SHAPE SENSITIVITY ANALYSIS IN NONLINEAR TRANSIENT HEAT TRANSFER

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Abstract. In the paper the 1D model of nonlinear transient heat transfer process is assumed. Thermophysical parameters of the material considered are strongly temperature dependent. The sensitivity analysis of this process with respect to the thickness of plate is presented. On the stage of numerical computations the finite difference method is used. In the final part of the paper the results obtained are shown.

1. Formulation of the problem

The heat transfer process in the plate of thickness L is described by following equation

$$C(T)\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T(x,t)}{\partial x}\right) \tag{1}$$

where $C(T)$ is the volumetric specific heat, $\lambda(T)$ is the thermal conductivity, T, x, t denote temperature, spatial coordinate and time. Equation (1) is supplemented by adequate boundary and initial conditions.

The aim of this paper is to estimate the change of temperature due to a change of the plate thickness. We assume that $b = L$ is the shape design parameter. Using the concept of material derivative we can write [1, 2]

$$\frac{\mathbf{D}T}{\mathbf{D}b}=\frac{\partial T}{\partial b}+\frac{\partial T}{\partial x}v \tag{2}$$

where $v=v(x,b)$ is the velocity associated with design parameter b.
Because (c.f. equation (2))

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial b}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial^2 T}{\partial x^2}v=\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b}\right)+\frac{\partial^2 T}{\partial x^2}v \tag{3}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)=\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b}+\frac{\partial T}{\partial x}\nu\right)=\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b}\right)+\frac{\partial^2 T}{\partial x^2}\nu+\frac{\partial T}{\partial x}\frac{\partial \nu}{\partial x} \tag{4}$$

therefore

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{\partial T}{\partial x}\frac{\partial \nu}{\partial x} \tag{5}$$

In similar way one obtains

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) \tag{6}$$

Using formula (5) we have

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right)=\frac{\mathrm{D}}{\mathrm{D}b}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\right]=\frac{\partial}{\partial x}\left[\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial x}\right)\right]-\frac{\partial^2 T}{\partial x^2}\frac{\partial \nu}{\partial x} \tag{7}$$

and next

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right)&=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{\partial T}{\partial x}\frac{\partial \nu}{\partial x}\right]-\frac{\partial^2 T}{\partial x^2}\frac{\partial \nu}{\partial x}=\\&=\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-2\frac{\partial^2 T}{\partial x^2}\frac{\partial \nu}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial^2 \nu}{\partial x^2}\end{aligned} \tag{8}$$

Taking into account the definition (2) we obtain

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial x}\right)^2=2\frac{\partial T}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{\partial T}{\partial x}\frac{\partial \nu}{\partial x}\right] \tag{9}$$

Additionally, we can check that

$$\frac{\mathrm{D}\lambda(T)}{\mathrm{D}b}=\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\frac{\mathrm{D}T}{\mathrm{D}b} \tag{10}$$

and

$$\frac{\mathrm{D}C(T)}{\mathrm{D}b}=\frac{\mathrm{d}C(T)}{\mathrm{d}T}\frac{\mathrm{D}T}{\mathrm{D}b} \tag{11}$$

Presented above formulas are necessary in order to realize the shape sensitivity analysis of heat transfer process.

2. Shape sensitivity analysis

The heat transfer equation (1) can be written in the following form

$$C(T)\frac{\partial T}{\partial t}=\lambda(T)\frac{\partial^2 T}{\partial x^2}+\frac{\mathrm{d}\lambda}{\mathrm{d}T}\left(\frac{\partial T}{\partial x}\right)^2 \tag{12}$$

