac{1}{2}}||^2.$$

The estimation using Cauchy-Schwartz inequality for the initial term is also,

$$\upsilon_{n-1}^{(\theta)}\left(\varepsilon^0,\varepsilon^{n-\frac{1}{2}}\right)\leq\frac{1}{2}\upsilon_{n-1}^{(\theta)}\left(||\varepsilon^0||^2+||\varepsilon^{n-\frac{1}{2}}||^2\right).$$

Using the summation by parts and under homogeneous boundary conditions, we obtain $\left(\mu\rho\delta_x^2\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)=-\mu\rho||\delta_x\varepsilon^{n-\frac{1}{2}}||^2\leq0,\ \left(\delta_x\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)=0.$
Thus, $\left(\mu\rho\delta_x^2\varepsilon^{n-\frac{1}{2}}-\mu\mathscr{V}\delta_x\varepsilon^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)\leq0$. The nonlinear term is approximated using Lipschitz continuity, and using Young inequality,

$$\Lambda_x\left(g(\frac{3}{2}\varepsilon^{n-1}-\frac{1}{2}\varepsilon^{n-2}),\varepsilon^{n-\frac{1}{2}}\right)\leq CL||\varepsilon^{n-1}||||\varepsilon^{n-\frac{1}{2}}||\leq C||\varepsilon^{n-1}||^2+\frac{1}{4}||\varepsilon^{n-\frac{1}{2}}||^2.$$

For the last term, the truncation error estimation using Young inequality can be written as $\left(T_{h\tau}^{n-\frac{1}{2}},\varepsilon^{n-\frac{1}{2}}\right)\leq||T^{n-\frac{1}{2}}||||\varepsilon^{n-\frac{1}{2}}||\leq C||\tau^{3-\theta}+h^4||^2+\frac{1}{4}||\varepsilon^{n-\frac{1}{2}}||^2$. At the end, the energy inequality for the collecting terms can be obtained as follows:

$$||\varepsilon^{n-\frac{1}{2}}||^2 \leq \sum_{k=1}^{n-1}\left(\upsilon_{n-k-1}^{(\theta)}-\upsilon_{n-k}^{(\theta)}\right)||\varepsilon^{n-\frac{1}{2}}||^2+\upsilon_{n-1}^{(\theta)}||\varepsilon^0||^2+C\left(\tau^{3-\theta}+h^4\right)^2.$$

Applying the fractional discrete Grönwall inequality [15] yields,

$$||\varepsilon^{n-\frac{1}{2}}|| \leq C\left(\tau^{3-\theta}+h^4\right),$$

This finishes the proof of the analysis. ■

4. Numerical results and discussion

Example 1 Consider the nonlinear time-fractional convection-diffusion equation over a finite domain:

$$\begin{cases} {}^cD_\tau^\theta u(x,t)+\mathscr{V}\dfrac{\partial u(x,t)}{\partial x}-\mathscr{D}\dfrac{\partial^2 u(x,t)}{\partial x^2}=g(u,x,t), \\ u(0,t)=0, u(1,t)=0, (x,t)\in(0,l')\times(0,T), \\ u(x,0)=0, x\in(0,l'). \end{cases}$$

One can confirm that this model has one analytical solution, $u(x,t)=t^{2+\theta}\sin(\pi x)$ when $g(u,x,t)=\dfrac{\Gamma(3+\theta)}{2}t^2\sin(\pi x)+\mathscr{V}t^{2+\theta}\pi\cos(\pi x)+\mathscr{D}t^{2+\theta}\pi^2\sin(\pi x)+u^2$.

Our numerical calculations are tested against this analytical solution. With the spatial step-sizes $M=4,8,16,32$ and the time-grid point $N=M^2$, Table 1 displays L_2 and L_∞-errors with scheme's spatial order for various fractional orders where $Order_s=\log_2\left(\dfrac{E(h,\tau)}{E(h/2,\tau)}\right)$. The time-order is also displayed in Table 2 for the time and space grid points $N=4,8,16,32$ and $M=1000$, respectively, where $Order_t=\log_2\left(\dfrac{E(h,\tau)}{E(h,\tau/2)}\right)$. Tables 1 and 2 demonstrate how the CPU-run time guarantees our solution's effectiveness and computational viability. In addition, since the scheme uses L2-1-approximation history dependent approximation of the Caputo-derivative, the cost at time level n generally depends on the previous time levels. Therefore, even though Table 1 has 64 spatial points, its 4096 time levels and long fractional-memory history can make it much more expensive than a calculation with $M=1000$, and $N=64$. Figure 1 displays the surface plot for the comparison of exact and numerical solutions, which are evaluated at space and time-grid point values (i.e. $N=1024$, $M=100$ with $\mathscr{V}=0.1, \mathscr{D}=1, T=1, \theta=0.8$. These solutions show that the numerical and the exact solutions are almost identical. Our current approach is more accurate with 4^{th}-order convergence in space at fractional orders $\theta=0.25$, 0.5, 0.75 than the method in [9], as demonstrated by the comparison values of L_2 errors described in Table 3 for $N=1000$, $M=4,8,16,32$.

