ON SETS OF SOME CLASSES AND THEIR TANGENCY

Tadeusz Konik

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. In this paper the problem of the tangency of sets of some classes having the Darboux property in a generalized metric space is considered. Some sufficient and necessary conditions for the tangency of sets of these classes have been given here.

1. Introduction

Let (E,l) be a generalized metric space. E denotes here an arbitrary non-empty set, and l is a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let k be any, but fixed positive real number, and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0 \tag{1}$$

Let $S_l(p,r)_u$ denote the so-called u-neighbourhood of the sphere $S_l(p,r)$ in the space (E,l) defined by the formula

$$S_l(p,r)_u = \begin{cases} \bigcup\limits_{q\in S_l(p,r)} K_l(q,u) & \text{for} \quad u > 0 \\ S_l(p,r) & \text{for} \quad u = 0 \end{cases} \tag{2}$$

We say that a pair (A,B) of sets of the family E_0 is (a,b)-clustered at the point p of the space (E,l), if 0 is the cluster point of the set of all numbers $r > 0$ such that the sets $A \cap S_l(p,r)_{a(r)}$ and $B \cap S_l(p,r)_{b(r)}$ are non-empty.

The tangency relation $T_l(a,b,k,p)$ of sets of the family E_0 in the generalized metric space (E,l) is defined as follows (see [12]):

$$T_l(a,b,k,p) = \{(A,B): \; A,B \in E_0, \text{ the pair } (A,B) \text{ is } (a,b)\text{-clustered}$$
$$\text{at the point } p \text{ of the space } (E,l) \text{ and}$$
$$\frac{1}{r^k} l(A\cap S_l(p,r)_{a(r)}, B\cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0\} \tag{3}$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is (a,b)-tangent (or briefly: is tangent) of order k to the set $B \in E_0$ at the point p of the space (E,l).

Let ρ be an arbitrary metric of the set E. We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A,B \in E_0$ in the metric space (E,ρ).

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0, such that $f(0)=0$. By $\mathfrak{F}_f$ we will denote the class of all functions l fulfilling the conditions:

1^0 $l : E_0 \times E_0 \longrightarrow \langle 0,\infty)$,

2^0 $f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A\cup B))$ for $A,B \in E_0$.

It is easy to check that every function $l \in \mathfrak{F}_f$ generates in the set E the metric l_0 defined by the formula:

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad x,y \in E \tag{4}$$

We say (see [6]) that the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), and we shall write this as: $A \in D_p(E,l)$, if there exists a number $\tau > 0$ such that $A \cap S_l(p,r) \ne \emptyset$ for $r \in (0,\tau)$.

In this paper we shall consider certain problems concerning the tangency of sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property at the point p of the generalized metric spaces (E,l), for $l \in \mathfrak{F}_f$. Some theorems for the tangency of sets of these classes have been given here.

2. On the tangency of sets of the classes $\widetilde{M}_{p,k}$

Let ρ be a metric of the set E, and let A be any set of the family E_0 of all non-empty subsets of the set E. By A' we shall denote the set of all cluster points of the set A of the family E_0.

The classes of sets $\widetilde{M}_{p,k}$, mentioned in the Introduction of this paper, are defined as follows (see [10]):

$$\begin{gathered}\widetilde{M}_{p,k} = \{A \in E_0 :\ p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \text{for every pair of points } (x,y) \in [A,p;\mu,k] \\ \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\}\end{gathered} \tag{5}$$

where

$$[A, p; \mu, k] = \{(x, y) : \ x \in E, \ y \in A \ \text{and} \ \mu\rho(x, A) < \rho^k(p, x) = \rho^k(p, y)\}$$

whereas $\rho(x, A)$ is equal to

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\} \quad \text{for} \quad x \in E \tag{6}$$

For an arbitrary $k > 0$ the classes of sets $\widetilde{M}_{p,k}$ contain, among other things, the classes $A^*_{p,k}$ defined in the paper [9]. If $k = 1$, then the classes of rectifiable arcs A_p and $\widetilde{A}_p$ are contained in the class of sets $\widetilde{M}_{p,1}$.

We say that the rectifiable arc A belongs to the class $\widetilde{A}_p$, if the point $p \in E$ is an origin of this arc, and

$$\lim_{A \ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p, x)} = g < \infty \tag{7}$$

where $\ell(\widetilde{px})$ denotes the lenght of the arc $\widetilde{px}$ with the ends p and x.

