SENSITIVITY ANALYSIS IN NON-LINEAR HEAT CONDUCTION

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Abstract. In the paper the sensitivity analysis is used in order to investigate the influence of external cooling conditions on the course of casting solidification. The model of thermal processes proceeding in the casting corresponds to the micro/macro one (the second generation model [1]). In particular we present the sensitivity of transient temperature field in casting volume on the changes of heat transfer coefficient given on its outer surface. On a stage of numerical modelling the control volume method [2, 3] has been used. In the final part of the paper the examples of numerical computations are shown.

1. Introduction

The thermal processes proceeding in the casting domain are determined by the Fourier equation in the form

$$P(x)\in\Omega \ : \ c(T)\frac{\partial T(x,t)}{\partial t}=\nabla\left[\lambda(T)\nabla T(x,t)\right]+L_V\frac{\partial f_s(x,t)}{\partial t} \tag{1}$$

where $T(x,t)$ is the casting temperature, $f_s(x,t)$ is the solid state fraction at the point x, $c(t)$ is the volumetric specific heat, $\lambda(t)$ is the thermal conductivity, L_V is the volumetric latent heat, x, t denote the spatial co-ordinates and time. Now, the problem of source function will be discussed. A temporary value of solid fraction f_S of the metal at point considered is given by the Johnson-Mehl-Avrami-Kolmogorov type equation [3-5]

$$f_s=1-\exp(-\omega) \tag{2}$$

where

$$\omega=\omega(x,t)=\frac{4}{3}\pi\,v\int_0^t\frac{\partial N}{\partial t}\left[\int_{t'}^{t}u\,d\tau\right]^3 dt \tag{3}$$

In equation (3) N is a number of nuclei (more precisely: density [nuclei/m^3]), v is the coefficient equal to 1 for the spherical grains and $v < 1$ for dendritic growth, u is a rate of solid phase growth, t' is a moment of crystallization process beginning. If we assume the constant number of nuclei then

$$\omega = \frac{4}{3}\pi\, v\, N\left[\int_{t'}^{t} u\, d\tau\right]^3 \tag{4}$$

The solid phase growth is determined by equation

$$u = \frac{\partial R}{\partial t} = \mu\, \Delta T^{p} \tag{5}$$

where R is a grain radius, μ is a growth coefficient, p is the exponent from the interval $p \in [1,2]$, and

$$\Delta T = \Delta T\left(x,t\right) = T^* - T\left(x,t\right) \tag{6}$$

is the undercooling below the solidification point T^*. Additionally we assume that for $T > T^*$: $\Delta T = 0$, in other words $u = 0$ and then the lower limit in integrals (3) and (4) equals $t' = 0$. So

$$\omega = \frac{4}{3}\pi\, v\, N\left[\int_{0}^{t} \mu \Delta T^{p} d\tau\right]^3 \tag{7}$$

Introducing (2) to the equation (1) one obtains

$$c(T)\,\frac{\partial T(x,t)}{\partial t} = \nabla\left[\lambda(T)\nabla T(x,t)\right] + L_V \exp(-\omega)\frac{\partial \omega}{\partial t} \tag{8}$$

and the source function is given by the formula

$$\begin{gathered} Q = L_V \exp(-\omega)\frac{\partial \omega}{\partial t} = \\ = 4\pi\, v\, N\, L_V\, \mu\, \Delta T^{p}\left(\int_{0}^{t} \mu\, \Delta T^{p} d\tau\right)^2 \exp\left[-\frac{4}{3}\pi v N\left(\int_{0}^{t} \mu\, \Delta T^{p} d\tau\right)^3\right] \end{gathered} \tag{9}$$

where $Q = Q(x,t)$. On the outer surface of the casting the Robin condition is assumed

$$P(x) \in \Gamma \ : \ -\lambda(T)\frac{\partial T(x,t)}{\partial n} = \alpha\left[T(x,t) - T_a\right] \tag{10}$$

where α is the substitute heat transfer coefficient between casting and environment, T_a is the ambient temperature, $\partial/\partial n$ is the normal derivative. The initial condition in the form

$$t = 0 \ : \ T(x,0) = T_0 \tag{11}$$

is also given.