Table 1. L_∞ and L_2-error with order of convergence in spatial for nonlocal order, and CPU-run time at $T=1, \mathscr{V}=0.1, \mathscr{D}=1, N=M^2$ for Example 1

θ	(M,N)	L_∞-error	$Order_s$	L_2-error	$Order_s$	CPU(s)
0.2	(4, 16)	1.2541e-03		8.8986e-04		0.0023
0.2	(8, 64)	7.6544e-05	4.0342	5.4314e-05	4.0342	0.0090
0.2	(16, 256)	4.757e-06	4.008	3.3758e-06	4.008	0.0673
0.2	(32, 1024)	2.9715e-07	4.0009	2.1086e-07	4.0009	0.9019
0.2	(64, 4096)	1.8595e-08	3.9983	1.3188e-08	3.999	14.463
0.8	(4, 16)	1.0935e-03		7.805e-04		0.0020
0.8	(8, 64)	6.4547e-05	4.0824	4.6094e-05	4.0817	0.0076
0.8	(16, 256)	3.8844e-06	4.0546	2.7755e-06	4.0538	0.0613
0.8	(32, 1024)	2.3522e-07	4.0456	1.6815e-07	4.0449	0.8803
0.8	(64, 4096)	1.4335e-08	4.0364	1.024e-08	4.0375	14.440

Table 2. L_∞ and L_2 error values with order of convergence in time for nonlocal orders, and CPU-run time at $T=1, \mathscr{V}=0.1, \mathscr{D}=1, M=1000$ for Example 1

θ	N	L_∞-error	$Order_t$	L_2-error	$Order_t$	CPU(s)
0.2	4	6.8943e-03		4.7656e-03		0.0033
0.2	8	1.7821e-03	1.9518	1.2319e-03	1.9517	0.0049
0.2	16	4.4993e-04	1.9858	3.1104e-04	1.9858	0.0098
0.2	32	1.1279e-04	1.9960	7.7975e-05	1.9960	0.0176
0.2	64	2.8210e-05	1.9994	1.9502e-05	1.9994	0.0380
0.8	4	4.5468e-03		3.0703e-03		0.0029
0.8	8	1.4051e-03	1.7000	9.5392e-04	1.7000	0.0040
0.8	16	3.8185e-04	1.8796	2.5981e-04	1.8764	0.0073
0.8	32	1.0051e-04	1.9257	6.8484e-05	1.9236	0.0142
0.8	64	2.6189e-05	1.9402	1.7867e-05	1.9385	0.0318

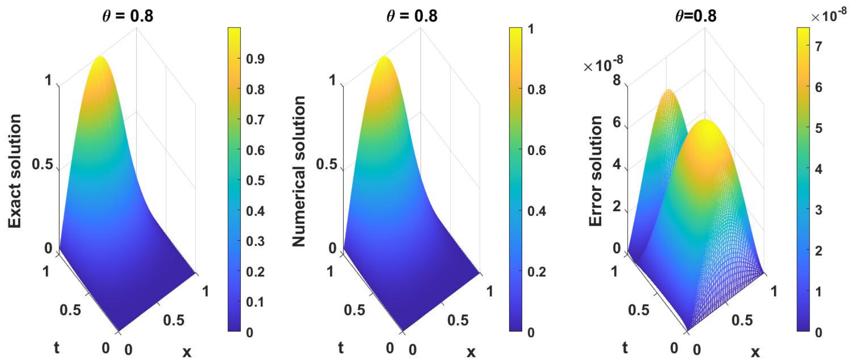

Fig. 1. The comparison of exact and numerical solutions for Example 1 with fractional order $\theta=0.8$ at different space and time-grid point values $M=100$, $N=1024$

Table 3. The comparison of L_2-errors with convergence order in space between present method and [9] for $N = 1000, T = 1, \mathscr{V} = 0.1, \mathscr{D} = 1$ for Example 1

