THE GREEN'S FUNCTIONS FOR VIBRATION PROBLEMS OF CIRCULAR PLATES WITH ELASTIC RING SUPPORTS

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Abstract. The paper deals with axisymmetric free vibration of a circular uniform plate. The formulation and solution of the problem take into account an arbitrary number of elastic concentric rings supporting the plate. The Green's function method was applied to solve the free vibration problem. The Green's functions were obtained to various boundary conditions. The obtained exact solution was used in a numerical analysis of an influence of the parameter characterizing the system on its free vibration frequencies.

1. Introduction

Free vibration problems of a circular plate with concentric circular supports are the subject of papers [1-5]. In the case of a plate with one supporting ring, two regions are distinguished in order to solve the problem: an inner circular region with the radius equal to the radius of the supporting ring and an outer annular region. The frequency equation is obtained by substituting the general solutions of the governing equation defined in the two regions into the boundary and continuity conditions. This approach was applied in paper [1]. The receptance method was adopted to solve the problem by Azimi in papers [2, 3]. Elastic edge supports [2] and elastic or rigid interior supports [3] have been considered. A method using the Rayleigh's optimization criterion to optimize the frequency coefficient has been proposed in the paper [4] by Laura *et al*. In paper [5] Gupta *et al.* have investigated the effect of elastically restrained edge conditions on the natural frequencies of axisymmetric vibrations of a circular plate. In the study the authors have employed the method of collocation by derivatives. The solution of the vibration problem of a multiple ring-supported circular plate obtained with the use of a numerical model based on a set of orthogonal plate functions was presented in paper [6].

This paper presents an application of the Green's function method in solving the problem of free axisymmetric vibration of a circular plate supported by concentric elastic rings. The Green's functions of the differential problem with classical boundary conditions are determined. In case of the considered circular plates this approach leads to an exact solution of the vibration problem (frequency and mode shape equations). The symbolic computations and numerical calculations have been performed with the use of the computer algebra system *Mathematica*.

2. Formulation and solution of the problem

Consider a solid circular plate with uniform thickness. The axisymmetric vibration of this plate is governed by a differential equation

$$D\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\}=qr-\bar{\rho}hr\frac{\partial^2 w}{\partial t^2} \tag{1}$$

where w is the displacement, r is the radial variable, t is the time variable, D is the bending rigidity of the plate, $\bar{\rho}$ is the mass per unit volume, q is the load per unit area and h is the plate thickness. If the plate is supported by elastic concentric rings with radii r_j (j = 1,…,n), then

$$q(r,t)=\sum_{j=1}^{n}k_j w(r,t)\delta\left(r-r_j\right)$$

where k_j are stiffness coefficients of the elastic rings and δ is the Dirac delta function. For free vibrations one assumes $w(r,t)=W(r)e^{i\omega t}$. Hence equation (1) takes the form

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right\}-\Omega^4W=\sum_{j=1}^{n}K_jW\left(r_j\right)\delta\left(r-r_j\right) \tag{2}$$

where $\Omega^4=\dfrac{\bar{\rho}h\omega^2}{D}$ and $K_j=\dfrac{k_j}{D}$. The problem of free axisymmetric vibration of a circular plate of radius a is considered with the following boundary conditions, at the edge $r = a$:

– a clamped plate

$$W=0,\quad \frac{\mathrm{d}W}{\mathrm{d}r}=0 \tag{3}$$

– a simply supported plate

$$W=0,\quad \frac{\mathrm{d}^2W}{\mathrm{d}r^2}+\nu\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}=0 \tag{4}$$

– a free plate

$$\frac{\mathrm{d}^2W}{\mathrm{d}r^2}+\nu\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}=0,\quad \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]=0 \tag{5}$$

where ν is the Poisson ratio.