2. Sensitivity model

The sensitivity analysis with respect to the external heat transfer conditions (in particular α) requires the differentiation of the equations forming the mathematical model with respect to α (the direct approach has been applied - see [6, 7]). Here we assume the constant value of the thermal conductivity $\lambda(T) = \lambda$ = const. So we have

$$\frac{\partial C(T)}{\partial \alpha}\frac{\partial T(x,t)}{\partial t} + C(T)\frac{\partial}{\partial \alpha}\left[\frac{\partial T(x,t)}{\partial t}\right] = \frac{\partial}{\partial \alpha}\left[\lambda \nabla^2 T(x,t)\right] + \frac{\partial Q(x,t)}{\partial \alpha} \tag{12}$$

Denoting $\partial T/\partial\alpha = U$ and using the Schwarz theorem one obtains

$$\frac{dC(T)}{d\alpha}U\frac{\partial T(x,t)}{\partial t} + C(T)\frac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) + \frac{\partial Q(x,t)}{\partial \alpha} \tag{13}$$

The Robin boundary condition for λ = const, takes a form

$$-\lambda\frac{\partial U(x,t)}{\partial n} = \alpha U(x,t) + T(x,t) - T_a = \alpha\left[U(x,t) - U_a\right] \tag{14}$$

where

$$U_a = \frac{T_a - T(x,t)}{\alpha} \tag{15}$$

The initial condition for the sensitivity model is $U(x,0) = 0$.
Now we denote

$$r_S = \int_0^t \mu \Delta T^p d\tau, \quad \rho_U = \int_0^t \mu \Delta T^{p-1} U d\tau \tag{16}$$

and then

$$\begin{gathered}\frac{\partial Q(x,t)}{\partial \alpha} = 4 p \pi v N L_V \exp\left(-\frac{4}{3}\pi v N r_S^3\right)\cdot \\ \left[4\pi v N \mu \Delta T^p \rho_U r_S^4 - 2\mu \Delta T^p \rho_U r_S - \mu U r_S^2 \Delta T^{p-1}\right]\end{gathered} \tag{17}$$

It should be pointed out that on the stage of numerical computations the value $p = 1$ has been assumed. Finally the sensitivity model with respect to the coefficient α can be written in the form

$$\begin{cases} P(x) \in \Omega \ : \ C(U)\dfrac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) + Q_U(x,t) \\ P(x) \in \Gamma \ : \ -\lambda \dfrac{\partial U(x,t)}{\partial n} = \alpha\left[U(x,t) - U_a\right] \\ t = 0 \ : \ U(x,0) = 0 \end{cases} \tag{18}$$

where (see (13) and (17))

$$Q_U(x,t) = \frac{\partial Q(x,t)}{\partial \alpha} - \frac{dC(T)}{dT} U \frac{\partial T(x,t)}{\partial t} \tag{19}$$

One can notice that the sensitivity model (19) is practically the same as the basic one. So, we can use the same numerical algorithm in order to solve both the basic and additional boundary initial problems (the different form of the source functions must be taken into account, of corse). It should be pointed out that the sensitivity model cannot be analyzed separately, because it is coupled with the solidification model by components Q_u and U_a. The problem discussed has been solved using the control volume method (CVM).

3. The control volume method

The control volume method (CVM) is a certain variant of the finite difference method quite commonly used in the calculations of heat and mass transfer. Using this approach the numerous very complex problems have been solved, and the high quality of results has been confirmed by experimental methods [2, 3].

The control volume method 'starts' directly from the energy balances for the control volumes with determined shapes and dimesions. In Figure 1 the selected 2D control volume is shown. The discretization is realized in this way that the sectors joining the central points of adjacent control volumes (0 and e) are perpendicular to the surfaces ΔA_e limiting the volume in e direction.