	M	L_2-error	$Order_s$	L_2-error	$Order_s$	L_2-error	$Order_s$
Method [9]	4	3.59e-2	–	1.82e-2		1.08e-2	-
	8	5.8e-3	2.5299	3.1e-3	2.5536	2e-3	2.4330
	16	1e-3	2.5361	5.56e-4	2.4779	3.86e-4	2.3729
	32	1.78e-4	2.4939	1.02e-4	2.4484	9.68e-5	1.9964
Our method	4	1.195e-3	–	1.14e-3		1.06e-3	-
	8	7.32e-5	4.0298	6.97e-5	4.0298	6.50e-5	4.0295
	16	4.47e-6	4.0315	4.25e-6	4.0364	3.96e-6	4.0376
	32	2.02e-7	4.4703	1.77e-7	4.5866	1.61e-7	4.5164

5. Conclusion

The numerical treatment of a nonlinear time-fractional convection-diffusion equation over a finite domain is the subject of this research. We have proposed a Crank--Nicolson scheme based L_{2-1} quadratic interpolation methodology for the Caputo--fractional derivative and compact exponential difference method for the approximation of convection-diffusion term. We used the energy method for the theoretical analysis of convergence order and the von Neumann-Fourier analysis method for the stability of the proposed scheme. Concerning the discrete norms, we have obtained that the suggested technique is unconditionally stable with 4^{th}-order convergent in space and an improved $(\min(2, 3-\theta)^{th})$-order convergent in time. Numerical tests provide strong confirmation of the theoretical convergence and stability conclusions. We will consider the solution and application for the nonlinear 2D-time fractional convection-diffusion problem using high-order schemes in the future.

Acknowledgements

This research was supported by the Fundamental Research Funds for the Central Universities from China.

References

[1] Owolabi, K.M., Jain, S., Pindza, E., & Mare, E. (2024). Comprehensive numerical analysis of time-fractional reactiondiffusion models with applications to chemical and biological phenomena. *Mathematics*, *12*(20), 3251.

[2] Jain, S., & Abuomar, M. (2025). Numerical investigation of the fractional Riccati equation: Analysis, discretization, and applications. *Commun. Non. Sci. Num. Simul.*, 109539.

[3] Owolabi, K.M., Pindza, E., & Jain, S. (2025). Complex spatiotemporal phenomena in multidimensional reactiondiffusion system. *Eur. Phys. J. Spec. Topics*, *234*(8), 2565-2599.

[4] Owolabi, K.M., & Jain, S. (2025). From spiral turbulence to spatiotemporal chaos: Numerical and analytical study of a novel CFGL system. *Chaos, Solitons & Fractals*, *200*, 117073.

[5] Khalouta, A. (2024). A new identification of Lagrange multipliers to study solutions of nonlinear Caputo-Fabrizio fractional problems. *Partial Diff. Equ. Appl. Math.*, *10*, 100711.

[6] Khalouta, A. (2025). A new coupling method of khalouta transform and residual power series method for solving nonlinear fractional hyperbolic-like equations with variable coefficients. *J. Appl. Math. Comput. Mech.*, *24*(1), 30-43.

[7] Khalouta, A. (2025). An accurate numerical technique for solving a special case of fractional differential equations using the Khalouta transform of two different fractional derivatives. *Iran. J. Numer. Anal. Optim.*, *15*(2), 625-644.

[8] Khalouta, A. (2025). *Conformable Fractional Taylor Series Algorithm for Solving Temporal Fractional Reaction-Diffusion-Convection Equation*. In Modeling of Discrete and Continuous Systems. (pp. 115-132). Singapore: Springer Nature.

[9] Basha, M., Anley, E.F., & Dai, B. (2023). Numerical solution for a nonlinear time-space fractional convection-diffusion equation. *ASME J. Comput. Non. Dyn.*, *18*, 011006.

[10] Anley, E.F., & Sun, C. (2025). Numerical solution of two-dimensional nonlinear time-space fractional reaction advection-diffusion equation with its application. *Int. J. Appl. Comput. Math.*, *11*, 1-26.

[11] Cui, M. (2014). A high-order compact exponential scheme for the fractional convection-diffusion equation. *J. Comput. Appl. Math.*, *255*, 404-416.

[12] Tian, Z.F., & Yu, P.X. (2011). A high-order exponential scheme for solving 1D unsteady convection-diffusion equations. *J. Comput. Appl. Math.*, *235*, 2477-2491.

[13] Zhang, J., Zhang, X., & Yang, B. (2018). An approximation scheme for the time fractional convection-diffusion equation. *Appl. Math. Comput.*, *335*, 305-312.

[14] Gao, G., Sun, Z.Z., & Zhang, H.W. (2014). A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. *J. Comput. Phys.*, *259*, 33-50.

[15] Quarteroni, A. & Valli, A. (1997). *Numerical Approximation of Partial Differential Equations*. New York: Springer-Verlag.