If the direct approach of sensitivity method is applied [1, 2] then the equation (12) is differentiated with respect to shape parameter b, and then

$$\begin{aligned}&\frac{\mathrm{D}C(T)}{\mathrm{D}b}\frac{\partial T}{\partial t}+C(T)\frac{\partial U}{\partial t}=\frac{\mathrm{D}\lambda(T)}{\mathrm{D}b}\frac{\partial^2 T}{\partial x^2}+\\&+\lambda(T)\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right)+\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\right)\left(\frac{\partial T}{\partial x}\right)^2+\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial T}{\partial x}\right)^2\end{aligned} \tag{13}$$

where $U=\mathrm{D}T/\mathrm{D}b$ is the sensitivity function.
Using previously presented dependencies one obtains

$$\begin{aligned}&\frac{\mathrm{d}C(T)}{\mathrm{d}T}U\frac{\partial T}{\partial t}+C(T)\frac{\partial U}{\partial t}=\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}U\frac{\partial^2 T}{\partial x^2}+\\&+\lambda(T)\left[\frac{\partial^2 U}{\partial x^2}-2\frac{\partial^2 T}{\partial x^2}\frac{\partial \nu}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial^2 \nu}{\partial x^2}\right]+2\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\frac{\partial T}{\partial x}\left[\frac{\partial U}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial \nu}{\partial x}\right]\end{aligned} \tag{14}$$

Taking into account the equation (1) we have

$$\begin{aligned}&C(T)\frac{\partial U}{\partial t}=\lambda(T)\frac{\partial^2 U}{\partial x^2}+2\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\frac{\partial T}{\partial x}\frac{\partial U}{\partial x}-\lambda(T)\frac{\partial T}{\partial x}\frac{\partial^2 \nu}{\partial x^2}+\\&+\left[\frac{1}{\lambda(T)}\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}C(T)U-2C(T)\frac{\partial \nu}{\partial x}-\frac{\mathrm{d}C(T)}{\mathrm{d}T}U\right]\frac{\partial T}{\partial t}-\\&-\frac{1}{\lambda(T)}\left[\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\right]^2\left(\frac{\partial T}{\partial x}\right)^2\end{aligned} \tag{15}$$

In similar way we differentiate the boundary and initial conditions.

So, the basic problem and additional one connected with the sensitivity function U should be solved using the numerical methods. The problems discussed are coupled and it is impossible to solve only the sensitivity one because in equation (15) the derivatives of temperature T with respect to x and t appear, so the knowledge of temporary temperature field is necessary.

3. Example of computations

The freezing process of biological tissue has been considered. The temperature dependent parameters $\lambda(T)$ and $C(T)$ of tissue have been described by the following formulas [3]

$$\lambda(T)=\begin{cases}2, & T<-8\\ 0.6192+0.2071\,T+0.1165\,T^2, & -8\le T\le -1\\ 0.52, & T>-1\end{cases} \tag{16}$$

and

$$C(T)=\begin{cases}1930000, & T<-8\\ 3.8975517\cdot 10^7+7.9613171\cdot 10^7\,T+ & \\ 5.3654283\cdot 10^7\,T^2+9.9711215\cdot 10^6\,T^3+ & \\ 5.5449217\cdot 10^5\,T^4, & -8\le T\le -1\\ 3600000, & T>-1\end{cases} \tag{17}$$

In Figures 1 and 2 the courses of functions $\lambda(T)$ [W/mK], $C(T)$ [MJ/m^{3}K] and courses of their derivatives $\mathrm{d}\lambda(T)/\mathrm{d}T$ [W/mK^2], $\mathrm{d}C(T)/\mathrm{d}T$ [MJ/m^{3}K^2] are shown.
It is assumed that L = 0.02 m. On the surface x = 0 the Dirichlet condition $T_b = -10$°C has been accepted, on the surface $x = L$ the Neumann condition $q_b = 0$ has been taken into account. Initial temperature in tissue domain: $T_0 = 37$°C.