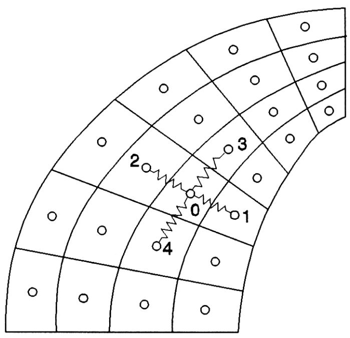

Fig. 1. Division of domain into control volumes

In the case of the transient heat transfer we introduce the time grid with time step $\Delta t = t^{f+1} - t^{f}$. The energy balance for domain ΔV_0 can be written in the form

$$\Delta H_0 = \Delta t \sum_e Q_e^s + \Delta t \int_{\Delta V_0} Q dV \tag{20}$$

where ΔH_0 is a change of control volume enthalpy, Q_e - the heat conducted at the time Δt from the nodes e to the node 0, Q - a capacity of internal heat sources in control volume ΔV_0, index s identifies the time t^s from the interval $[t^f, t^{f+1}]$. The successive components of energy balance result from the following formulas

$$\Delta H_0 = C_0^f \left(T_0^{f+1} - T_0^f \right) |\Delta V_0| \tag{21}$$

while

$$Q_e^s \Delta t = \frac{T_e^s - T_0^s}{R_e^f} \Delta A_e \Delta t \tag{22}$$

where

$$R_e^f = \frac{h_{0e}}{\lambda_0^f} + \frac{h_{e0}}{\lambda_e^f} \tag{23}$$

or

$$R_e^f = \frac{h_{0e}}{\lambda_0^f} + \frac{1}{\alpha}, \quad T_e^f = T_a \tag{24}$$

In the last formulas h_{0e} is the distance between node 0 and surface ΔA_0, h_{e0} is the distance between node e and surface ΔA_e. The thermal resistance given by equation (23) correspond to the two internal nodes, the thermal resistance given by the next equation corresponds to the boundary control volume and the direction to the external boundary. In order to avoid the non-linearities in the energy balances the volumetric specific heat and the thermal conductivity are taken for time t^f (the beginning of time interval considered). The source term can be written in the form

$$\int_{\Delta V_0} Q dV = Q_m |\Delta V_0| \tag{25}$$

where Q_m is a mean capacity of Q in the control volume ΔV_0.

As in the case of FDM, it is necessary to decide for what moment of time we shall assume the quantities Q_e. If we assume that the heat fluxes flowing to the element are proportional to the temperature differences at the moment $t = t^f$, then we shall obtain a solving system of the type 'explicit scheme'. Let us write the balance equation in explicit scheme

$$C_0^f\left(T_0^{f+1}-T_0^f\right)\Delta V_0 = \\ \sum_e \frac{T_e^f - T_0^f}{R_e^f}\Delta t\,\Psi_e + \sum_e \frac{T_a - T_0^f}{R_0^f}\Delta t\left(1-\Psi_e\right) + \Delta V_0\, Q_m\, \Delta t \tag{26}$$

where $\Psi_e = 1$ for directions to internal nodes and $\Psi_e = 0$ for directions to the external boundary. Hence we calculate the temperature T_0^{f+1}

$$T_0^{f+1} = T_0^f + \frac{\Delta t}{C_0^f \Delta V_0}\left[\sum_e \frac{T_e^f - T_0^f}{R_e^f}\Psi_e + \sum_e \frac{T_a - T_0^f}{R_e^f}\left(1-\Psi_e\right)\right] + \frac{Q_m \Delta t}{C_0^f} \tag{27}$$

or

$$T_0^{f+1} = T_0^f + \sum_e A_e T_e^f + B \tag{28}$$

where A_e and B result directly from (28). The stability condition of the scheme presented is the same as in the case of FDM [3]. Introducing in a place of temperature T the sensitivity U and taking into account the form of source function Q_U we can use the equations (26) for numerical modelling of transient sensitivity field.