Having a Green's function $G(r,\rho)$, the solution of this problem can be presented in the form

$$W(r)=\sum_{j=1}^{n} K_j W(r_j) G(r,r_j) \tag{6}$$

Substituting $r = r_j$, $j = 1,2,...,n$ into (6) we obtain a set of equations, which can be written in the matrix form

$$\mathbf{AW}=\mathbf{0} \tag{7}$$

where $\boldsymbol{A}=\left[a_{ij}\right]_{1\le i,j\le n}$, $\boldsymbol{W}=\left[W(r_1)\dots W(r_n)\right]^{\mathrm{T}}$ and $a_{ij}=K_i G(r_i,r_j)-\delta_{ij}$. A nontrivial solution of equation (7) exists only if the following condition is fulfilled

$$\det \boldsymbol{A}=0 \tag{8}$$

Equation (8) is a characteristic equation of the problem. This equation is then solved numerically with respect to the natural frequencies Ω.

3. Green's functions

We use the Green's function G to obtain the solution of problem (2) with boundary conditions (3) or (4) or (5). This function satisfies the below equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}G(r,\rho)\right)\right]\right\}-\Omega^4 G(r,\rho)=\delta(r-\rho) \tag{9}$$

The function G can be represented as a sum

$$G(r,\rho)=G_0(r,\rho)+G_1(r,\rho)H(r-\rho) \tag{10}$$

where G_0 is a general solution of the homogeneous equation (9) and $G_1(r,\rho)H(r-\rho)$ is a particular solution of equation (9). At the same time the function G_1 satisfies the following conditions:

$$G_1\Big|_{r=\rho}=\frac{\partial G_1}{\partial r}\Big|_{r=\rho}=\frac{\partial^2 G_1}{\partial r^2}\Big|_{r=\rho}=0 \quad \text{oraz} \quad \frac{\partial^3 G_1}{\partial r^3}\Big|_{r=\rho}=1 \tag{11}$$

It may be proved that the function G_1 is a solution of the homogeneous equation (9). The general solution of this equation has the form [4]

$$G_1(r,\rho)=c_1 J_0(r\Omega)+c_2 I_0(r\Omega)+c_3 Y_0(r\Omega)+c_4 K_0(r\Omega) \tag{12}$$

Constants c_1, c_2, c_3, c_4 are calculated with the use of conditions (11). Transforming (12), one obtains

$$G_1(r) = \frac{\rho}{2\Omega^2}\Big(I_0(r\Omega)K_0(\rho\Omega) - I_0(\rho\Omega)K_0(r\Omega) + \frac{\pi}{2}\big(J_0(r\Omega)Y_0(\rho\Omega) - J_0(\rho\Omega)Y_0(r\Omega)\big)\Big) \tag{13}$$

Function G_0 from equation (10) can be presented in the form

$$G_0(r,\rho) = C_1 J_0(r\Omega) + C_2 I_0(r\Omega) \tag{14}$$

Constants C_1 and C_2 are calculated with the use of equations (10) (13), (14) and boundary conditions (3) or (4) or (5). Introducing the notations: $\bar{I}_0 = I_0(a\Omega)$, $\bar{J}_0 = J_0(a\Omega)$, $\bar{K}_0 = K_0(a\Omega)$, $\bar{Y}_0 = Y_0(a\Omega)$, $\bar{\bar{I}}_0 = I_0(\rho\Omega)$, $\bar{\bar{J}}_0 = J_0(\rho\Omega)$, $\bar{\bar{K}}_0 = K_0(\rho\Omega)$ and $\bar{\bar{Y}}_0 = Y_0(\rho\Omega)$, the constants C_1 and C_2 in the considered cases of boundary conditions can be written in the form:

– for a clamped plate:

$$C_1 = \frac{\pi\ \rho}{4\Omega^2}\left(-\bar{\bar{Y}}_0 + \frac{1}{M}\left(\frac{2}{\pi a\Omega}\bar{\bar{I}}_0 + \bar{\bar{J}}_0\left(\bar{I}_1\bar{Y}_0 + \bar{I}_0\bar{Y}_1\right)\right)\right)$$

$$C_2 = \frac{\rho}{2\Omega^2}\left(-\bar{\bar{K}}_0 + \frac{1}{M}\left(\frac{1}{a\Omega}\bar{\bar{J}}_0 + \bar{\bar{I}}_0\left(\bar{J}_1\bar{K}_0 + \bar{J}_0\bar{K}_1\right)\right)\right)$$

where $M = \bar{I}_1\bar{J}_0 + \bar{I}_0\bar{J}_1$;