If $g = 1$, then we say that the rectifiable arc A has the *Archimedean* property at the point p of the metric space (E, ρ), and is the arc of the class A_p.

From the above it follows that all results obtained in this paper for the sets of the classes $\widetilde{M}_{p,k}$ will be true, in particular, for arbitrary sets of the classes A_p, $\widetilde{A}_p$ and $A^*_{p,k}$.

In the paper [10] was proved the following:

Theorem 2.1. *In the classes $\widetilde{M}_{p,k}$ of sets on the Darboux property at the point p of the metric space (E, ρ) the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \tag{8}$$

is necessary and sufficient to

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p, r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{9}$$

Using this theorem we shall prove:

Theorem 2.2. *If for an arbitrary function $l \in \mathfrak{F}_f$ the pair of sets $(A, B) \in T_l(a, b, k, p)$ for $A, B \in D_p(E, l)$, then*

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \tag{10}$$

Proof. We assume that $(A, B) \in T_l(a, b, k, p)$ for $A, B \in D_p(E, l)$ and $l \in \mathfrak{F}_f$. From here, putting $l = f \circ d_\rho$ (see definition of the class $\mathfrak{F}_f$), we obtain

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))) \xrightarrow[r \to 0^+]{} 0 \tag{11}$$

Because

$$d_\rho(A \cap S_l(p,r)_{a(r)}) \leq d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))$$

and

$$d_\rho(B \cap S_l(p,r)_{b(r)}) \leq d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))$$

then from here, from (11) and from the properties of the function f follows

$$\frac{1}{r^k} f(d_\rho(A \cap S_l(p,r)_{a(r)})) \xrightarrow[r \to 0^+]{} 0 \tag{12}$$

and

$$\frac{1}{r^k} f(d_\rho(B \cap S_l(p,r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0 \tag{13}$$

Hence and from the equality

$$f(d_\rho A) = d_l A \quad \text{for} \quad A \in E_0 \tag{14}$$

we obtain

$$\frac{1}{r^k} d_l(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{15}$$

and

$$\frac{1}{r^k} d_l(B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{16}$$

Because every function $l \in \mathfrak{F}_f$ generates in the set E the metric l_0, then from here, from (15), (16) and from Theorem 2.1 of the paper [10] follows the thesis of this theorem.

Theorem 2.2 of the present paper has fundamental meaning for the tangency of sets of the classes $\widetilde{M}_{p,k}$ in the generalized metric spaces (E, l). In many earlier papers were considered, among other things, the problems concerning: compatibility, equivalence, additivity and homogeneity of the tangency relation $T_l(a, b, k, p)$ of sets of the classes $\widetilde{M}_{p,k}$ in the spaces (E, l), for $l \in \mathfrak{F}_f$. In these considerations the condition (10) is sufficient condition for above mentioned problems. From Theorem 2.2 it follows that (10) is also necessary condition for these and other problems relating to the tangency of sets. It is shown below:

Theorem 2.3. *If* $l \in \mathfrak{F}_f$*, the sets* A, B *on the Darboux property at the point* p *of the space* (E, l) *are subsets of a certain set* $C \in \widetilde{M}_{p,k}$*, then* $(A, B) \in T_l(a, b, k, p)$ *if and only if the functions* a, b *fulfil the condition* (10).

`Proof.` Because the sets $A, B \in D_p(E, l)$ are subsets of the set $C \in \widetilde{M}_{p,k}$, then the set C has also the *Darboux* property at the point p of this space, in other words, $C \in \widetilde{M}_{p,k} \cap D_p(E, l)$.

