DISTRIBUITION OF MANHATTAN DISTANCE IN SQUARE AND TRIANGULAR LATTICES

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Abstract. This paper presents exact formulas to find the number of Manhattan distances in square and triangular lattices (with the step $\delta = 1$) as a function of number of nodes, L. In the limit of $L \to \infty$ and $\delta \to 0$, we provide probability density functions for distances in unit squares and triangles. These formulas are useful in the fields of statistical physics and computer science.

The geometrical properties of networks have attracted much attention due to progress in the fields of computer science, mathematical biology and statistical physics. In papers recently published in *Journal of Physics A: Math. Gen.* the authors examined different problems such as the optimal shape of a city [1], properties of polymers on directed lattices [2] or quantum localization problems in the context of a network model of disordered superconductors embedded on the Manhattan lattice [3]. Much attention has been dedicated to the statistics of random walk path and end-to-end distance distributions on regular networks [4-6].

The common question of the above mentioned problems is how many pairs of points separated by a given number (q) of steps can be found in a bounded region of a regular lattice. Such number q is referred to as the so-called Manhattan distance. For a square lattice the Manhattan distance is defined as the sum of the horizontal and the vertical distance (see Figure 1). Similarly, for the triangular lattice we can define the Manhattan distance as the sum of the distances along directions parallel to the edges of the triangle. This paper focuses on geometry but the knowledge of the number of Manhattan distances in a particular lattice can be useful for studying many quantities of physical importance.

First, we consider the square lattice. From Figure 1 it is easy to see that the number of two-point segments A-A separated by a given length q measured in steps δ, is equal to

$$2 \times \sum_{j=0}^{j=q-1} (N - q + j)(N - j) \tag{1}$$

Multiplication by 2 in above equation comes from segments obtained by a 90° counter-clockwise rotation of the A-A segments. The number of B-B segments is equal to

$$2 \times \sum_{j=1}^{j=p+1} j(p-j+2), \quad \text{with } q = 2(N-1) - p \tag{2}$$

where an auxiliary quantity $p = 0,1, \ldots , N-2$ measures the distance between the right end of the B-B segment and the top-right corner of the square. After adding Equations (1) and (2), we obtain the following expression for the number of distances q on the square

$$\Delta_s(q) = \begin{cases} 2N(N-q)q + \frac{1}{3}(q-1)q(q+1) & , \text{ for } q = 1,2,\ldots N-1 \\ \frac{1}{3}(2N-q-1)(2N-q)(2N-q+1) & , \text{ for } q = N, N+1, \ldots 2N-2 \end{cases} \tag{3}$$

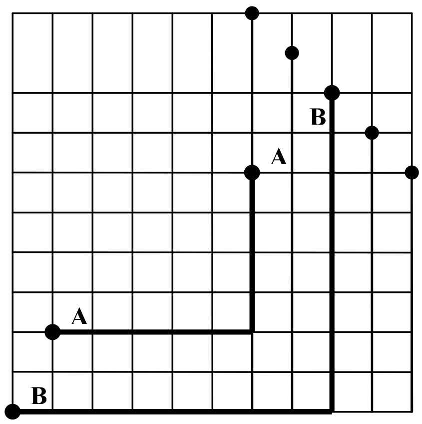

Fig. 1. A-A, B-B are pairs of points in a square lattice with $N = 11$. The Manhattan distances: $1 \le q(A,A) < N$ and $N \le q(B < B) < 2N - 2$

Note, that with the help of the normalization condition

$$\sum_{q=1}^{q=2N-2} \Delta_s(q) = \frac{1}{2} N^2 (N^2 - 1) \tag{4}$$

Equation (3) can be written in the form of a probability distribution function for the discrete sets of distances $x_q = q/N$ in the unit square grid with the step $\delta = 1/N$. In the limit of $N \to \infty$ we get a continuous limit with the following density function

$$D_s(x) = \begin{cases} 4x(1-x) + \frac{2}{3}x^3 & \text{for } 0 \le x \le 1 \\ \frac{2}{3}(2-x)^3 & \text{for } 1 \le x \le 2 \end{cases} \tag{5}$$

Similarly, we derive $\Delta(q)$ and $\Delta(x)$ for the triangular lattice:

$$\Delta_t(q) = \begin{cases} \frac{3}{2} q(N-q)(N-q+1) \text{ for } q = 1,2,\dots N-1 \end{cases} \tag{6}$$

and

$$D_t(x) = 12x(1-x)^2,\ 0 \le x \le 1 \tag{7}$$

Subscripts *s* and *t* stand for square and triangular lattices, respectively.
From Equations (5) and (7) it is easy to calculate moments $\int_{\Re} x^k D(x)dx$ of the corresponding densities $D_s(x)$ and $D_t(x)$. They are given by the following equations:

$$m_s^{(k)} = \frac{2^{k+6} - 8(k+5)}{(k+1)(k+2)(k+3)(k+4)} \tag{8}$$

$$m_t^{(k)} = \frac{24}{(k+2)(k+3)(k+4)} \tag{9}$$

Thus, in the case of the square, the moments diverge, $m_s^{(k)} \to \infty$ with $k \to \infty$, and they asymptotically decay for the triangle $m_t^{(k)} \to \infty$. The mean distance and the variance $\sigma^2 = m^{(2)} - (m^1)^2$ are equal to:

$$m_s^1 = \frac{2}{3},\quad \sigma_s^2 = \frac{1}{9} \tag{10}$$

$$m_t^1 = \frac{2}{5},\quad \sigma_t^2 = \frac{1}{25} \tag{11}$$

In the different way, the same distance value $m_s^{(1)} = 2/3$ for the square lattice, was obtained in [1].

It is interesting to note that Equations (5) and (7) give the distribution of distances between two consecutive steps of a random walker allowed to jump to any point within the unit square or unit triangle, whereas the distribution of distances between this walker and a given fixed corner of his walking area is equal to $d_s(x) = 1 - |1-x|, 0 \le x \le 2$ and $d_t(x) = x, 0 \le x \le 1$ for the square and the triangular lattices, respectively.

In conclusion, we have derived the probability density functions for the Manhattan distance within the square and triangular geometries. We have also calculated the moments of these distributions and found that for the triangular lattice the moments asymptotically vanish whereas for the square lattice they diverge.

The probability density functions obtained for square and triangular geometries give the probability weight of class q containing pairs of points with given distance q. Thus, they may contain valuable information related to the directed walk models, such as Dyck or Motzkin paths [7].

References

[1] Bender C.M. et al., J. Phys. A: Math. Gen. 2004, 37, 147-159.
[2] Janse van Rensburg E.J., J. Phys. A: Math. Gen. 2003, 36, R11-R61.
[3] Bearnond E.J., Owczarek A.L., Cardy J., J. Phys. A: Math. Gen. 2003, 36, 10251-10267.
[4] Everaers R., New Journal of Physics 1999, 1, 12.1-12.54.
[5] Jensen J., J. Phys. A: Math. Gen. 2004, 37, 5503-5524.
[6] Knieževič M., Knieževič D., Spasojevič D., J. Phys. A: Math. Gen. 2004, 37, 1-8.
[7] Orlandini E., Whittington S.G., J. Phys. A: Math. Gen. 2004, 37, 5305-5314.