– for a simply supported plate:

$$C_1 = \frac{\pi\ \rho}{4\Omega^2}\left(-\bar{\bar{Y}}_0 + \frac{1}{M}\left(\frac{2(\nu-1)}{\pi a\Omega}\bar{\bar{I}}_0 + \bar{\bar{J}}_0\left(2a\Omega\bar{I}_0\bar{Y}_0 + (\nu-1)\left(\bar{I}_1\bar{Y}_0 + \bar{I}_0\bar{Y}_1\right)\right)\right)\right)$$

$$C_2 = \frac{\rho}{2\Omega^2}\left(-\bar{\bar{K}}_0 + \frac{1}{M}\left(\frac{\nu-1}{a\Omega}\bar{\bar{J}}_0 + \bar{\bar{I}}_0\left(2a\Omega\ \ \bar{J}_0\bar{K}_0 + (\nu-1)\left(\bar{J}_1\bar{K}_0 + \bar{J}_0\bar{K}_1\right)\right)\right)\right)$$

where $M = 2a\Omega\bar{I}_0\bar{J}_0 + (\nu-1)\left(\bar{I}_1\bar{J}_0 + \bar{I}_0\bar{J}_1\right)$;

– for a free plate:

$$C_1 = \frac{\pi\ \rho}{4\Omega^2}\left(-\bar{\bar{Y}}_0 - \frac{1}{M}\left(\frac{2}{\pi}\bar{\bar{I}}_0 - \bar{\bar{J}}_0\left(2(\nu-1)\bar{I}_1\bar{Y}_1 + a\Omega\left(\bar{I}_1\bar{Y}_0 + \bar{I}_0\bar{Y}_1\right)\right)\right)\right)$$

$$C_2 = \frac{\rho}{2\Omega^2}\left(-\bar{\bar{K}}_0 - \frac{1}{M}\left(\bar{\bar{J}}_0 + \bar{\bar{I}}_0\left(2\ \ (\nu-1)\bar{J}_1\bar{K}_1 - a\Omega\left(\bar{J}_1\bar{K}_0 - \bar{J}_0\bar{K}_1\right)\right)\right)\right)$$

where $M = 2(\nu-1)\bar{I}_1\bar{J}_1 + a\Omega\left(\bar{I}_1\bar{J}_0 + \bar{I}_0\bar{J}_1\right)$.

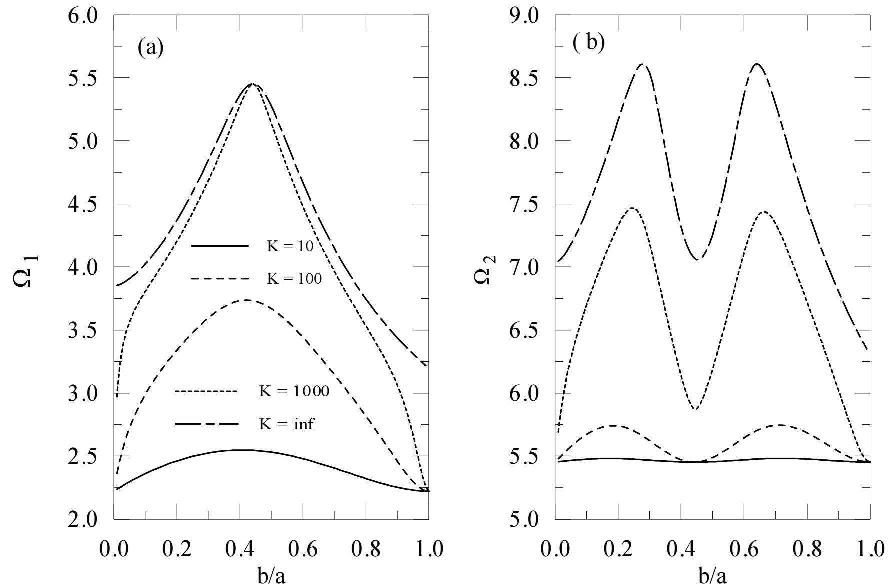

Fig. 1. Frequency parameter values Ω_i for the first two modes of axisymmetric vibration of a simply supported circular plate with an elastic ring as functions of the ratio *b*/*a* for various values of the stiffness coefficient *K*