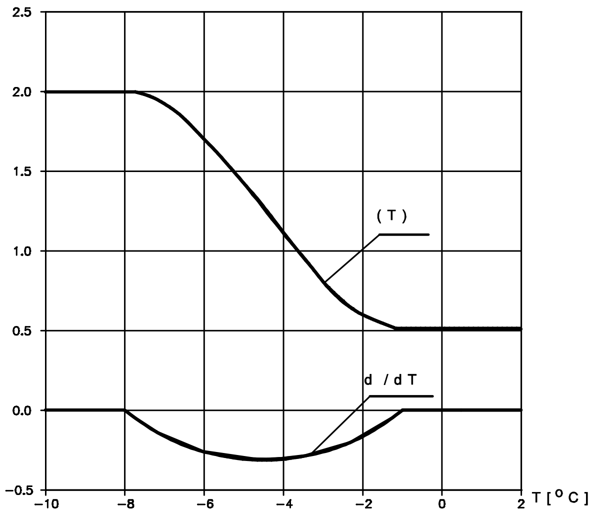

Fig. 1. Function $\lambda(T)$ [W/mK] and its derivative

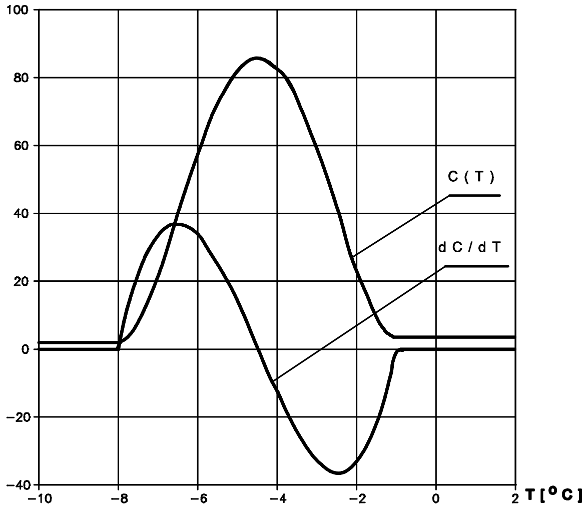

Fig. 2. Function $C(T)$ [MJ/m^{3}K] and its derivative

The velocity associated with design parameter b has been defined as $\nu = \nu(x, b) = (L - x)/(L - b)$. The problem has been solved using the finite difference method [4]. In Figure 3 the temperature distribution for $x \in [0,\ 0.005\,\text{m}]$ for times 2, 4, 6, 8 and 10 s under the assumption that b = 0 is presented. Figure 4 shows the changes of temperature due to a change of b for time 2, 4, 6, 8 and 10 s. It is assumed that b = 0.002 m and

$$\Delta T(x, t) = U(x, t)\, b \tag{18}$$

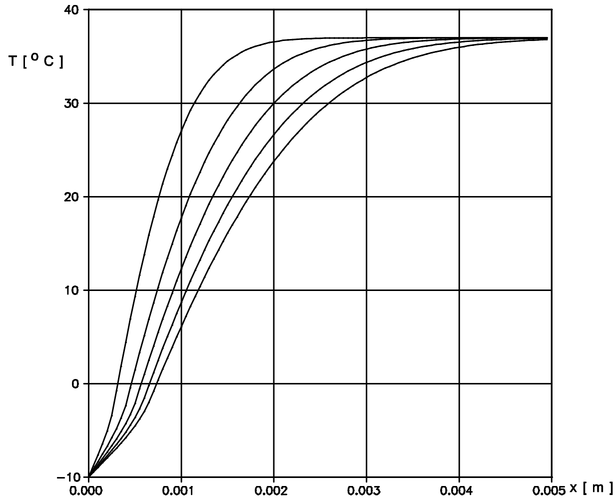

Fig. 3. Temperature distribution

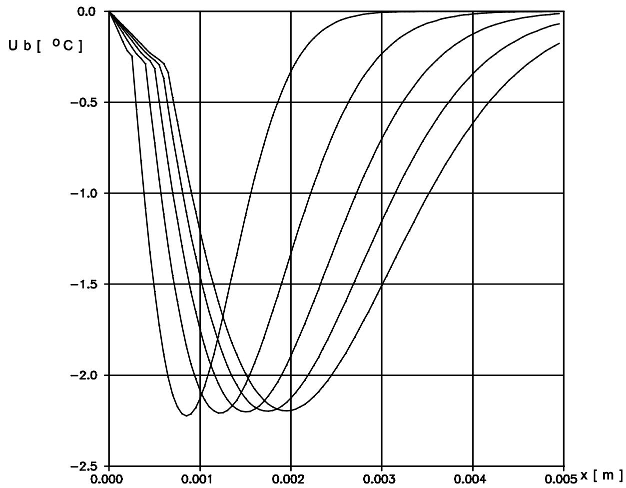

Fig. 4. Distribution of function *U·b*

Summing up, the shape sensitivity analysis allows, among others, to estimate the changes of temperature in the case when the geometry of the domain considered is changed.

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