We assume that the functions a, b fulfil the condition (10). Let $\alpha = \max(a, b)$. Hence, from (10), from Theorem 2.1 of the paper [10] and from the fact that every function $l \in \mathfrak{F}_f$ generates in the set E the metric l_0 it follows that

$$\frac{1}{r^k} d_l(C \cap S_l(p, r)_{\alpha(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{17}$$

Because from the assumptions of this theeorem on the sets $A, B, C \in \widetilde{M}_{p,k}$

$$A \cap S_l(p, r)_{a(r)} \subset C \cap S_l(p, r)_{\alpha(r)} \tag{18}$$

and

$$B \cap S_l(p, r)_{b(r)} \subset C \cap S_l(p, r)_{\alpha(r)} \tag{19}$$

then from here and from (17) follows the conditions:

$$\frac{1}{r^k} d_l(A \cap S_l(p, r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{20}$$

and

$$\frac{1}{r^k} d_l(B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{21}$$

Let by the definition $\rho_l(A, B)$ denote the distance of sets $A, B \in E_0$ in the generalized metric space (E, l), i.e.,

$$\rho_l(A, B) = \inf\{l_0(x, y) :\ x \in A,\ y \in B\} \quad \text{for} \quad A, B \in E_0 \tag{22}$$

From the equality (4) and from the properties of the function f follows

$$\begin{aligned} f(\rho(A, B)) &= f(\inf\{\rho(x, y) :\ x \in A,\ y \in B\}) \\ &= \inf\{f(\rho(x, y)) :\ x \in A,\ y \in B\} \\ &= \inf\{l_0(x, y) :\ x \in A,\ y \in B\} = \rho_l(A, B) \end{aligned}$$

that is to say

$$\rho_l(A,B) = f(\rho(A,B)) \quad \text{for} \quad A,B \in E_0 \tag{23}$$

From (18) and (19) it follows that

$$(A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}) \subset C \cap S_l(p,r)_{\alpha(r)}$$

Hence we get the inequality

$$\begin{aligned} &\rho_l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \\ &\leq d_l((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)})) \leq d_l(C \cap S_l(p,r)_{\alpha(r)}) \end{aligned} \tag{24}$$

From here, from (18), (24), from the properties of the function f and from the inequality

$$d_\rho(A \cup B) \leq d_\rho A + d_\rho B + \rho(A,B) \quad \text{for} \quad A,B \in E_0 \tag{25}$$

we obtain

$$\begin{aligned} 0 &\leq l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \\ &\leq f(d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))) \\ &\leq f(d_\rho(A \cap S_l(p,r)_{a(r)}) + d_\rho(B \cap S_l(p,r)_{b(r)}) \\ &\quad +\rho(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)})) \\ &\leq f(d_\rho(A \cap S_l(p,r)_{a(r)})) + f(d_\rho(B \cap S_l(p,r)_{b(r)})) \\ &\quad +f(\rho(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)})) \\ &= d_l(A \cap S_l(p,r)_{a(r)}) + d_l(B \cap S_l(p,r)_{b(r)}) \\ &\quad +\rho_l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \\ &\leq 3d_l(C \cap S_l(p,r)_{\alpha(r)}) \end{aligned}$$

in other words,

$$l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \leq 3d_l(C \cap S_l(p,r)_{\alpha(r)}) \tag{26}$$

Hence and from the condition (17) we get

$$\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{27}$$

From the Darboux property of sets $A, B \in \widetilde{M}_{p,k}$ at the point p of the space (E, l) it follows that the pair (A, B) is (a, b)-clustered at the point p of this space.
Hence and from (27) it results that $(A, B) \in T_l(a, b, k, p)$. This ends the proof of the sufficient condition.

The necessary condition of this theorem follows from the assumptions of the theorem and from Theorem 2.2 of this paper.

From Theorem 2.3 the following corollaries are easily derived:

Corollary 2.1. *If* $l \in \mathfrak{F}_f$ *and* $A, B \in D_p(E, l)$, $A \cup B \in \widetilde{M}_{p,k}$, *then* $(A, B) \in T_l(a, b, k, p)$ *if and only if the functions* a, b *fulfil the condition* (10)

Corollary 2.2. *If* $l \in \mathfrak{F}_f$ *and* $A \in D_p(E, l)$ *is subsets of the set* $B \in \widetilde{M}_{p,k} \cap D_p(E, l)$, *then* $(A, B) \in T_l(a, b, k, p)$ *if and only if the functions* a, b *fulfil the condition* (10)

Corollary 2.3. *If* $A \in \widetilde{M}_{p,k} \cap D_p(E, l)$ *and* $l \in \mathfrak{F}_f$, *then* $(A, A) \in T_l(a, b, k, p)$, *in other words, the tangency relation* $T_l(a, b, k, p)$ *is reflexive in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E, l)$ *if and only if the functions* a, b *fulfil the condition* (10).

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