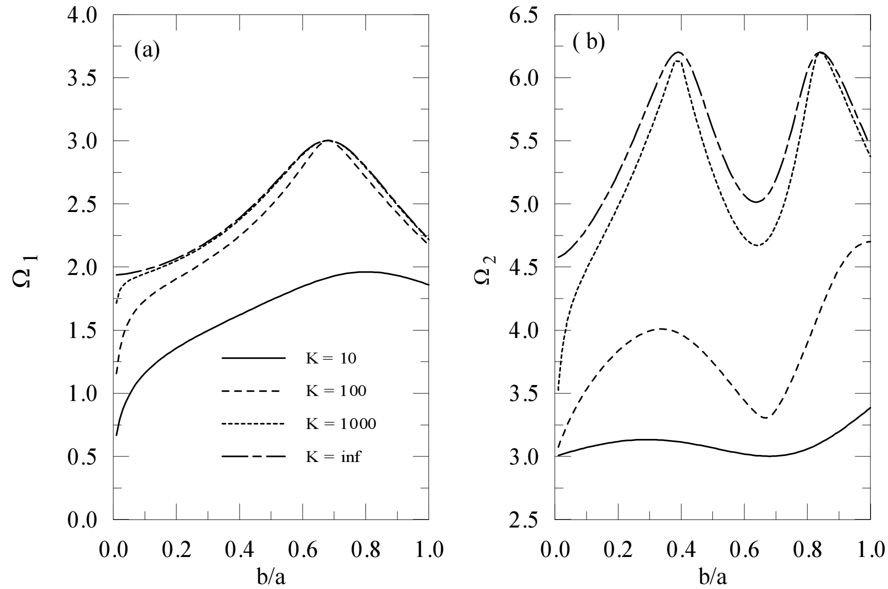

Fig. 2. As Figure 1, but for a free circular plate

4. Numerical example

A numerical example is presented here to demonstrate the Green's function method. Numerical calculations deal with free vibrations of circular plates with one supporting elastic ring. In this case the frequency equation (8) takes the form

$$KG(b,b;\Omega)+1=0 \tag{15}$$

where b is the radius of the elastic ring. Frequency parameter values Ω_i for the first two modes of axisymmetric vibration of circular plates with an elastic ring are shown on Figures 1 and 2 as functions of the ratio b/a for various values of the stiffness coefficient K. The calculations was performed for a simply supported plate (Fig. 1) and a free plate (Fig. 2). The Poisson ratio was assumed as ν = 0.3.

The results have shown that both the radius of the supporting ring and the stiffness coefficient of the support have a significant effect on free vibration frequencies of the plate.

5. Conclusions

The exact solution of the free vibration problem of a uniform plate supported by elastic concentric rings has been obtained with the application of the Green's function method. The Green's functions corresponding to the plate were derived to three cases of boundary conditions. Although the numerical examples deal with free vibration of plates with one supporting ring, the solution can be used to analyze the vibration of any circular plate with an arbitrary number of ring supports.

References

[1] Wang C.Y., Wang C.M., Fundamental frequencies of circular plates with internal elastic ring support, Journal of Sound and Vibration 2003, 263, 1071-1078.

[2] Azimi S., Free vibration of circular plates with elastic edge supports using the receptance method, Journal of Sound and Vibration 1988, 120(1), 19-35.

[3] Azimi S., Free vibration of circular plates with elastic or rigid interior support, Journal of Sound and Vibration 1988, 120(1), 37-52.

[4] Laura P.A.A., Gutiérrez R.H., Vera S.A., Vega D.A., Transverse vibrations of circular plate with a free edge and a concentric circular support, Journal of Sound and Vibration 1999, 223(5), 843-845.

[5] Gupta U.S., Jain S.K., Jain D., Method of collocation by derivatives in the study of axisymmetric vibrations of circular plates, Computers & Structures 1995, 57, 841-845.

[6] Liew K.M., Lam K.Y., Transverse vibration of solid circular plates continuous over multiple concentric annular supports, ASME Journal of Applied Mechanics 1993, 60, 208-210.