4. Example of computations

The quarter of aluminium casting (0.15×0.15 m) has been considered. The pouring temperature T_0 = 700°C, while T^* = 660°C. The following parameters of the metal has been introduced: $\lambda_L = \lambda_S$ = 150 W/mK, $c_L = 3.070 \cdot 10^6$ J/m^{3}K, $c_S = 2.943 \cdot 10^6$ J/m^{3}K, $L_V = 9.75 \cdot 10^8$ J/m^3, $N = 10^{11}$ nuclei/m^3, $\mu = 3 \cdot 10^{-6}$ m/sK, $v = 1$, $p = 1$. The substitute heat transfer coefficient α = 1200 W/m^{2}K, ambient temperature T_a = 30°C (the big value of α corresponding to the conditions of the continuous casting technology [8] has been assumed in order to assure the essential changes of temperature in the domain considered and the big temperature gradients near the external boundry). Because the volumetric specific heat has been approximated by a piece-wise function therefore in numerical realization we have assume that dC/dT = 0. For the above input data the basic problem has been solved. Next the sensitivity $U(x,t)$ field has been found. The influence of the local temporary sensitivity on the changes of temperature field result from the Taylor formula

$$T\left(x_i, t^f, \alpha + \Delta\alpha\right) - T\left(x_i, t^f, \alpha\right) \approx \frac{\partial T\left(x_i, t^f\right)}{\partial \alpha}\Delta\alpha = U\left(x_i, t^f\right)\Delta\alpha \tag{29}$$

In this way the basic solution can be rebuilt on the other one corresponding to the new value of heat transfer coefficient. So the new coefficient α was equal to α = 1400 W/m^{2}K. The boundary initial problem has been solved again for this value if α and next the same solution has been found using the sensitivity approach. In Figure 2 the temperature field (t = 12 s) found directly, while in Figure 3 the temperature field calculated on the basis of sensitivity field are shown.

One can notice that the both results are practically the same. The possibility of the basic numerical solution transformation on the solution for the others cooling conditions is one of the essential practical applications of the sensitivity analysis.

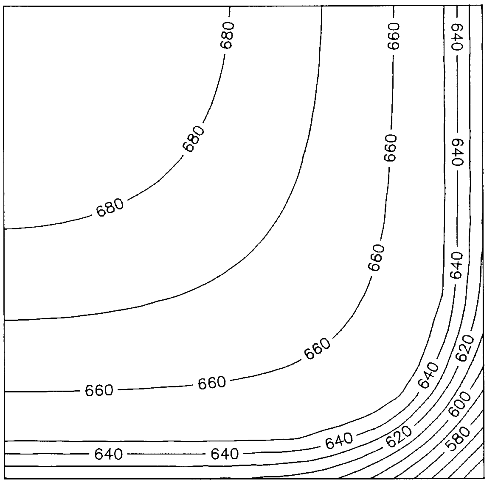

Fig. 2. Direct solution (α = 1400)

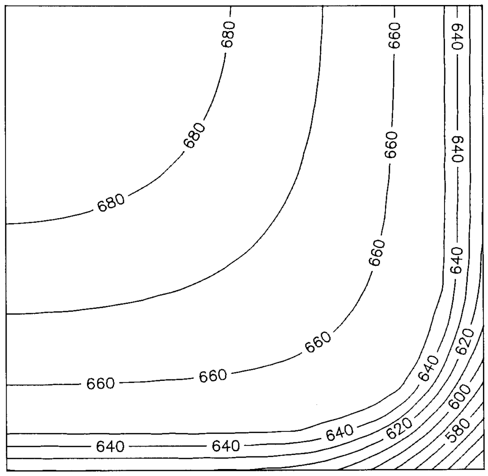

Fig. 3. Indirect solution (α = 1400)

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